Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

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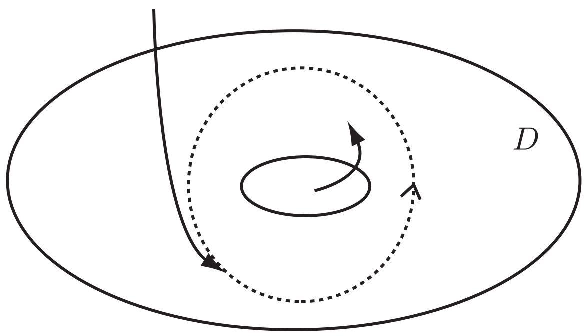
2.2 Poincaré-Bendixson Theorem

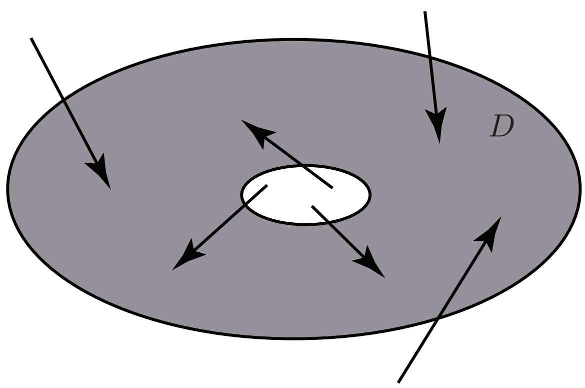
It is generally difficult to establish the existence of a limit cycle. In 2-D one has the following useful theorem:

Theorem: Suppose that there exists a bounded region D of phase-space such that any trajectory entering D cannot leave D. If there are no fixed points in D then there exists at least one periodic orbit in D.

Typically, D will be an annular region with an unstable focus or node in the hole in the middle (so trajectories enter the inner boundary) and all trajectories cross the outer boundary inwards.



The standard trick to apply the Poincaré-Bendixson theorem is to construct a *trapping region* D, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of D.



The Poincaré-Bendixson theorem tells us that the dynamics of planar systems is severely limited — if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. There is no CHAOS for planar systems!

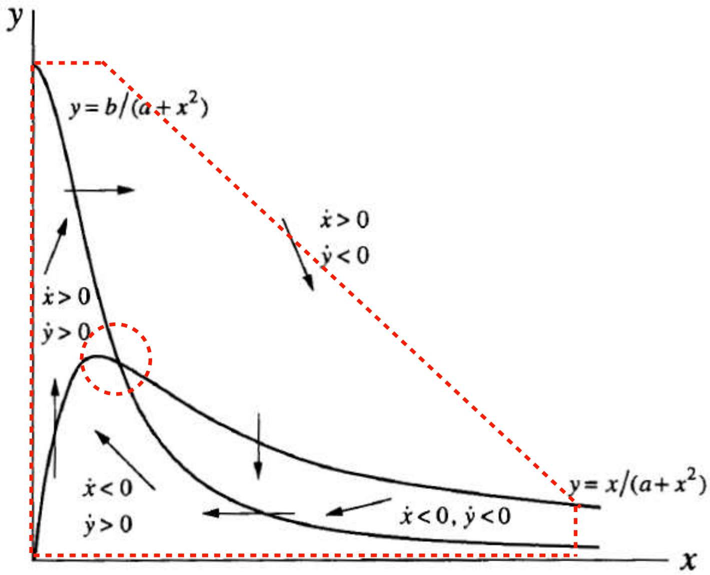
In higher-dimensional systems (in $\mathbb{R}^n,\ n \geq 3$) the Poincaré-Bendixson theorem no longer applies and trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a *strange attractor*.

Example 1. In a fundamental biochemical process called glycolysis, living cells obtain energy by breaking down sugar. In yeast cells, for example, glycolysis can proceed in an oscillatory fashion, with concentrations of intermediate products varying periodically. A model of this process is given by

$$\dot{x} = -x + ay + x^2 y \equiv f(x, y)$$
$$\dot{y} = b - ay - x^2 y \equiv g(x, y)$$

where x and y are concentrations of ADP (adenosine phosphate) and F(6)P (Fructose-6 phosphate) and $a, b > 0$ are kinetic parameters. Construct a trapping region for this system.

Solution: First find the nullclines ($f(x, y) = 0 = g(x, y)$)



and then show that all trajectories are *inwards* in some region. To construct the bounding region consider large x and y. Then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$, so $\mathrm{d}y/\mathrm{d}x \approx -1$ along trajectories. Hence, the vector field at large x is parallel to the diagonal, which suggests comparing the sizes of $\dot{x}$ and $-\dot{y}$. So, consider

$$\dot{x} - (-\dot{y}) = -x + ay + x^2 y + (b - ay - x^2 y) = b - x$$

Hence $-\dot{y} > \dot{x}$ if $x > b$. This implies that the vector field points inward on the diagonal line (of the above figure) because $\mathrm{d}y/\mathrm{d}x$ is more negative than -1 and therefore the vectors are steeper than the diagonal — we have a trapping region! We must now find under those conditions which make the fixed point unstable (so as to repel orbits). Linearisation:

$$A = \begin{bmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{bmatrix}$$

Fixed point

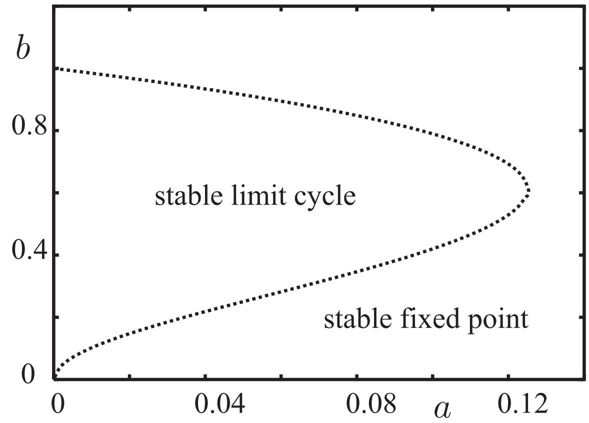
$$\overline{x} = b, \qquad \overline{y} = \frac{b}{a + b^2}$$

Determinant $\det A = a + b^2 > 0$ and

$$\mathrm{Tr}\, A = -\frac{b^4 + (2a - 1)b^2 + (a + a^2)}{a + b^2}$$

Hence the fixed point is unstable for Tr $A > 0$ and stable for Tr $A < 0$. The border of stability Tr $A = 0$ occurs when

$$b^2 = \frac{1}{2}\left[1 - 2a \pm \sqrt{1 - 8a}\right]$$



Numerical integration shows that there is one stable limit cycle in the parameter regime which guarantees an unstable fixed point.

2.3 Relaxation oscillators

Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

for the special case that $\mu \gg 1$ (strongly nonlinear limit). Using

$$\ddot{x} + \mu(x^2 - 1)\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x} + \mu(x^3/3 - x)\right]$$

and introducing

$$F(x) = \frac{x^3}{3} - x, \qquad w = \dot{x} + \mu F(x)$$

we may write

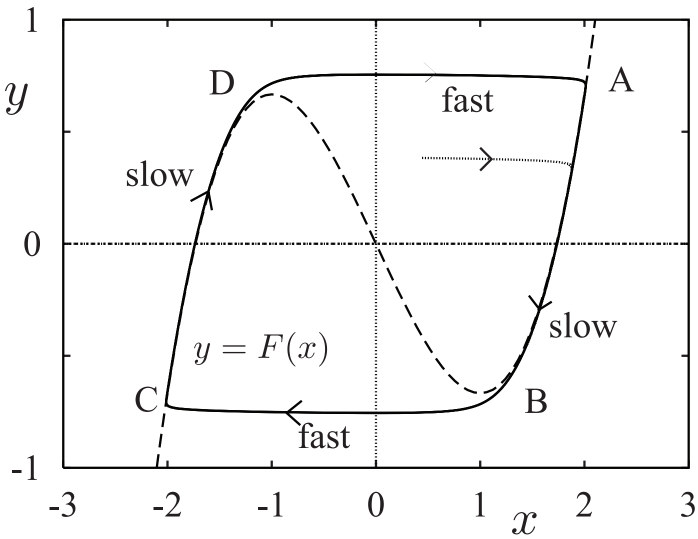
$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x$$

Hence, the van der Pol system has a planar representation:

$$\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x \end{aligned}$$

With the re-scaling $y = w/\mu$ we have

$$\begin{aligned} \dot{x} &= \mu[y - F(x)] \\ \dot{y} &= -\frac{1}{\mu}x \end{aligned}$$



Suppose that the initial condition is not too close to the cubic nullcline, i.e. $y - F(x) \sim O(1)$. Then $|\dot{x}| \sim O(\mu) \gg 1$ and $|\dot{y}| \sim O(1/\mu) \ll 1$; hence the velocity is large in the horizontal direction and small in the vertical direction, so trajectories move horizontally. If the initial condition is above the cubic nullcline then $y - F(x) > 0$ so $\dot{x} > 0$; the trajectory moves sideways towards the right-hand branch of the nullcline. Once the trajectory gets so close that $y - F(x) \sim O(1/\mu^2)$, then $\dot{x}$ and $\dot{y}$ become comparable (both being $O(1/\mu)$). The trajectory crosses the nullcline vertically (see the figure) and then moves slowly down the branch with a velocity $O(1/\mu)$, until it reaches the knee and can jump sideways.

The system has two widely separated time scales. The jumps take a time $O(1/\mu)$ and the crawls a time $O(\mu)$. The period of oscillation can be approximated by the time spent on the slow branches:

$$T \approx \int_{t_A}^{t_B} \mathrm{d}t + \int_{t_C}^{t_D} \mathrm{d}t = 2\int_{t_A}^{t_B} \mathrm{d}t \quad \text{by symmetry}$$

On the slow branch $y = F(x)$ so

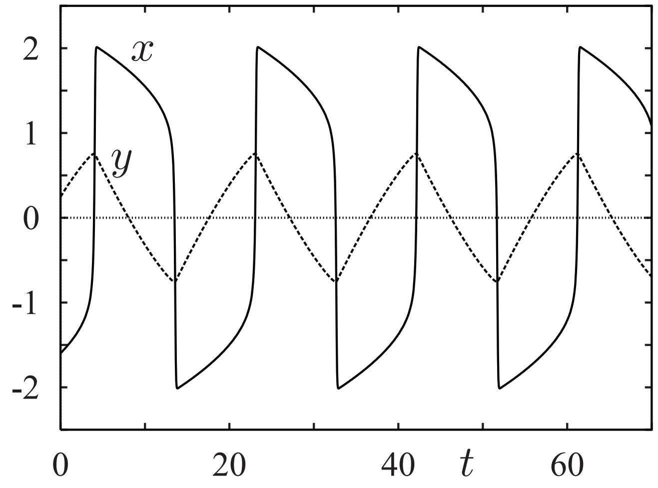
$$\dot{y} \approx \frac{\mathrm{d}y}{\mathrm{d}x}\dot{x} = F'(x)\dot{x} = (x^2 - 1)\dot{x}$$

Using $\dot{y} = -x/\mu$ we have that $\dot{x} = -x/[\mu(x^2 - 1)]$, so

$$\mathrm{d}t \approx -\frac{\mu(x^2 - 1)}{x}\mathrm{d}x$$

Now $x_A = 2$ and $x_B = 1$ (check this for yourselves) so

$$T \approx 2\int_2^1 -\frac{\mu}{x}(x^2 - 1)\mathrm{d}x = \mu[3 - 2\ln 2]$$

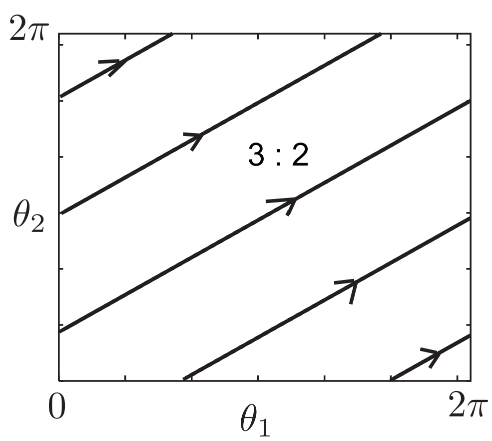


2.4 Coupled oscillators

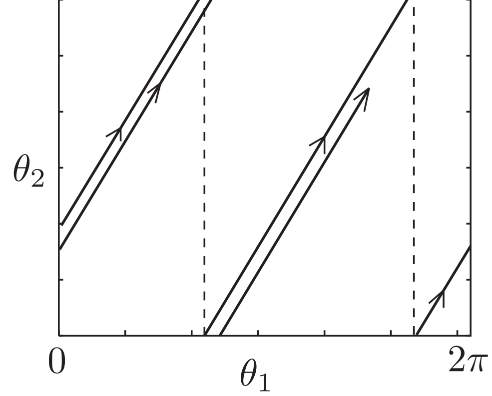
Consider the model

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

In the uncoupled state ($K_1 = K_2 = 0$) we have $\theta_1(t) = \theta_1(0) + \omega_1 t$ and $\theta_2(t) = \theta_2(0) + \omega_2 t$ such that $\mathrm{d}\theta_2/\mathrm{d}\theta_1 = \omega_2/\omega_1$. If the slope is rational, $\omega_2/\omega_1 = p/q$, $p, q \in \mathbb{Z}$, then all trajectories lie on closed orbits of the torus (with coords (θ_1, θ_2)).



For irrational slopes the flow is said to be **quasiperiodic**. Each trajectory is dense on the torus (i.e. comes arbitrarily close to any given point).



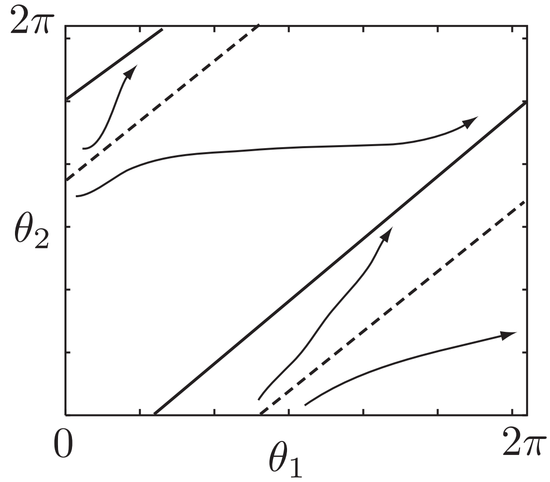
Introducing $\phi = \theta_1 - \theta_2$ the coupled system takes the form

$$\dot{\phi} = \omega_1 - \omega_2 - (K_1 + K_2)\sin\phi$$

There are two fixed points if $|\omega_1 - \omega_2| < K_1 + K_2$, defined by $\sin\phi^* = (\omega_1 - \omega_2)/(K_1 + K_2)$, and a saddle-node (tangent) bifurcation occurs when $|\omega_1 - \omega_2| = K_1 + K_2$. In this case $\dot{\phi} = 0$ so that $\dot{\theta}_1 = \dot{\theta}_2 = \text{constant} = \omega^*$, where

$$\omega^* = \omega_2 + K_2 \sin\phi^* = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$$

We may regard ω^* as a co-operative frequency that is an emergent property of the coupled system. When no-cooperative frequency can be established the two oscillators cannot phase-lock (although they may still frequency lock).

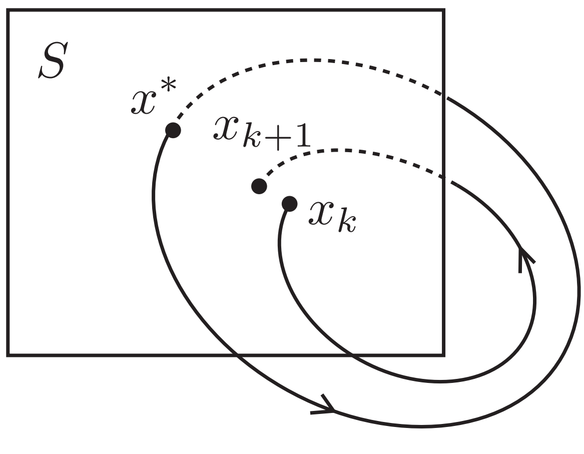


2.5 Poincaré maps

Poincaré maps are useful for studying the flows near a periodic orbit. Consider an n-dimensional system

$$\dot{x} = f(x).$$

Let S is an $n-1$ dimensional surface of section. S is required to be transverse to the flow, i.e. all trajectories starting on S flow through it (not parallel to it).



The **Poincaré map** is a mapping from S to itself, obtained by following trajectories from one intersection with S to the next. If $x_k \in S$ denotes the kth intersection, then the Poincaré map is defined by

$$x_{k+1} = P(x_k).$$

Suppose that x^* is a fixed point of P, i.e. $P(x^*) = x^*$. Then a trajectory starting at x^* returns to x^* after some time T, and is therefore a closed orbit for the original system $\dot{x} = f(x)$.

Linear stability of limit cycle

Consider a system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n,$$

with a closed orbit. To ask whether the orbit is stable or not, we ask whether the corresponding fixed point x^* of the Poincaré map is stable. Consider $x^* + v_0$ in S, where v_0 is a perturbation. Then after the first return to S

$$x^* + v_1 = P(x^* + v_0) = P(x^*) + [DP(x^*)]v_0 + \text{small terms},$$

where $DP(x^*)$ is an $(n-1) \times (n-1)$ matrix called the **linearised Poincaré map** at x^*. Since $x^* = P(x^*)$, we have

$$v_1 = DP(x^*)v_0.$$

The stability criterion is expressed in terms of the eigenvalues λ_j of $DP(x^*)$: The closed orbit is linearly stable if and only if $|\lambda_j| < 1$ for all $j = 1, ..., n-1$.

(From the expression $v_k = \sum_{j=1}^{n-1} c_j(\lambda_j)^k e_j$, where e_j - eigenvectors and c_j - some scalars).
λ_j are called the **characteristic** or **Floquet multipliers** of the periodic orbit. In general, the characteristic multipliers can only be found by numerical integration.

Example.

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

Let S be the positive x-axis. Compute the Poincaré map and show that the system has a unique periodic orbit and determine its stability. Find the characteristic multipliers for the limit cycle.

Let r_0 be the initial condition on S. Since $\dot{\theta} = 1$, the first return to S occurs after a period $T = 2\pi$. Then $r_1 = P(r_0)$ where

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r(1-r^2)} = \int_0^{2\pi} \mathrm{d}t = 2\pi$$

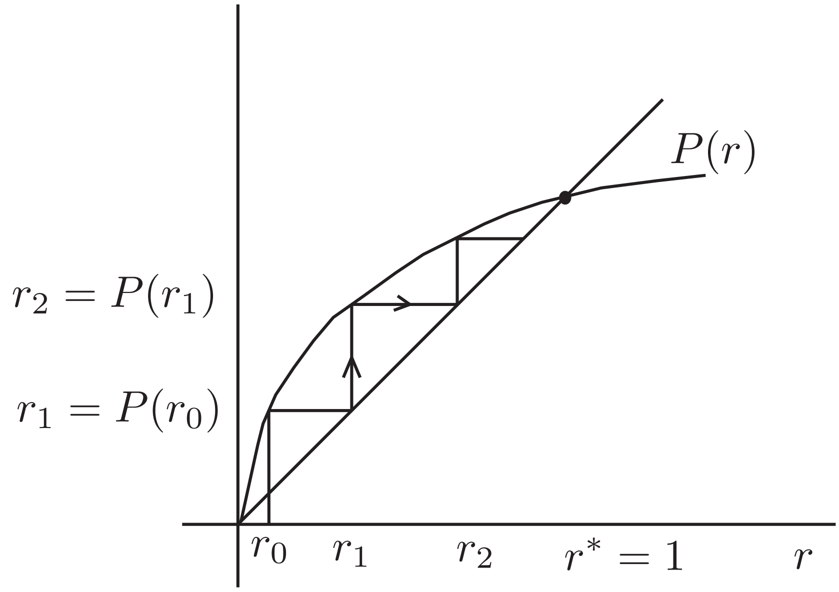
so

$$r_1 = \left[1 + \mathrm{e}^{-4\pi}\left(\frac{1}{r_0^2} - 1\right)\right]^{-1/2}$$

Therefore

$$P(r) = \left[1 + \mathrm{e}^{-4\pi}\left(\frac{1}{r^2} - 1\right)\right]^{-1/2}$$

We can show graphically that P has a unique stable fixed point at $r^* = 1$.



$$\frac{\mathsf{d}P(r)}{\mathsf{d}r} = \mathsf{e}^{-4\pi} r^{-3}[1 + \mathsf{e}^{-4\pi}(r^{-2} - 1)]^{-3/2} \qquad \Rightarrow$$

$$\left.\frac{\mathsf{d}P(r)}{\mathsf{d}r}\right|_{r^*=1} = \mathsf{e}^{-4\pi} \qquad \text{– Floquet multiplier}$$

$|\mathsf{e}^{-4\pi}| < 1 \qquad \Rightarrow \qquad$ the closed orbit is linearly stable.