Approximate comparison of distance automata

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**Distance automata**

**Distance automaton:** Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition. 

\[(\mathbb{A}, Q, I, T, E) \text{ with } E \subseteq (Q \times \mathbb{A} \times \mathbb{N} \times Q)\]

![Diagram of a distance automaton](image)
Distance automata

Distance automaton: Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition.

\[(A, Q, I, T, E) \text{ with } E \subseteq (Q \times A \times \mathbb{N} \times Q)\]

Weight of a run: 
sum of the weights of the transitions

Diagram:
- **q1**: Transition on 'a' labeled with 0 and transition on 'b' labeled with 1.
- **q2**: Transitions on 'a' and 'b' both labeled with 0.
- **q3**: Transition on 'a' labeled with 1.
- **q4**: Transition on 'a' and 'b' both labeled with 0.

Computed function:
\[A^* \rightarrow \mathbb{N} \cup \{+\infty\}\]
\[w \mapsto \text{minimum of the weights of the runs labeled by } w \text{ going from an initial state to a final state} (+\infty \text{ if no such run})\]
Distance automaton: Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition.

\[(\mathbb{A}, Q, I, T, E) \text{ with } E \subseteq (Q \times \mathbb{A} \times \mathbb{N} \times Q)\]

**Weight of a run:** sum of the weights of the transitions

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Distance automata

$q_1$

$q_2\xrightarrow{b:0} q_3\xrightarrow{b:0} q_4$

$q_1\xrightarrow{a:0} \xrightarrow{b:1}$

$q_2\xrightarrow{a,b:0}$

$q_3\xrightarrow{a:1}$

$q_4\xrightarrow{a,b:0}$
Distance automata

\[
a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)
\]
Decision problems on comparison

$f, g$ computed by distance automata: $A^* \rightarrow \mathbb{N} \cup \{+\infty\}$

$f \leq g$ if for all words $w$, $f(w) \leq g(w)$
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$f, g$ computed by distance automata: $A^* \rightarrow \mathbb{N} \cup \{+\infty\}$

$f \leq g$ if for all words $w$, $f(w) \leq g(w)$

**Undecidable [Krob, 92]**

Given $f, g$ computed by distance automata, is $f \leq g$?

**Decidable [Colcombet, 09]**

Is there a polynomial $P$ s.t $f \leq P \circ g$ ?

(context of cost functions)

Generalisation of results by Hashiguchi, Leung and Simon
Theorem of affine domination

Proposition

Given \( f, g \) computed by distance automata, the two assertions are equivalent:

1. There is a polynomial \( P \) s.t. \( f \leq P \circ g \).
2. There is an integer \( a \) s.t. \( f \leq ag + a \).

Theorem

Given \( f, g \) computed by distance automata, one can decide if there is an integer \( a \) s.t. \( f \leq ag + a \).
Theorem of approximate comparison

Input: $f, g$ computed by distance automata and $\varepsilon > 0$

\[ g(1 + \varepsilon)g \leq f \Rightarrow \text{YES} \]

\[ g(1 + \varepsilon)g \not\leq f \Rightarrow \text{NO} \]

\[ g(1 + \varepsilon)g \leq f \Rightarrow \text{YES or NO} \]

\[ g(1 + \varepsilon)g \not\leq f \Rightarrow \text{NO} \]

\[ g(1 + \varepsilon)g \leq f \Rightarrow \text{YES or NO} \]

**Theorem:** Existence of an algorithm having this behaviour.
Conclusion and further questions

**Undecidable [Krob, 92]**

\[ f \leq g \]?

\[ \downarrow \]

**Algorithm of approximate comparison**

**EXPSPACE**

(problem PSPACE-hard)

**Decidable [Colcombet, 09]**

Is there a polynomial \( P \) s.t

\[ f \leq P \circ g \]?

\[ \uparrow \]

**Decidable**

Is there an integer \( a \) s.t

\[ f \leq ag + a \]?

**Next steps**

Capture other kinds of asymptotic behaviours

Case of max-+ automata