Chapter 4

Hyperbolic PDE's

4.1 The Advection Equation Revisited

The linear wave equation is the archetypal hyperbolic equation. In this chapter we want to devise methods to solve

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \tag{4.1}$$

on the interval $[x_L, x_R]$ subject to periodic, Dirichlet or Neumann boundary conditions as we did for parabolic equations in Chap. 3. Since Eq. (4.1) is second order in time, we require two pieces of initial data to specify the problem:

$$\begin{aligned} v(x,0) &= V(x) \\ \frac{\partial v}{\partial x}(x,0) &= V'(x). \end{aligned} \tag{4.2}$$

Hyperbolic equations are generally much tougher to solve numerically than their parabolic counterparts for reasons which we will now explore. Let us begin by writing Eq. (4.1) as a system of first order PDEs. Define

$$\begin{aligned} v_1(x,t) &= \frac{\partial v}{\partial t} \\ v_2(x,t) &= c\frac{\partial v}{\partial x}. \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{\partial v_1}{\partial t} &= \frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2} = c\frac{\partial}{\partial x}\left(c\frac{\partial v}{\partial x}\right) = c\frac{\partial v_2}{\partial x} \\ \frac{\partial v_2}{\partial t} &= c\frac{\partial^2 v}{\partial t \partial x} = c\frac{\partial}{\partial x}\left(c\frac{\partial v}{\partial t}\right) = c\frac{\partial v_1}{\partial x}, \end{aligned}$$

so that the two can be compactly written as

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{\partial}{\partial x}\left[\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right]. \tag{4.3}$$

In many applications, Eq. (4.3) are actually the primary equations from which Eq. (4.1) is derived. The initial conditions are then specified for v_1 and v_2. We shall adopt this perspective since it will sidestep unnecessary complications relating v_1 and v_2 to v. From this point of view, the wave equation provides a simple example of a (vector–valued) conservation law:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial}{\partial x}\left[\mathbf{F}(\mathbf{v})\right] \tag{4.4}$$

where $\mathbf{v} = (v_1, v_2)$ and the flux, $\mathbf{F}(\mathbf{v})$, is given by

$$\mathbf{F}(\mathbf{v}) = C\,\mathbf{v}$$

where C is a matrix

$$C = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}.$$

Many hyperbolic equations can be thought of as conservation laws so we will study them from this perspective. Although not strictly a hyperbolic equation, the scalar advection equation, Eq. (2.6), which we studied in Chap. 2 is an even simpler example of a conservation law. It can be written as

$$\frac{\partial v}{\partial t} = -\frac{\partial F(v)}{\partial x} \qquad F(v) = c\, v. \tag{4.5}$$

Many of the issues which arise in the numerical solution of general hyperbolic equations also manifest themselves at the level of Eq. (4.5) so we shall, for clarity and simplicity, derive algorithms which solve this equation and then extend them to include vector-valued cases like Eq. (4.3).

An obvious starting point (forgetting, for now, that we know the Method of Characteristics) is to derive the analogue of the FTCS scheme for Eq. (4.5). As in Sec. 3.1 we use a centred difference approximation for the spatial derivative and a forward difference approximation for the time derivative:

$$\begin{aligned} -\frac{\partial F}{\partial x}(x_i, t_j) &\approx -\frac{F_{i+1,j} - F_{i-1,j}}{2\,\Delta x} = -\frac{c}{2\,\Delta x}\,(v_{i+1,j} - v_{i-1,j}) \\ \frac{\partial v}{\partial t}(x_i, t_j) &\approx \frac{v_{i,j+1} - v_{i,j}}{h}. \end{aligned}$$

Putting these together in Eq. (4.5) we arrive at the FTCS scheme for the advection equation:

$$v_{i,j+1} = v_{i,j} - \frac{1}{2}\,\gamma\,(v_{i+1,j} - v_{i-1,j}), \tag{4.6}$$

where

$$\gamma = \frac{c\,h}{\Delta x}. \tag{4.7}$$

What is the pointwise accuracy of this scheme? As usual, we can determine this by comparing the Taylor expansions of the left and right sides of Eq. (4.6) and seeing at what order they differ. Beginning with the left hand side:

$$\begin{aligned} v_{i,j+1} &= v_{i,j} + h\,\left.\frac{\partial v}{\partial t}\right|_{i,j} + \frac{1}{2}\,h^2\,\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j} + O(h^3) \\ &= v_{i,j} - c\,h\,\left.\frac{\partial v}{\partial x}\right|_{i,j} + \frac{1}{2}\,c^2 h^2\,\left.\frac{\partial^2 v}{\partial x^2}\right|_{i,j} + O(h^3) \qquad \text{Using Eq. (4.6)} \end{aligned} \tag{4.8}$$

Now the right hand side:

$$\begin{aligned} & v_{i,j} - \frac{ch}{2\Delta x}\,(v_{i+1,j} - v_{i-1,j}) \\ = \;& v_{i,j} - \frac{ch}{2\Delta x}\Bigg(v_{i,j} + \Delta x\,\left.\frac{\partial v}{\partial x}\right|_{i,j} + \frac{1}{2}\,(\Delta x)^2\,\left.\frac{\partial^2 v}{\partial x^2}\right|_{i,j} + \frac{1}{6}\,(\Delta x)^3\,\left.\frac{\partial^3 v}{\partial x^3}\right|_{i,j} + ((\Delta x)^4) \\ & - \left[v_{i,j} - \Delta x\,\left.\frac{\partial v}{\partial x}\right|_{i,j} + \frac{1}{2}\,(\Delta x)^2\,\left.\frac{\partial^2 v}{\partial x^2}\right|_{i,j} - \frac{1}{6}\,(\Delta x)^3\,\left.\frac{\partial^3 v}{\partial x^3}\right|_{i,j} + ((\Delta x)^4) \right] \Bigg) \\ = \;& v_{i,j} - c\,h\,\left.\frac{\partial v}{\partial x}\right|_{i,j} + \frac{1}{6}\,c\,(\Delta x)^2\,h\,\left.\frac{\partial^3 v}{\partial x^3}\right|_{i,j} + O(h(\Delta x)^3). \end{aligned} \tag{4.9}$$

Comparing Eq. (4.8) and Eq. (4.9), we conclude that the pointwise errors for the FTCS scheme, Eq. (4.6), are $O(h^2)$ and $O((\Delta x)^2 h)$.

Recall that the FTCS scheme derived for the diffusion equation was conditionally stable. We were required to choose $\delta < \frac{1}{2}$ in order to maintain stability. What is the corresponding stability condition for Eq. (4.6)? Let is perform a Neumann stability analysis. Consider a trial solution

$$v(x,t) = a(t)\,\mathrm{e}^{i\,k\,x},$$

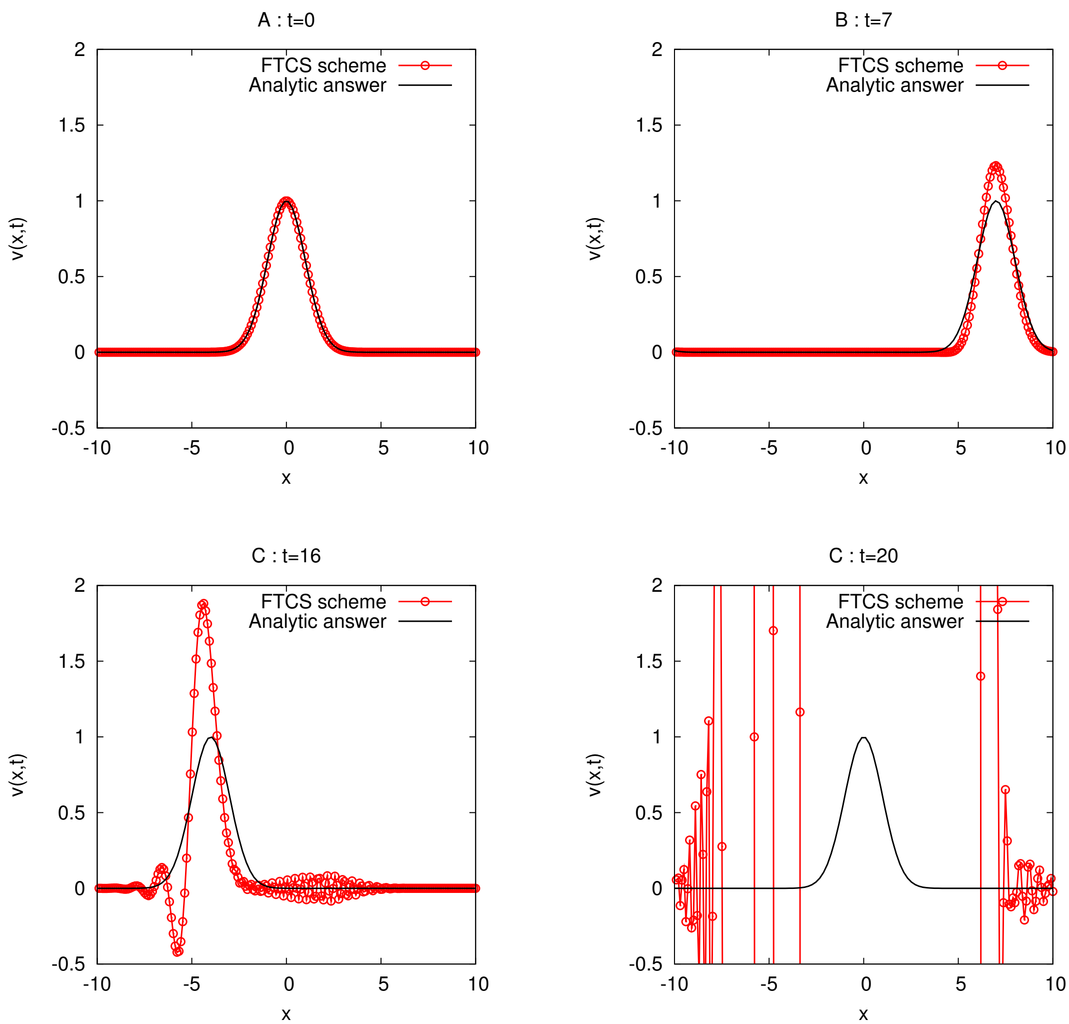

Figure 4.1: Instability of the FTCS scheme applied to the advection equation.

and substitute it into the finite difference scheme. We obtain

$$\begin{aligned} a(t_{j+1})\, e^{i\,k\,x} &= a(t_j)\, e^{i\,k\,x} - \frac{1}{2}\,\gamma\, a(t_j)\, \left[e^{i\,k\,(x_i+\Delta x)} - e^{i\,k\,(x_i-\Delta x)} \right] \\ \Rightarrow \frac{a(t_{j+1})}{a(t_j)} &= 1 - -\frac{1}{2}\,\gamma\, \left[e^{i\,k\,\Delta x} - e^{-i\,k\,\Delta x} \right] \\ &= 1 - \gamma\, i\, \sin(k\,\Delta x). \end{aligned}$$

For stability we need

$$\begin{aligned} & \left| \frac{a(t_{j+1})}{a(t_j)} \right| < 1 \\ \Rightarrow\ & |1 - \gamma\, i\, \sin(k\,\Delta x)| < 1 \\ \Rightarrow\ & 1 + \gamma^2\, \sin^2(k\,\Delta x) < 1. \end{aligned}$$

This is clearly impossible for any real γ. We conclude therefore that the FTCS scheme applied to the advection equation is unconditionally unstable. The results of attempting to apply the method is shown in Fig. 4.1. As was the case for the diffusion equation, the numerical scheme produces unphysical oscillations which grow exponentially in time and swamp the solution. The problem can be postponed by taking smaller timesteps but ultimately renders the FTCS scheme unsuitable for solving hyperbolic problems.

4.2 Lax Method and CFL Criterion

The problem of unconditional instability for the FTCS scheme is easily fixed although fixing it does, as we shall see, introduce new difficulties. The simplest fix is to use the Lax scheme. The Lax scheme modifies the FTCS scheme, Eq. (4.6), in a seemingly trivial way: the $v_{i,j}$ term on the RHS is replaced by the average, $(v_{i+1,j}+v_{i-1,j})/2$, of the solution at the two adjacent grid points. The resulting scheme is

$$v_{i,j+1} = \frac{1}{2}(v_{i+1,j} + v_{i-1,j}) - \frac{1}{2}\,\gamma\,(v_{i+1,j} - v_{i-1,j}). \tag{4.10}$$

A similar analysis to that which we applied to the FTCS scheme in the previous section shows that the leading pointwise errors for this scheme are $O(h^2)$ and $O((\Delta x)^2)$. Thus, on the face of it, the Lax scheme, Eq. (4.10), is *less* accurate than Eq. (4.6). Its stability properties are much improved however. To perform a Neumann stability analysis, we consider a trial solution of the usual form, $v(x,t) = a(t)\, e^{i\,k\,x}$, and substitute it into Eq. (4.10). After some manipulations we find

$$\frac{a(t_{j+1})}{a(t_j)} = \cos(k\,\Delta x) - \gamma\, i\, \sin(k\,\Delta x).$$

For stability we require

$$\begin{aligned} & \left| \frac{a(t_{j+1})}{a(t_j)} \right| < 1 \\ \Rightarrow\ & \cos^2(k\,\Delta x) + \gamma^2\, \sin^2(k\,\Delta x) < 1 \\ \Rightarrow\ & (\gamma^2 - 1)\, \sin^2(k\,\Delta x) < 0 \\ \Rightarrow\ & \gamma = \frac{c\,h}{\Delta x} < 1. \end{aligned} \tag{4.11}$$

The condition, $\frac{c\,h}{\Delta x} < 1$, is known as the CFL condition (Courant - Friedrichs - Lewy). It is a very common constraint arising in explicit numerical schemes for hyperbolic equations: a finer spatial grid requires smaller timesteps in order to maintain stability. Note, however, that the restriction is not as bad as for the FTCS scheme for the diffusion equation where we required $h \sim (\Delta x)^2$ to maintain stability. Remember that hyperbolic equations generally describe the propagation of waves (in the advection equation, the propagation speed is c). The CFL condition has an intuitive interpretation:

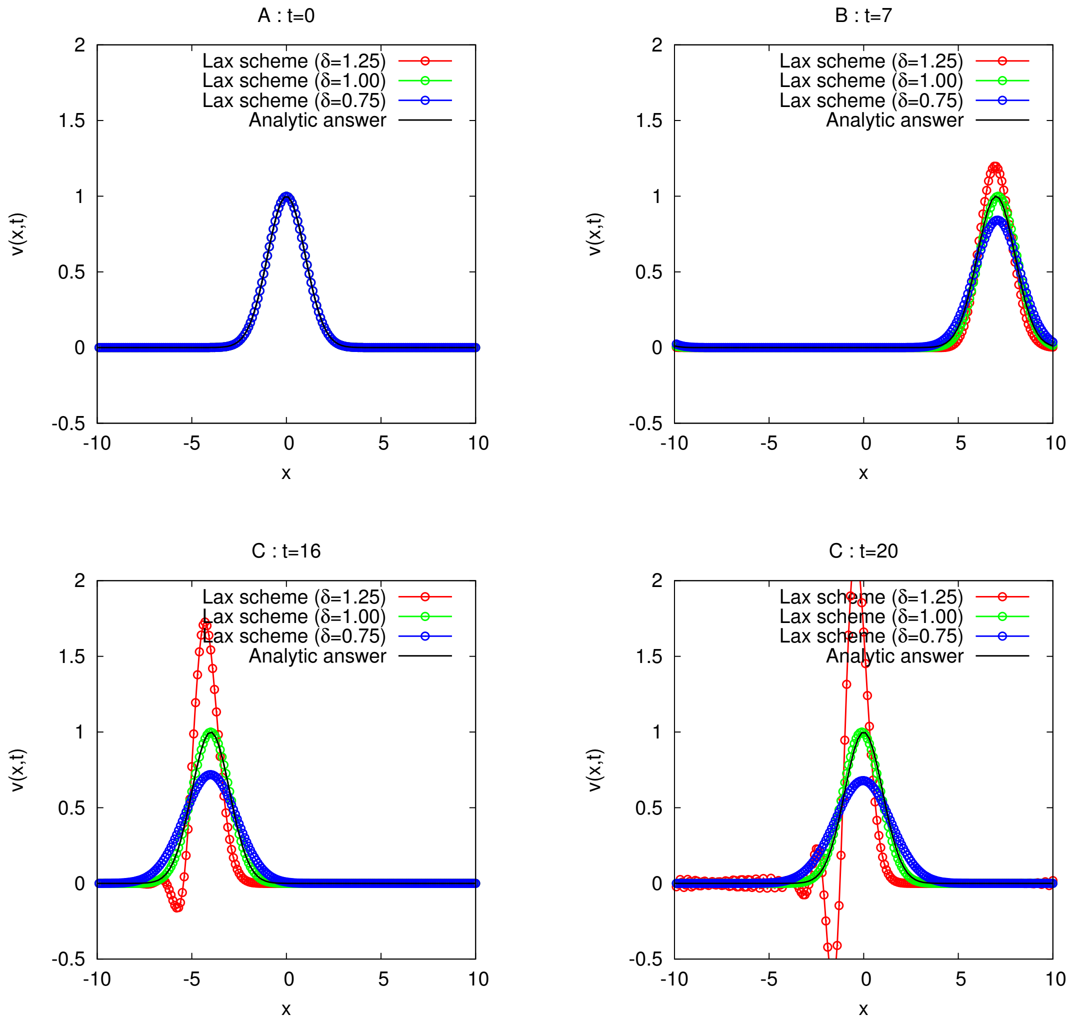

Figure 4.2: Lax scheme applied to the advection equation.

the timestep must be sufficiently small that the distance, $c\,h$, travelled by the wave in a single timestep does not exceed the grid spacing, Δx.

Fig. 4.2 shows snapshots of the results obtained by applying Eq. (4.10) to Eq. (4.5) with a gaussian initial condition for three different values of γ set by fixing $c = 1$, $\Delta x = 0.1$ and varying h. For $\gamma = 1.25$, the numerical scheme is clearly unstable, producing exponentially growing oscillations similar to those in Fig. 4.1. For $\gamma = 1$, the scheme seems to work well as we would expect from our stability analysis. For $\gamma = 0.75$ we observe, perhaps surprisingly, that the scheme again fails. The numerical solution decays in time whereas the analytic solution does not. This runs counter to our expectation that a smaller timestep should always produce a more accurate solution. What is going on?

With a little thought, it is clear that the Lax method, Eq. (4.10), is equivalent to the FTCS scheme applied to a modified equation:

$$\frac{\partial v}{\partial t} = -c\,\frac{\partial v}{\partial x} + D\frac{\partial^2 v}{\partial x^2} \qquad D = \frac{(\Delta x)^2}{2\,h}. \tag{4.12}$$

Thus the Lax scheme works by introducing an effective diffusion term into the advection scheme. This diffusion damps out the oscillations which would otherwise grow and destroy the solution. The artificial diffusion which occurs in the Lax scheme is called *numerical diffusion*. The amount of numerical diffusion is inverselt proportional to h. This explains why the Lax scheme fails when $\gamma \ll 1$: the numerical diffusion is so strong that it damps out the solution. The price to be paid for stabilising the FTCS scheme is that we must accept some artificial spreading and damping of the numerically computed wave - an effect known as *numerical dispersion*. The trade-off between numerical stability and dispersion is a common issue in the numerical study of hyperbolic problems. The Lax scheme works best when $\gamma = 1$ which is the least amount of diffusion sufficient to stabilise the solution. Note that the diffusion is not zero in this case - the numerical solution will decay eventually even if it is not very evident from Fig. 4.2.

4.3 Conservation Laws and Lax–Wendroff Method

The Lax-Wendroff scheme will be our third and final numerical algorithm to derive for the advection equation. It builds on the success of the Lax method in suppressing the instability of the FTCS scheme via numerical diffusion but, by keeping more terms in the relevant Taylor expansion, it reduces the rate at which the numerical solution is damped out by the artificial diffusion.

We will derive Lax-Wendroff for a general conservation law,

$$\frac{\partial v}{\partial t} = -\frac{\partial F(v)}{\partial x}, \tag{4.13}$$

since the thinking which underpins the method is clearer there. We shall then specialise our formulae to the particular case of Eq. (refeq-scalarAdvectionEqn). The idea behind the method is very similar to the Taylor series methods for ODE's which we briefly discussed for ODEs in Chap. 1. We start from the Taylor expansion for $v_{i,j+1}$:

$$v_{i,j+1} = v_{i,j} + h\,\left.\frac{\partial v}{\partial t}\right|_{i,j} + \frac{1}{2}h^2\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j} + O(h^3). \tag{4.14}$$

We now use Eq. (4.13) to express the time derivatives in terms of the flux, F:

$$\left.\frac{\partial v}{\partial t}\right|_{i,j} = -\left.\frac{\partial F}{\partial x}\right|_{i,j} \tag{4.15}$$

$$\begin{aligned}
\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j} &= -\frac{\partial}{\partial t}\left[\frac{\partial F}{\partial x}\right]\Bigg|_{i,j} \\
&= -\frac{\partial}{\partial x}\left[\frac{dF}{dv}\,\frac{\partial v}{\partial t}\right]\Bigg|_{i,j} \\
&= \frac{\partial}{\partial x}\left[F'(v)\,\frac{\partial F}{\partial x}\right]\Bigg|_{i,j}
\end{aligned} \tag{4.16}$$

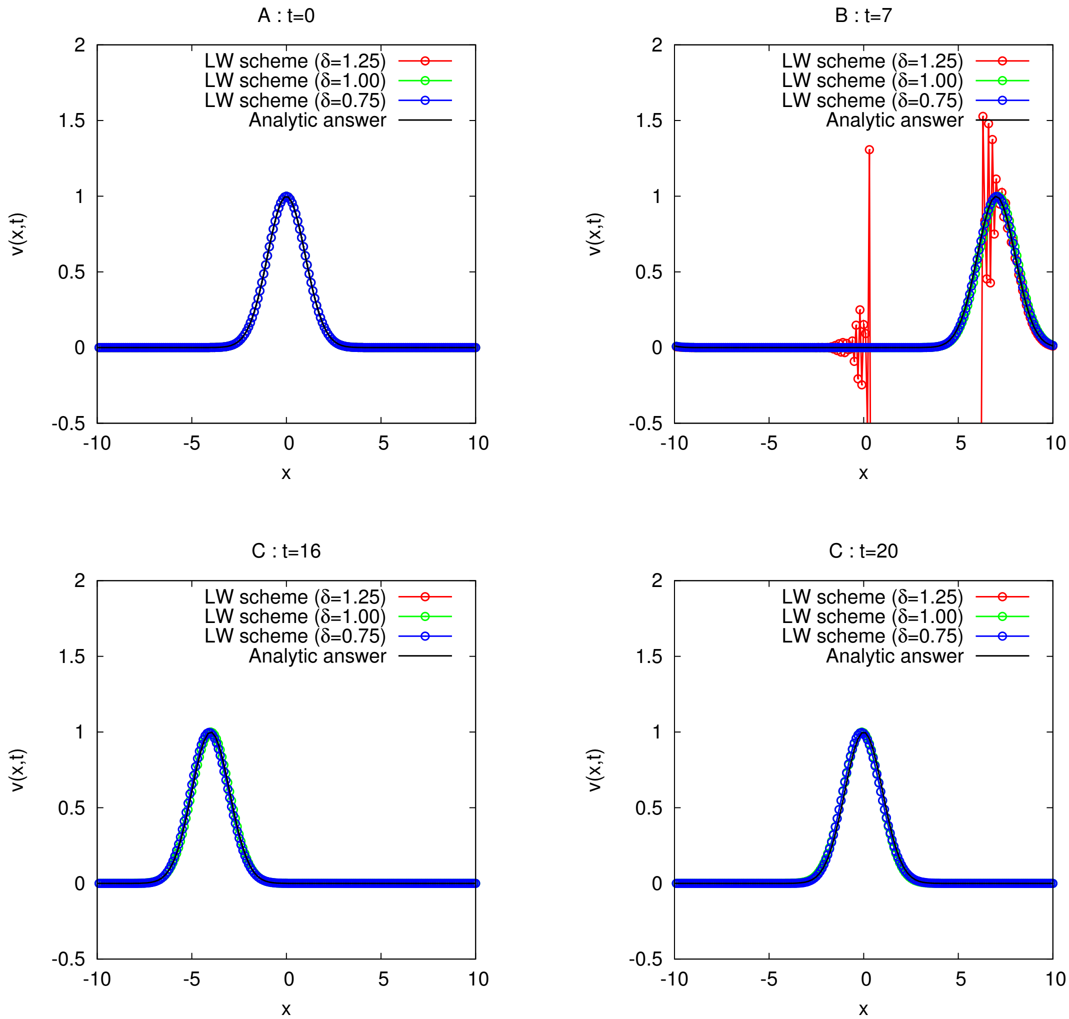

Figure 4.3: Lax Wendroff scheme applied to the advection equation.

Putting Eqs. (4.15) and (4.16) together with Eq. (4.14) we have the essence of the Lax-Wendroff scheme:

$$v_{i,j+1} = v_{i,j} - h\left.\frac{\partial F}{\partial x}\right|_{i,j} + \frac{1}{2}h^2\frac{\partial}{\partial x}\left[F'(v)\frac{\partial F}{\partial x}\right]\Bigg|_{i,j}. \tag{4.17}$$

The tricky bit is discretising the spatial derivatives on the RHS of this formula. The first term is easy - we just use a centred difference approximation as we have done for the FTCS and Lax methods previously:

$$\left.\frac{\partial F}{\partial x}\right|_{i,j} \approx \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} \tag{4.18}$$

For the second term, we discretise the outer derivative using a centred difference formula involving the points $x_i - \frac{\Delta x}{2}$, x_i and $x_i + \frac{\Delta x}{2}$:

$$\frac{\partial}{\partial x}\left[F'(v)\frac{\partial F}{\partial x}\right]\Bigg|_{i,j} = \frac{1}{\Delta x}\left[F'(v)\big|_{i+\frac{1}{2},j}\left.\frac{\partial F}{\partial x}\right|_{i+\frac{1}{2},j} - F'(v)\big|_{i-\frac{1}{2},j}\left.\frac{\partial F}{\partial x}\right|_{i-\frac{1}{2},j}\right]. \tag{4.19}$$

We now estimate $F'(v)$ at the intermediate gridpoints by linear interpolation:

$$\begin{aligned} F'(v)\big|_{i+\frac{1}{2},j} &= F'\left(\frac{1}{2}\left[v_{i+1,j} + v_{i,j}\right]\right) \equiv F'_{i+\frac{1}{2},j} \\ F'(v)\big|_{i-\frac{1}{2},j} &= F'\left(\frac{1}{2}\left[v_{i,j} + v_{i-1,j}\right]\right) \equiv F'_{i-\frac{1}{2},j}, \end{aligned}$$

where we have introduced the notation $F'_{i+\frac{1}{2},j}$ and $F'_{i-\frac{1}{2},j}$ to simplify the subsequent formulae. We approximate the inner derivatives again with centred difference formulae with spacing $\Delta x/2$:

$$\begin{aligned} \left.\frac{\partial F}{\partial x}\right|_{i+\frac{1}{2},j} &\approx \frac{F_{i+1,j} - F_{i,j}}{\Delta x} \\ \left.\frac{\partial F}{\partial x}\right|_{i-\frac{1}{2},j} &\approx \frac{F_{i,j} - F_{i-1,j}}{\Delta x} \end{aligned}$$

Putting all this together we arrive at the general Lax Wendroff scheme:

$$v_{i,j+1} = v_{i,j} - \frac{h}{2\Delta x}\left[F_{i+1,j} - F_{i-1,j}\right] + \frac{1}{2}\frac{h^2}{\Delta x}\left[F'_{i+\frac{1}{2},j}\left(\frac{F_{i+1,j} - F_{i,j}}{\Delta x}\right) - F'_{i-\frac{1}{2},j}\left(\frac{F_{i,j} - F_{i-1,j}}{\Delta x}\right)\right]. \tag{4.20}$$

For the specific case of the advection equation, $F(v) = c\,v$ and $F'(v) = c$. Then Eq. (4.20) reduces to

$$v_{i,j+1} = v_{i,j} - \frac{c\,h}{2\Delta x}\left[v_{i+1,j} - v_{i-1,j}\right] + \frac{c^2h^2}{2(\Delta x)^2}\left[v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\right]. \tag{4.21}$$

Fig. 4.3 shows the application of this scheme to the advection equation. Although the method is still unstable for $\gamma > 1$, the damping of the solution is greatly reduced for the case $\gamma = 0.75$ as compared to the Lax scheme.

4.4 The Wave Equation

Let us now return and adapt the methods which we have developed for the advection equation to the wave equation. In fact we shall write our formulae for a general K-component hyperbolic conservation law:

$$\frac{\partial v^{(k)}}{\partial t} = -\frac{\partial}{\partial x}\left[F^{(k)}(\mathbf{v})\right] \qquad k = 1\ldots K, \tag{4.22}$$

and then specialise our results to the case of Eq. (4.3). Life is complicated by the fact that we are now required to keep track of an additional index - that labelling the components of $\mathbf{v}$. Just to be clear, the notation $w_{i,j}^{(k)}$ denotes the k^{th} component of a K-vector $\mathbf{w}$ evaluated at the space-time point (x_i, t_j). The Lax scheme, Eq. (4.10), simply generalises component-wise:

$$v_{i,j+1}^{(k)} = \frac{1}{2}(v_{i+1,j}^{(k)} + v_{i-1,j}^{(k)}) - \frac{h}{2\Delta x}\,(F_{i+1,j}^{(k)} - F_{i-1,j}^{(k)}). \tag{4.23}$$

For the wave equation, $K = 2$ with

$$\mathbf{v} = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} \tag{4.24}$$

$$\mathbf{F} = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -c\,v^{(2)} \\ -c\,v^{(1)} \end{pmatrix} \tag{4.25}$$

so that Eq. (4.23) becomes:

$$\begin{aligned} v_{i,j+1}^{(1)} &= \frac{1}{2}(v_{i+1,j}^{(1)} + v_{i-1,j}^{(1)}) + \frac{h\,c}{2\,\Delta x}\left[v_{i+1,j}^{(2)} - v_{i-1,j}^{(2)}\right] \\ v_{i,j+1}^{(2)} &= \frac{1}{2}(v_{i+1,j}^{(2)} + v_{i-1,j}^{(2)}) + \frac{h\,c}{2\,\Delta x}\left[v_{i+1,j}^{(1)} - v_{i-1,j}^{(1)}\right]. \end{aligned} \tag{4.26}$$

The Lax-Wendroff scheme for the vectorial case requires considerable care. The complications arise when calculating the analogue of Eq. (4.16) which, after careful application of the multi-variable chain rule, comes out as

$$\left.\frac{\partial^2 v^{(k)}}{\partial t^2}\right|_{i,j} = \sum_{l=1}^{K} \frac{\partial}{\partial x}\left[\frac{\partial F^{(k)}}{\partial v^{(l)}}\,\frac{\partial F^{(l)}}{\partial x}\right]\Bigg|_{i,j}. \tag{4.27}$$

The resulting Lax-Wendroff scheme is

$$v_{i,j+1}^{(k)} = v_{i,j}^{(k)} - \frac{h}{2\Delta x}\left[F_{i+1,j}^{(k)} - F_{i-1,j}^{(k)}\right] \tag{4.28}$$

$$+\frac{1}{2}\frac{h^2}{\Delta x}\sum_{l=1}^{K}\left[\left.\frac{\partial F^{(k)}}{\partial v^{(l)}}\right|_{i+\frac{1}{2},j}\left(\frac{F_{i+1,j}^{(l)} - F_{i,j}^{(l)}}{\Delta x}\right) - \left.\frac{\partial F^{(k)}}{\partial v^{(l)}}\right|_{i-\frac{1}{2},j}\left(\frac{F_{i,j}^{(l)} - F_{i-1,j}^{(l)}}{\Delta x}\right)\right]. \tag{4.29}$$

For the wave equation, this rather daunting formula simplifies considerably due to the fact that

$$\begin{aligned} \frac{\partial F^{(1)}}{\partial v^{(1)}} &= 0 & \frac{\partial F^{(1)}}{\partial v^{(2)}} &= -c \\ \frac{\partial F^{(2)}}{\partial v^{(1)}} &= -c & \frac{\partial F^{(2)}}{\partial v^{(2)}} &= 0, \end{aligned}$$

which reduces Eq. (4.28) to

$$\begin{aligned} v_{i,j+1}^{(1)} &= v_{i,j}^{(1)} + \frac{h\,c}{2\,\Delta x}\left[v_{i+1,j}^{(2)} - v_{i-1,j}^{(2)}\right] + \frac{1}{2}\frac{h^2c^2}{(\Delta x)^2}\left[v_{i+1,j}^{(1)} - 2\,v_{i,j}^{(1)} + v_{i-1,j}^{(1)}\right] \\ v_{i,j+1}^{(2)} &= v_{i,j}^{(2)} + \frac{h\,c}{2\,\Delta x}\left[v_{i+1,j}^{(1)} - v_{i-1,j}^{(1)}\right] + \frac{1}{2}\frac{h^2c^2}{(\Delta x)^2}\left[v_{i+1,j}^{(2)} - 2\,v_{i,j}^{(2)} + v_{i-1,j}^{(2)}\right]. \end{aligned} \tag{4.30}$$

Notice how the higher order term in h again introduces numerical diffusion.

4.5 Boundary Conditions for the Wave Equation

We now consider the wave equation on a finite interval, $[x_L, x_R]$. Periodic boundary conditions, as always, are very straightforward to incorporate into the algorithms we have developed. Implementing other boundary conditions can be tricky in general. Here we shall look explicitly at the case of Dirichlet conditions.

Consider the wave equation in the form

$$\frac{\partial v^{(1)}}{\partial t} = c\frac{\partial v^{(2)}}{\partial x} \tag{4.31}$$

$$\frac{\partial v^{(2)}}{\partial t} = c\frac{\partial v^{(1)}}{\partial x} \tag{4.32}$$

on the interval $[x_L, x_R]$ with initial conditions:

$$\begin{aligned} v^{(1)}(x,0) &= V_1(x) \\ v^{(2)}(x,0) &= V_2(x) \end{aligned} \tag{4.33}$$

and Dirichlet boundary conditions:

$$\begin{aligned} v^{(1)}(x_L,t) &= D_L(t) \\ v^{(1)}(x_R,t) &= D_R(t). \end{aligned} \tag{4.34}$$

From what we have learned already, our first thought is to introduce false points, x_{-1} and x_N and proceed as before. We quickly encounter a problem, however. Eqs. (4.34) only specify the behaviour of $v^{(1)}$ at the false points. What about $v^{(2)}$? Luckily, this problem is rather easily solved: the PDE's themselves can be used to determine the behaviour of $v^{(2)}$ at the boundaries given that we know the behaviour of $v^{(1)}$.

At $x = x_L$, Eq. (4.31) implies

$$\begin{aligned} \frac{\partial v^{(1)}}{\partial t}(x_L,t) &= c\,\frac{\partial v^{(2)}}{\partial x}(x_L,t) \\ \Rightarrow \frac{\partial v^{(2)}}{\partial x}(x_L,t) &= \frac{1}{c}\frac{dD_L}{dt} \equiv D'_L(t). \end{aligned} \tag{4.35}$$

Likewise, at $x = x_R$:

$$\begin{aligned} \frac{\partial v^{(1)}}{\partial t}(x_R,t) &= c\,\frac{\partial v^{(2)}}{\partial x}(x_R,t) \\ \Rightarrow \frac{\partial v^{(2)}}{\partial x}(x_R,t) &= \frac{1}{c}\frac{dD_R}{dt} \equiv D'_R(t). \end{aligned} \tag{4.36}$$

From Eqns. (4.35) and (4.36), we conclude that Dirichlet conditions for $v^{(1)}$ imply Neumann conditions for $v^{(2)}$. We can now determine the behaviour of the solution at the false points as we did for the diffusion equation in Sec. 3.2. For $v^{(1)}$ the boundary conditions determine the solution at the false points directly:

$$\begin{aligned} v^{(1)}_{-1}(t_j) &= D_L(t_j) \\ v^{(1)}_{N}(t_j) &= D_R(t_j). \end{aligned} \tag{4.37}$$

For $v^{(2)}$ we wish to choose, $v^{(2)}_{-1,j}$ and $v^{(2)}_{N,j}$ so that

$$\begin{aligned} \frac{\partial v^{(2)}}{\partial x}(x_L,t_j) &= \approx \frac{v^{(2)}_{1,j} - v^{(2)}_{-1,j}}{2\Delta x} = D'_L(t_j) \\ \frac{\partial v^{(2)}}{\partial x}(x_R,t_j) &= \approx \frac{v^{(2)}_{N,j} - v^{(2)}_{N-2,j}}{2\Delta x} = D'_R(t_j) \end{aligned}$$

from which we are led to choose

$$\begin{aligned} v^{(2)}_{-1,j} &= v^{(2)}_{1,j} - 2\,\Delta x\, D'_L(t_j) \\ v^{(2)}_{N,j} &= v^{(2)}_{N-2,j} + 2\,\Delta x\, D'_R(t_j). \end{aligned} \tag{4.38}$$

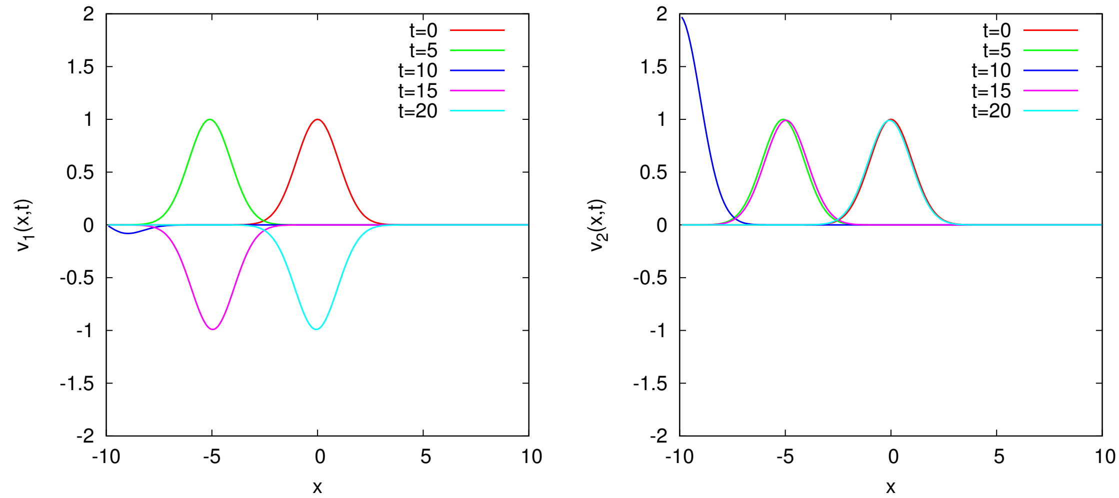

Figure 4.4: A wave reflecting from a boundary. The wave equation was solved with zero Dirichlet conditions at the boundaries and integrated using the Lax scheme as detailed in the text. The left panel shows the evolution of $v^{(1)}(x,t)$ and the right panel shows the evolution of $v^{(2)}(x,t)$.

Eqns. (4.37) and (4.38) together with Eq. (4.26) or Eq. (4.30) now determine the finite difference algorithm completely.

Imposing boundary conditions on wave equations usually introduces a new phenomenon into the game: wave reflections. Fig. 4.4 illustrates this effect for a numerical solution of the wave equation on the spatial domain $[-10:10]$ with initial conditions

$$\begin{aligned} V_1(x) &= \mathrm{e}^{-x^2} \\ V_2(x) &= \mathrm{e}^{-x^2} \end{aligned}$$

and Dirichlet boundary conditions $v^{(1)}(-10,t) = v^{(1)}(10,t) = 0$. The Lax scheme was used with $c = 1$, $\Delta x = 0.1$ and $h = 0.1$. Notice how the pulse is reflected from the boundary with a change of sign.