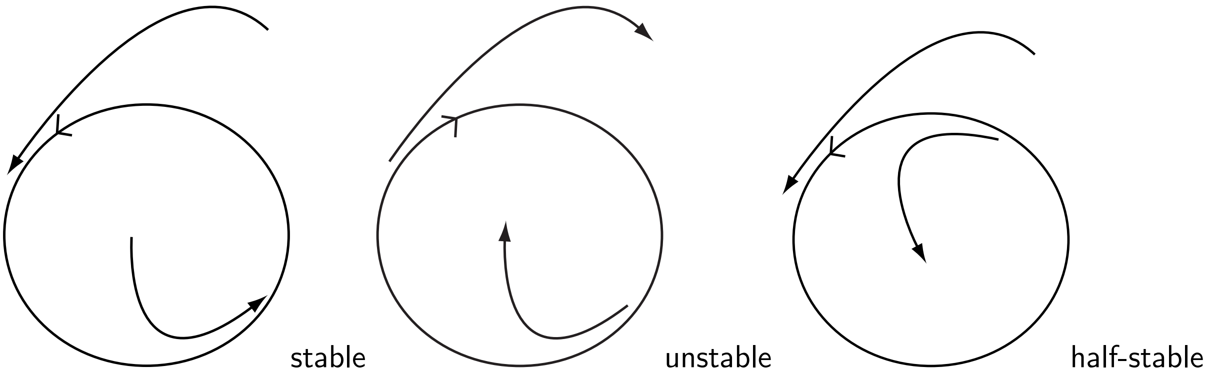
2 Nonlinear oscillations

2.1 Limit cycles in $\mathbb{R}^2$

A limit cycle is an isolated closed trajectory.



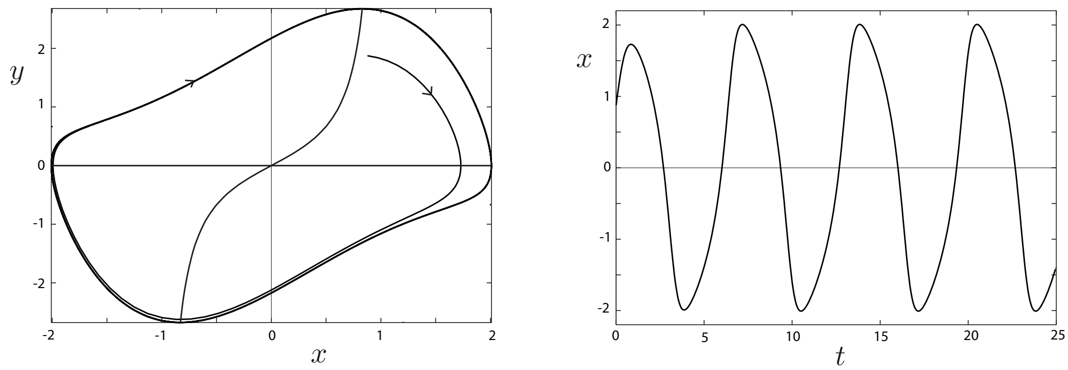
Limit cycles are often found in models that exhibit self-sustained oscillations. There are many example throughout the applied sciences: beating of a heart, chemical reactions, daily (circadian) rhythms in human body temperature and hormone secretion, dangerous self-excited oscillations in bridges (Takoma Narrows) and airplane wings.

Limit cycles are inherently a nonlinear phenomenon — a linear system can have closed orbits but they are not isolated (instead they foliate the phase-plane).

Example 1. Van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$\mu \geq 0$ is a parameter. Historically, this equation arose in connection with the nonlinear electrical circuits used in the first radios. This equation looks like a simple harmonic oscillator, but with a nonlinear damping term $\mu(x^2 - 1)\dot{x}$. This term acts like ordinary positive damping for $|x| > 1$, but like negative damping for $|x| < 1 \Rightarrow$ it causes large-amplitude oscillations to decay, but it pumps them back up if they become too small.

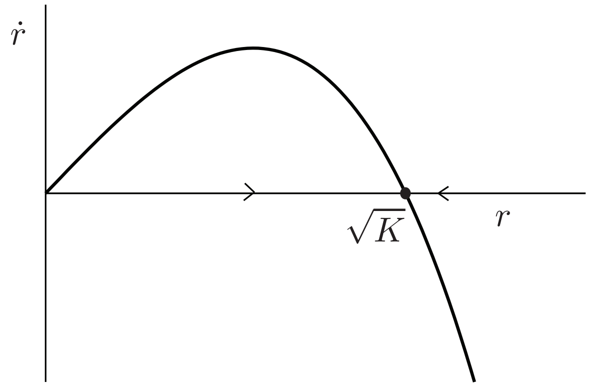


The van der Pol equation has a unique, stable limit cycle for each $\mu > 0$.

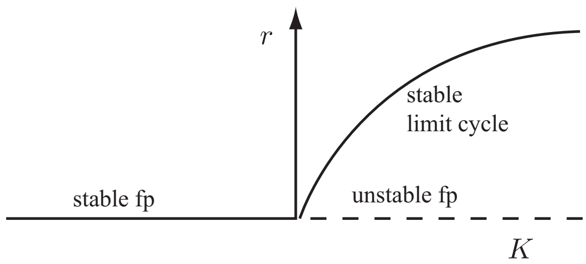
Example 2. Consider the following 2D system in polar coordinates

$$\boxed{\dot{r} = r(K - r^2), \qquad \dot{\theta} = 1}$$

For $K > 0$ there exists an unstable fixed point at the origin and a stable limit cycle at $r = \sqrt{K}$.



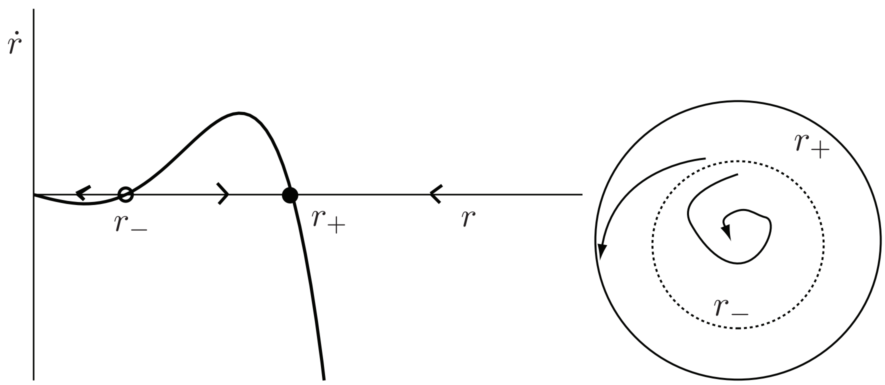
For $K < 0$ there is no limit cycle but only a stable fixed point at the origin.
We can represent qualitative change in the dynamics as the parameter K varies by a **bifurcation diagram**.



This is an example of a **super-critical Hopf bifurcation**. Since an arbitrarily small perturbation of the origin will produce a self-sustained oscillation the system is said to exhibit a **soft excitation**.

Example 3. Now consider the following example

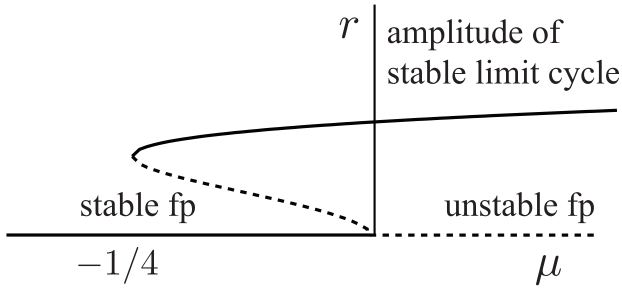
$$\boxed{\dot{r} = \mu r + r^3 - r^5, \qquad \dot{\theta} = 1}$$



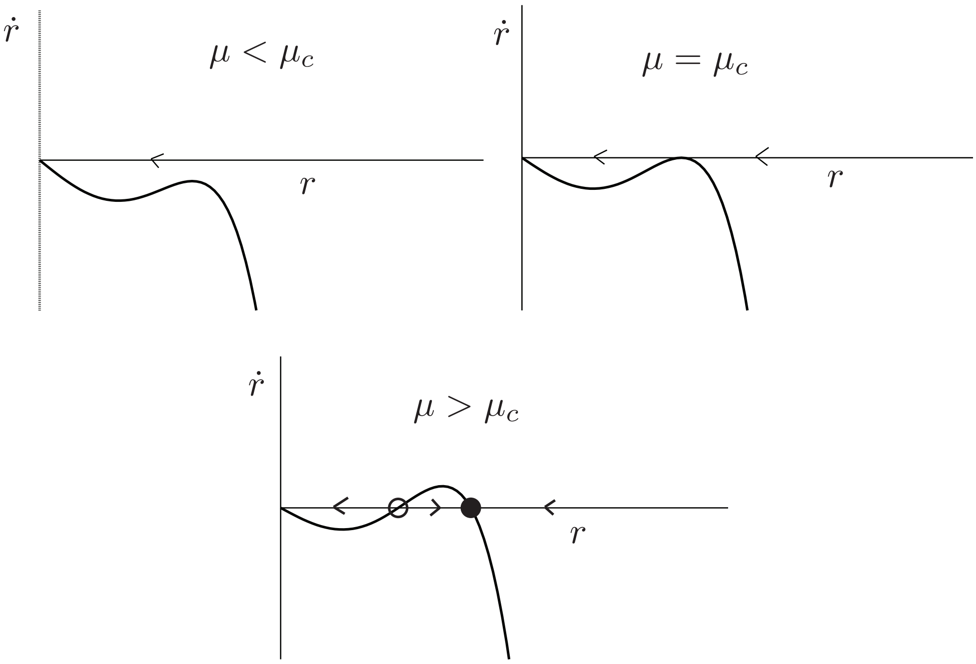
For $\mu < 0$ there exists a stable fixed point at the origin, an unstable limit cycle at $r = r_-$ and a stable limit cycle at $r = r_+$ where

$$r_\pm^2 = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\mu}\right]$$

For $\mu > 0$ there exists an unstable fixed point at the origin and a stable limit cycle at $r = r_+$.



At $\mu = 0$ there is a **sub-critical Hopf bifurcation**. The system also exhibits hysteresis; once large amplitude oscillations have begun they cannot be turned off by bringing μ back to zero. In fact they persist until $\mu = -1/4$ where the stable and unstable cycles collide and annihilate in a **saddle-node bifurcation** of limit cycles.



When the system passes through the sub-critical Hopf bifurcation it jumps from the fixed point to a large amplitude oscillation — termed a **hard excitation**. This is potentially dangerous in engineering applications.