Weighted automata and Identities

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A natural and fundamental question: 

\[ A \mathcal{A} \mathcal{A} = ? \]

Which pairs of inputs can be distinguished by a given computational model?
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Given a class $C$ of weighted automata:

1. For all $u \neq v$, is there $A \in C$ which distinguishes $u$ and $v$?

2. Is there $A \in C$ which distinguishes all pairs $u \neq v$?

3. Minimal size to distinguish two given input words?
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Boolean Automata

$[\mathcal{A}] : A^* \rightarrow \{\text{Acc, Rej}\}$

For all $u \neq v$, is there $A \in C$ which distinguishes $u$ and $v$?

→ Yes

Is there $A \in C$ which distinguishes all pairs $u \neq v$?

→ No

Minimal size to distinguish two given input words?

→ Profinite theory...
For all $u \neq v$, is there $\mathcal{A} \in \mathcal{C}$ which distinguishes $u$ and $v$?

→ Yes
Boolean Automata

\[ [A] : A^* \rightarrow \{ \text{Acc, Rej} \} \]

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   \( \rightarrow \) Yes

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   \( \rightarrow \) No
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Weighted automata \([\text{Schützenberger}]\)

\([\mathcal{A}] : A^* \rightarrow S\)
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Semiring \((S, \oplus, \otimes)\): transitions are weighted by elements of \(S\)
Weighted automata [Schützenberger]

\[ [A] : A^* \rightarrow S \]

Semiring \((S, \oplus, \otimes)\): transitions are weighted by elements of \(S\)

Paths: \(\otimes\)  
Non-determinism: \(\oplus\)
Weighted automata \[\textbf{[Schützenberger]}\]

\[\mathcal{A} : A^* \rightarrow S\]

Semiring \((S, \oplus, \otimes)\): transitions are weighted by elements of \(S\)

Paths: \(\otimes\)  

Non-determinism: \(\oplus\)

\[\mathcal{A} : w \mapsto \bigoplus_{\rho \text{ accepting path labelled by } w} (\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{|w|})\]
Automata weighted over \((\mathbb{R}, +, \times)\)

\[ \mathcal{A} : A^* \rightarrow \mathbb{R} \]

An example with \(A = \{0, 1\}\)

\[
\begin{array}{c}
0, 1 : 2 \\
\end{array}
\]

\[
\begin{array}{c}
0 : 0 \\
1 : 1 \\
\end{array}
\]

\[
\begin{array}{c}
0, 1 : 1 \\
\end{array}
\]

For all \(u \neq v\), is there \(A \in \mathcal{C}\) which distinguishes \(u\) and \(v\)?

\(\rightarrow\) Yes

Is there \(A \in \mathcal{C}\) which distinguishes all pairs \(u \neq v\)?

\(\rightarrow\) Yes

Minimal size to distinguish two given input words?

1 or 2 states
Automata weighted over \((\mathbb{R}, +, \times)\)

\[ A^* \rightarrow \mathbb{R} \]

An example with \( A = \{0, 1\} \)

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\begin{align*}
0, 1 : 2 & \quad 0 : 0 \\
0, 1 : 1 & \quad 1 : 1
\end{align*}
\]

100101 \mapsto 2^0 + 0 + 0 + 2^3 + 0 + 2^5
Automata weighted over \((\mathbb{R}, +, \times)\)

\([\mathcal{A}] : A^* \rightarrow \mathbb{R}\)

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3. Minimal size to distinguish two given input words?
   \(\rightarrow\) 1 or 2 states
Max-plus automata

Semiring \((\mathbb{N} \cup \{-\infty\}, \text{max}, +)\)

\([A] : A^* \rightarrow \mathbb{N} \cup \{-\infty\}\)

\([A] : w \mapsto \max_{\rho \text{ accepting path labelled by } w} (\rho_1 + \rho_2 + \cdots + \rho_{|w|})\)
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   \(\rightarrow\) Yes

2. Is there \(\mathcal{A} \in \mathcal{C}\) which distinguishes all pairs \(u \neq v\)?
   \(\rightarrow\) No

3. Minimal size to distinguish two given input words?
   \(\rightarrow\) ???????
Given a positive integer $n$, are there $u \neq v$ such that for all max-plus automata $A$ with at most $n$ states:

$$[A](u) = [A](v)$$

?
If $n = 1$.

$$A = \{a, b\}$$
If $n = 1$

$A = \{a, b\}$

$w \mapsto \alpha|w|_a + \beta|w|_b$
If $n = 1$

$A = \{a, b\}$

Max-plus automata with one state can distinguish words with different contents (in particular different lengths), and only these ones.
If \( n = 2 \) or \( n = 3 \)

There exist pairs of distinct words with the same values for all automata with at most 3 states...

But we do not know much more.
If $n = 2$ or $n = 3$

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2 states [Izhakian, Margolis] - words of length 20
If $n = 2$ or $n = 3$

There exist pairs of distinct words with the same values for all automata with at most 3 states...
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2 states [Izhakian, Margolis] - words of length 20

3 states [Shitov] - words of length 1795308
Theorem [Izhakian]

For all $n$, there exist a pair of distinct words $u \neq v$ such that for all triangular automata $A$ with at most $n$ states,

$$[A](u) = [A](v)$$
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For all $n$, there exist a pair of distinct words $u \neq v$ such that for all triangular automata $A$ with at most $n$ states,

$$[[A](u) = [[A](v)$$
Let’s go back to automata with 2 states

\[ A = \{a, b\} \]
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First attempt: Restrict the class of automata we have to consider
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First attempt: Restrict the class of automata we have to consider

\[ \mathbb{R} \rightarrow \mathbb{Q} \rightarrow \mathbb{Z} \rightarrow \mathbb{N} \]
Let’s go back to automata with 2 states

\[ A = \{a, b\} \]

First attempt: Restrict the class of automata we have to consider

- \(\mathbb{R} \rightarrow \mathbb{Q} \rightarrow \mathbb{Z} \rightarrow \mathbb{N}\)
- Complete automaton
Let’s go back to automata with 2 states

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First attempt: Restrict the class of automata we have to consider

- \( \mathbb{R} \rightarrow \mathbb{Q} \rightarrow \mathbb{Z} \rightarrow \mathbb{N} \)
- Complete automaton
- Only one initial and one final states
Let’s go back to automata with 2 states

\[ A = \{a, b\} \]

First attempt: Restrict the class of automata we have to consider

- \( \mathbb{R} \rightarrow \mathbb{Q} \rightarrow \mathbb{Z} \rightarrow \mathbb{N} \)
- Complete automaton
- Only one initial and one final states
- Reduce the number of parameters
Let’s go back to automata with 2 states

Second attempt: Give a list of criteria which can be checked

There are two pairs of distinct words of minimal length which cannot be distinguished by an $\text{max-plus}$ automata with two states:

\[
\begin{align*}
\text{a}_2 \text{b}_3 \text{a}_3 \text{ba}_2 = & \text{a}_2 \text{b}_3 \text{ab}_3 \text{a}_4 \text{b}_3 \text{a}_2 \\
\text{ab}_3 \text{a}_4 \text{b}_3 \text{a}_2 = & \text{ab}_3 \text{a}_2 \text{b}_3 \text{a}_2 \\
\end{align*}
\]

Eliminate the shortest pairs by using the list of criteria

Checking the pairs directly using the restrictions
Let’s go back to automata with 2 states

Second attempt: Give a list of criteria which can be checked

- Content, length
Let's go back to automata with 2 states

Second attempt: Give a list of criteria which can be checked

- Content, length
- ...

Theorem [D., Johnson] - counter-example to a conjecture of Izhakian

There are two pairs of distinct words of minimal length which cannot be distinguished by an $\mathit{max}$-plus automata with two states:

\begin{align*}
\text{a}_2 \text{b}_3 \\
\text{a}_3 \text{bab}_2 \text{a}_3 = \\
\text{ab}_3 \text{a}_4 \text{baba}_2 \text{b}_3 \text{a}_4 \\
= \\
\text{ab}_3 \text{a}_2 \text{baba}_4 \text{b}_3 \text{a}_4
\end{align*}
Let’s go back to automata with 2 states

Second attempt: Give a list of criteria which can be checked

- Content, length
- ...

Theorem [D., Johnson] - counter-example to a conjecture of Izhakian

There are two pairs of distinct words of minimal length which cannot be distinguished by any max-plus automata with two states:

\[ a^2 b^3 a^3 babab^3 a^2 = a^2 b^3 ababa^3 b^3 a^2 \] and \[ ab^3 a^4 baba^2 b^3 a = ab^3 a^2 baba^4 b^3 a \]
Second attempt: Give a list of criteria which can be checked

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**Theorem [D., Johnson]** - counter-example to a conjecture of Izhakian

There are two pairs of distinct words of minimal length which cannot be distinguished by any max-plus automata with two states:

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\[ \rightarrow \] Eliminate the shortest pairs by using the list of criteria
\[ \rightarrow \] Checking the pairs directly using the restrictions
A closer look at the list of criteria
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- First and last blocks
A closer look at the list of criteria

- First and last blocks
- Bloc-permutation

![Transition diagram with states and transitions](attachment:transition_diagram.png)

- `a`: 0
- `b`: 0
- `b`: \(-m\)
- `a`: 1

**"Counting modulo 2" criteria**

Triangular automata with two states
A closer look at the list of criteria

- First and last blocks
- Bloc-permutation
- “Counting modulo 2” criteria

Number of a’s after an even number of b’s

![Diagram](image)
A closer look at the list of criteria

- First and last blocks
- Bloc-permutation
- “Counting modulo 2” criteria
- Triangular automata with two states
And now?
Ultimate (very far away) goal:
Characterize all the identities holding for the class of max-plus
automata with at most $n$ states, for all $n$...
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Characterize all the identities holding for the class of max-plus automata with at most \( n \) states, for all \( n \)...

- Is there a strict subset of max-plus automata containing all their computational power?
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Ultimate (very far away) goal:
Characterize all the identities holding for the class of max-plus automata with at most $n$ states, for all $n$...

- Is there a strict subset of max-plus automata containing all their computational power?
- Link with decidability/undecidability of the equivalence problem?