Asymptotic Behaviour of Max-Plus Automata and Size-Change Abstraction

Laure Daviaud
joint work with Thomas Colcombet and Florian Zuleger

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Max-Plus Automata

Non deterministic finite automaton for which each transition is also labelled by a non-negative integer (= weight of the transition).

\[ a: 0 \]
\[ b: 0 \]
\[ a, b: 0 \]

\[ q_1 \]
\[ q_2 \]
\[ q_3 \]
\[ q_4 \]

Weight of a run
Sum of the weights of the transitions.

Computed function
\[ A^* \rightarrow N \cup \{-\infty\} \]
\[ w \mapsto \text{maximum of the weights of the runs labelled by } w \text{ going from an initial state to a final state} \] (\(-\infty\) if no such run)

\[ a_{n_0} b_{n_1} a_{n_2} b_{n_3} \mapsto \max(n_0, n_1, \ldots, n_k, k) \]
Max-Plus Automata

Non deterministic finite automaton for which each transition is also labelled by a non-negative integer (= weight of the transition).

Weight of a run

Sum of the weights of the transitions.
Max-Plus Automata

Non deterministic finite automaton for which each transition is also labelled by a non-negative integer (= weight of the transition).

Weight of a run
Sum of the weights of the transitions.

Computed function
\[ A^* \rightarrow \mathbb{N} \cup \{-\infty\} \]
\[ w \mapsto \text{maximum of the weights of the runs labelled by } w \text{ going from an initial state to a final state} \]
\[ (-\infty \text{ if no such run}) \]
Max-Plus Automata

Non deterministic finite automaton for which each transition is also labelled by a non-negative integer (\(=\) weight of the transition).

Weight of a run
Sum of the weights of the transitions.

Computed function
\(A^* \rightarrow \mathbb{N} \cup \{-\infty\}\)
\(w \mapsto \) maximum of the weights of the runs labelled by \(w\) going from an initial state to a final state (\(-\infty\) if no such run)

\(a^{n_0}ba^{n_1}b \cdots ba^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)\)
Main result

\[ f : \mathbb{A}^* \rightarrow \mathbb{N} \cup \{-\infty\} \] computed by a max-plus automaton
**Main result**

\[ f : \mathbb{A}^* \rightarrow \mathbb{N} \cup \{-\infty\} \] computed by a max-plus automaton

\[ f_{\text{min}} : \mathbb{N} \rightarrow \mathbb{N} \cup \{-\infty\} \]

\[ n \mapsto \min\{f(w) \mid |w| \geq n\} \]

**Theorem [Krob]**

There exists effectively \( \alpha \in (\mathbb{Q} \cap [0,1]) \cup \{-\infty\} \) such that

\[ f_{\text{min}}(n) = \Theta(n^\alpha) \]

**Theorem**

Length of the longest word having value at most \( n \):

\[ \Theta(n^{3/5}) \]
Main result

\[ f : \mathbb{A}^* \rightarrow \mathbb{N} \cup \{-\infty\} \text{ computed by a max-plus automaton} \]

\[
\begin{align*}
     f_{\text{min}} : \mathbb{N} & \rightarrow \mathbb{N} \cup \{-\infty\} \\
     n & \mapsto \min\{f(w) \mid |w| \geq n\}
\end{align*}
\]

Theorem [Krob]

Undecidable: for all \( n \), \( f_{\text{min}}(n) \leq n \).
Main result

\[ f : \mathbb{A}^* \to \mathbb{N} \cup \{ -\infty \} \] computed by a max-plus automaton

\[ f_{\min} : \mathbb{N} \to \mathbb{N} \cup \{ -\infty \} \]
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Theorem [Krob]

Undecidable:
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**Theorem [Krob]**

Undecidable:

for all \( n \), \( f_{\min}(n) \leq n \).

**Theorem**

There exists effectively \( \alpha \in (\mathbb{Q} \cap [0, 1]) \cup \{-\infty\} \) such that \( f_{\min}(n) = \Theta(n^{\alpha}) \).

Length of the longest word having value at most \( n \):

\( \Theta(n^{1/\alpha}) \).
Size-Change Abstraction

\[ t_1: x \geq x', \ y > y' \]

\[ t_2: x > x' \]

Variables: \( x \) and \( y \)
Size-Change Abstraction

\[ t_1: x \geq x', y > y' \]

\[ t_2: x > x' \]

Variables: \( x \) and \( y \)

\( (5, 5) \xrightarrow{t_1} (5, 4) \xrightarrow{t_2} (4, 8) \xrightarrow{t_1} (3, 6) \xrightarrow{t_1} (3, 2) \xrightarrow{t_2} (2, 10) \ldots \)
Size-Change Abstraction

\[ t_1: x \geq x', y > y' \]

\[ t_2: x > x' \]

Variables: \( x \) and \( y \)

Terminating SCA instance:
no infinite trace.

\((5, 5) \xrightarrow{t_1} (5, 4) \xrightarrow{t_2} (4, 8) \xrightarrow{t_1} (3, 6) \xrightarrow{t_1} (3, 2) \xrightarrow{t_2} (2, 10) \ldots\)
Size-Change Abstraction

$t_1: x \geq x', y > y'$

$t_2: x > x'$

Variables: $x$ and $y$

Terminating SCA instance: no infinite trace.

Theorem [Lee, Jones, Ben-Amram]
It is decidable whether a given SCA instance is terminating.

\[(5, 5) \xrightarrow{t_1} (5, 4) \xrightarrow{t_2} (4, 8) \xrightarrow{t_1} (3, 6) \xrightarrow{t_1} (3, 2) \xrightarrow{t_2} (2, 10) \ldots\]
Size-Change Abstraction

$t_1$: $x \geq x'$, $y > y'$

$t_2$: $x > x'$

Variables: $x$ and $y$

Terminating SCA instance:
no infinite trace.

Restriction to $[0, n]$:

**Theorem**

Given a terminating SCA instance, there is $\beta \geq 1$, rationnal, computable such that the longest trace is of order $\Theta(n^\beta)$.

$(5, 5) \xrightarrow{t_1} (5, 4) \xrightarrow{t_2} (4, 8) \xrightarrow{t_1} (3, 6) \xrightarrow{t_1} (3, 2) \xrightarrow{t_2} (2, 10)\ldots$
Conclusion and further questions

- What about min-plus automata?

- Complexity?