Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

1.5 Nonlinear systems in $\mathbb{R}^2$ (in $\mathbb{R}^n$)

We shall consider equations of the form

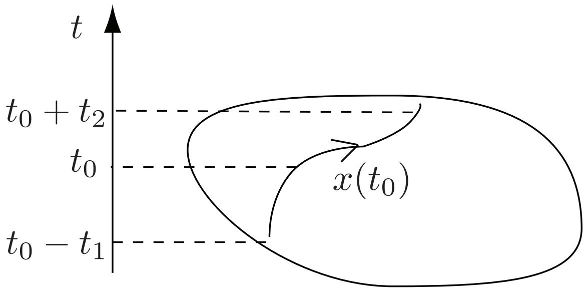
$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2).$$

This system can be wirtten in vector notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x}(x_1, x_2)$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$. $\mathbf{x}$ represents a point in the phase plane, and $\dot{\mathbf{x}}$ is the velocity vector at that point.

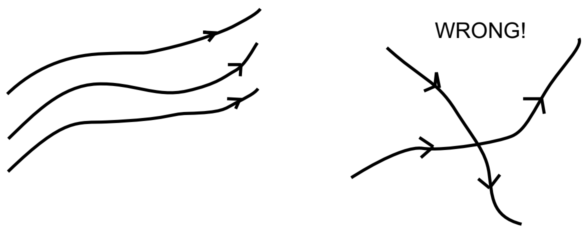
Existence and uniqueness theorem (in $\mathbb{R}^n$): Suppose $\dot{x} = f(x)$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable (i.e. $\partial f_i/\partial x_j$, $i,\ j = 1, ..., n$ exist and are continuous for all x). Then there exits $t_1 > 0$ and $t_2 > 0$ such that the solution with $x(t_0) = x_0$ exists and is unique for all $t \in (t_0 - t_1, t_0 + t_2)$.



Phase-space and flows. Refer to local solution through x_0 as a *solution curve* or *trajectory*. Suppose that $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$. We define a flow $\phi(x, t)$ such that $\phi(x, t)$ is the solution of the ODE at time t with initial value x_0 at $t = 0$. The solution $x(t)$ with $x(0) = x_0$ is now written as $\phi(x_0, t)$

$$\frac{\mathrm{d}\phi(x,t)}{\mathrm{d}t} = f(\phi(x,t)), \qquad \phi(x,0) = x_0$$

By varying initial condition x_0 we generate a family of trajectories called the *flow* generated by Φ.



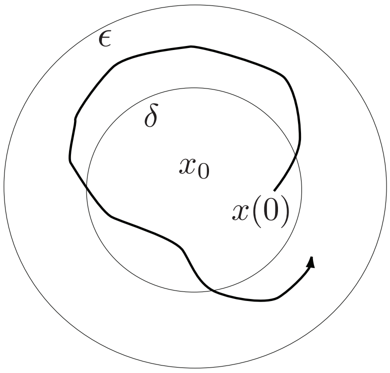
Note that uniqueness imples that trajectories cannot cross.

An **equilibrium** or fixed point satisfies $\Phi(x, t) = x$ for all t. Thus $f(x) = 0$. An important feature of nonlinearities is that there can exist more than one (isolated) fixed point.

Stability

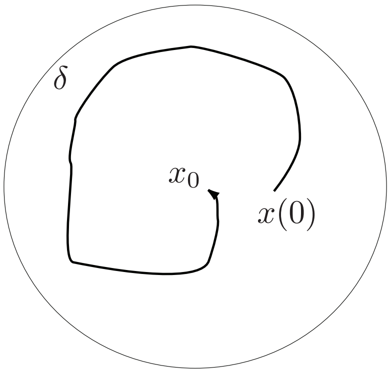
A fixed point x_0 is an attracting fixed point if all trajectories that start near x_0 approach it as $t \to \infty$. If x_0 attracts all trajectories it is called globally attracting.

A fixed point x_0 is **Lyapunov** (neutrally) stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x(0) - x_0| < \delta$ implies that $|x(t) - x_0| < \epsilon$ for all $t > 0$.



In other words, if a solution starts near an equilibrium x_0 then it stays near x_0 (for example harmonic oscillator).

A fixed point is **asymptotically stable** if it is Lyapunov stable and there exists $\delta > 0$ such that if $|x(0) - x_0| < \delta$ then $|x(t) - x_0| \to 0$ as $t \to \infty$.



Linearisation

Consider the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y)$$

and suppose that (x^*, y^*) is a fixed point. Considering a small disturbance from the fixed point

$$u = x - x^*, \qquad v = y - y^*$$

we have (by Taylor series expansion)

$$\dot{u} = \dot{x} = f(u + x^*, v + y^*) = f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv).$$

This leads to

$$\dot{u} = \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv)$$

and similarly

$$\dot{v} = \left.\frac{\partial g}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial g}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv).$$

Hence

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \quad - \text{ the linearised system}$$

with

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)} \quad - \text{ \underline{the Jacobian matrix}}$$

Theorem (linear stability): Suppose that $\dot{x} = f(x)$ has an equilibrium at x^* and the linearisation $\dot{x} = Ax$. If A has no zero or purely imaginary eigenvalues then the local stability of the fixed point (which is called **hyperbolic** in this case) is determined by the linear system. In particular, if all eigenvalues have a negative real part Re $(\lambda_i) < 0$ for all $i = 1, \ldots, n$ then the fixed point is asymptotically stable.

Hartman-Grobman theorem: The local phase-portrait near a hyperbolic fixed point is topologically equivalent to the phase-portrait of the linearisation.

Structural stability A phase portrait is structurally stable if its topology cannot be changed by an arbitrarily small perturbation to the vector field, i.e. a system is structurally stable if it is topologically equivalent to any ϵ-perturbation

$$\dot{x} = f(x) + \epsilon p(x)$$

where $\epsilon \ll 1$ and p is smooth enough. For example, the phase portrait of a saddle is structurally stable, but that of a centre is not: an arbitrarily small amount of damping converts the center to a spiral.

Exercise. Consider the system

$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2) \\ \dot{y} &= x + ay(x^2 + y^2), \end{aligned}$$

where a is a parameter. Show that the linearised system incorrectly predicts that the origin is a centre for all values a. (Hint: rewite the system in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$)

Example 1. To illustrate some of the principles covered let us do a phase-plane analysis of the Lotka-Volterra model of population dynamics of two competing species. Assume i) each species grows in the absence of the other with logistic growth ($\dot{x} = x(1-x)$) and ii) when both species are present they compete for food such that one may go hungry. A particular model of rabbits (r) and sheep (s):

$$\begin{aligned} \dot{r} &= r(3 - r - 2s) \equiv f(r, s) \\ \dot{s} &= s(2 - r - s) \equiv g(r, s) \end{aligned}$$

Fixed points defined by $\dot{r} = \dot{s} = 0$. One finds $(\overline{r}, \overline{s}) = (0,0), (0,2), (3,0), (1,1)$. To classify them we compute

$$A = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial s} \end{bmatrix} = \begin{bmatrix} 3 - 2r - 2s & -2r \\ -s & 2 - r - 2s \end{bmatrix}$$

1. $(\overline{r}, \overline{s}) = (0,0)$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues are both positive so $(0,0)$ is an unstable node. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $(0,1)$.

2. $(\overline{r}, \overline{s}) = (0,2)$

$$A = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Hence $(0,2)$ is a stable node. Slow eigendirection is $(1,-2)$.

3. $(\overline{r}, \overline{s}) = (3,0)$

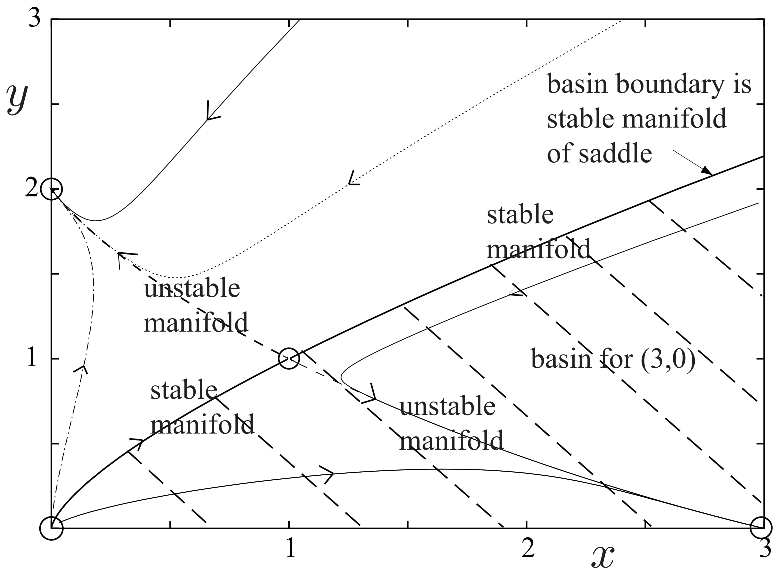
$$A = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence $(3,0)$ is a stable node. Slow eigendirection is $(3,-1)$.

4. $(\overline{r}, \overline{s}) = (1,1)$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 + \sqrt{2} & 0 \\ 0 & -1 - \sqrt{2} \end{bmatrix}$$

Hence, $(1,1)$ is a saddle



The above example nicely illustrates the notion of a **basin of attraction**. Given an attracting fixed point $\overline{x}$ we define its basin of attraction to be the set of initial conditions x_0 such that $x(t) \to \overline{x}$ as $t \to \infty$. For instance the basin of attraction for the node at $(3,0)$ consists of all points lying below the stable manifold of the saddle. Because the stable manifold separates the basins of two nodes, it is called the **basin boundary**.

1.6 Lyapunov function

Lyapunov theorem: Suppose that x^* is a fixed point for the differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Then x^* is Lyapunov stable if there exists a (continuously differentiable) function $L(x)$ (called a Lyapunov function) with the following properties in some neighbourhood of x^*:

1. $L(x)$ and its partial derivatives are continuous
2. $L(x) > 0$ for all $x \neq x^*$ and $L(x^*) = 0$
3. $\dot{L} \leq 0$ for all $x \neq x^*$

Note that $\dot{L}$ is determined by the chain-rule

$$\dot{L} = \sum_i \frac{\partial L}{\partial x_i} \dot{x}_i = \sum_i \frac{\partial L}{\partial x_i} f(x_i)$$

Example 2. Show that $L(x, y) = x^2 + 4y^2$ is a Lyapunov function for

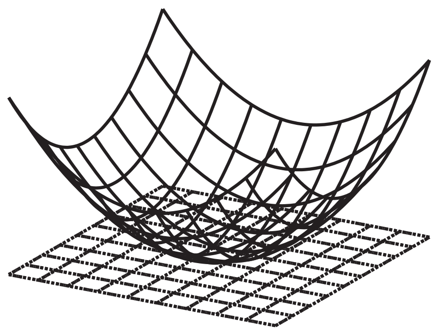
$$\dot{x} = -x + 4y, \qquad \dot{y} = -x - y^3$$

The fixed point is at $(0, 0)$.

1. $L(x, y)$ is continuously differentiable.
2. $L(x, y) > 0$, $L(0, 0) = 0$.
3. $$\dot{L} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = -2x^2 - 8y^4 < 0$$

$\Rightarrow L(x, y)$ - Lyapunov function

Heuristic picture: sufficiently close to the fixed point, L forms a bowl and L decreases along trajectories.



Main difficulty of this method for checking stability is finding an appropriate Lyapunov function.