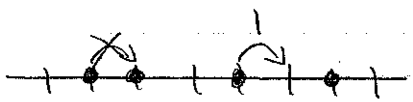
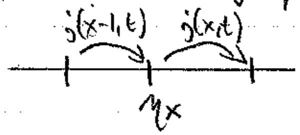
Stoch. particle systems & Burgers equation

TASEP:  $\Lambda = \mathbb{T}_L$ Statespace $S = \{0,1\}^\Lambda$ States $\eta \in S$

rates: $c(\eta, \eta^{x,x+1}) = \eta_x (1-\eta_{x+1})$ (exclusion IA)

ME: $\frac{d}{dt}\pi_t(\eta) = \sum_{x\in\Lambda} \pi_t(\eta^{x,x-1})\,\eta_x(1-\eta_{x-1}) - \sum_{x\in\Lambda}\pi_t(\eta)\,\eta_x(1-\eta_{x+1})$

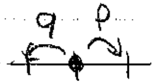
Lattice continuity eqn. $\rho(x,t) := \mathbb{E}_{\pi_t}(\eta_x)$ $\quad j(x,t) := \mathbb{E}_{\pi_t}\big(c(\eta,\eta^{x,x+1})\big)$



$\frac{d}{dt}\rho(x,t) = j(x-1,t) - j(x,t)$

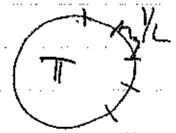
$\Rightarrow \boxed{\frac{d}{dt}\rho(x,t) + \nabla_x j(x,t) = 0}$

$\frac{d}{dt}\sum_{x\in\Lambda}\rho(x,t) = -\sum_{x\in\Lambda}\nabla_x j(x,t) = 0 \to$ conservation law

[more general: $j(x,t) = \mathbb{E}_{\pi_t}\big[p\,\eta_x(1-\eta_{x+1}) - q(1-\eta_x)\eta_{x+1}\big]$ 

for $p=q=1$ cancellation $\to \frac{d}{dt}\rho(x,t) = \Delta_x \rho(x,t)$ diff. eqn.]

How to close eqn. for $p=1$, $q=0$? $\quad j(x,t) = f(\rho(x,t))$?

Scaling: $y = \frac{x}{L} \in \mathbb{T}$, $\quad \hat{\rho}(y,t) = \rho([yL],t)$, $\hat{j}(y,t) = j([yL],t)$ 

$j(x-1,t) = \hat{j}(y-1/L,t) = \hat{j}(y,t) - \frac{1}{L}\partial_y\hat{j}(y,t) + \frac{1}{2L^2}\partial_y^2\hat{j}(y,t)$

$\Rightarrow \frac{d}{dt}\hat{\rho}(y,t) + \frac{1}{L}\partial_y\hat{j}(y,t) + o(\frac{1}{L}) = 0 \to$ speed up process $s = t/L$

$\Rightarrow \boxed{\frac{d}{ds}\hat{\rho}(y,s) + \partial_y\hat{j}(y,s) = 0}$ as $L\to\infty$

$\hat{j}(y,s) \simeq \mathbb{E}_{\pi_{sL}}\big(\eta_{yL}(1-\eta_{yL+1})\big)$ as $L\to\infty$

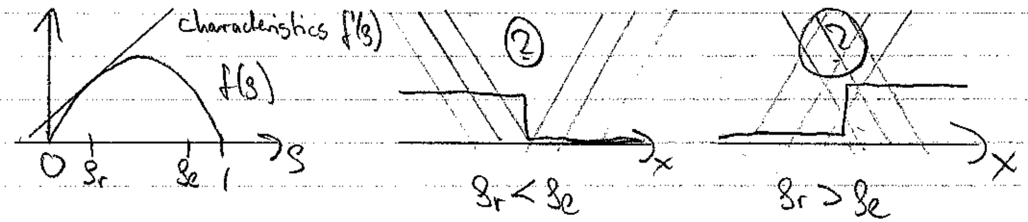
$\llcorner$ replace by stat. distribution (convergence to eqm.)

π^* for ASEP: $\pi^*_\rho(\eta) = \prod_{x\in\Lambda}\rho^{\eta_x}(1-\rho)^{1-\eta_x}$ iid $Be(\rho)$ with $\rho\in[0,1]$

$\Rightarrow \hat{j}(y,s) = \mathbb{E}_{\pi^*_{\hat{\rho}(y,s)}}\big[\eta_0(1-\eta_1)\big] = \hat{\rho}(y,s)(1-\hat{\rho}(y,s)) = f(\hat{\rho}(y,s))$

$\llcorner$ local equilibrium

$\Rightarrow \boxed{\frac{d}{ds}\hat{\rho}(y,s) + \partial_y f(\hat{\rho}(y,s)) = 0}$ (kind of) Burgers (inviscid)



How to get a reasonable unique solution?

$\to$ keep viscosity term l.h.s. $= \underset{\varepsilon\to 0}{\frac{1}{2L}}\partial_y^2 f(\hat{\rho}(y,s))$ from Taylor exp.

$\to$ parabolic eqn. $\xrightarrow{\exists!}\hat{\rho}^\varepsilon(y,s) \xrightarrow[\varepsilon\to 0]{} \hat{\rho}(y,s)$ entropy solution

$\Rightarrow \rho_r < \rho_e$ rarefaction fan, $\rho_r > \rho_e$ shock with speed $v = \frac{f(\rho_r)-f(\rho_e)}{\rho_r-\rho_e}$

[Thm: [Rost '81]  'discrete density'

empirical measure $\mu^L_t = \frac{1}{L}\sum_{x\in\Lambda_L}\eta_x(t)\,\delta_{x/L}$ on torus $\mathbb{T}$

Then $\mu^L_{sL} \to \hat{\rho}(\cdot,s)\,dy$ (weakly & in prob.), $\hat{\rho}$ is entr. sol. of Burgers]