Joint spectral radius: the power of automata

Laure Daviaud

LIP, ENS Lyon
joint work with Pierre Guillon and Glenn Merlet
(I2M, Marseille)

Louvain, 12th July 2016
Two points of view...
The model under study: Max-plus automata

Syntax: Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).
Semantic: Weight of a run = sum of the weights of the transitions.

A* \rightarrow N \cup \{-\infty\}
w \mapsto \text{Maximum of the weights of accepting runs labelled by } w (\text{−∞ if no such run})

The model under study: Max-plus automata
The model under study: Max-plus automata

Syntax:

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

Semantic:

Weight of a run = sum of the weights of the transitions.

A \rightarrow N \cup \{-\infty\}

w \mapsto \text{Maximum of the weights of accepting runs labelled by } w (\text{−}\infty \text{ if no such run})

The model under study: Max-plus automata
The model under study: Max-plus automata

Syntax:

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

Semantic:

Weight of a run = sum of the weights of the transitions.

\[ A^* \rightarrow \mathbb{N} \cup \{ -\infty \} \]

\[ w \mapsto \text{Maximum of the weights of accepting runs labelled by } w \]

\((-\infty \text{ if no such run})\)
The model under study: Max-plus automata

Syntax:

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

Semantic:

Weight of a run $= \text{sum of the weights of the transitions.}$

$A^* \rightarrow \mathbb{N} \cup \{-\infty\}$

$w \mapsto \text{Maximum of the weights of accepting runs labelled by } w$

$(-\infty \text{ if no such run})$
In fact, these are matrices...

![Diagram](image-url)
In fact, these are matrices...
In fact, these are matrices...

\[
\begin{pmatrix}
0 & \infty & \infty & \infty \\
\infty & 0 & \infty & \infty \\
\infty & \infty & 1 & \infty \\
\infty & \infty & \infty & 0
\end{pmatrix} = \mu(a)
\]

\[
\begin{pmatrix}
1 & \infty & \infty & \infty \\
\infty & 0 & 0 & \infty \\
\infty & \infty & \infty & 0 \\
\infty & \infty & \infty & 0
\end{pmatrix} = \mu(b)
\]
In fact, these are matrices...

\[
I = \begin{pmatrix} 0 & 0 & 0 & \infty \end{pmatrix} \quad F = \begin{pmatrix} \infty \\ 0 \\ 0 \end{pmatrix}
\]

\[
\mu(a) = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \\
1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix}
\]

\[
= \mu(b)
\]
In fact, these are matrices...

\[
\mu(a_1 \cdots a_n) = \mu(a_1) \cdots \mu(a_n)
\]

\[
q_1 = \begin{pmatrix}
0 & \infty & \infty & \infty \\
\infty & 0 & \infty & \infty \\
\infty & \infty & 1 & \infty \\
\infty & \infty & \infty & 0
\end{pmatrix}
\]

\[
q_2 = \mu(a)
\]

\[
q_3 = \begin{pmatrix}
1 & \infty & \infty & \infty \\
\infty & 0 & 0 & \infty \\
\infty & \infty & \infty & 0 \\
\infty & \infty & \infty & 0
\end{pmatrix}
\]

\[
q_4 = \mu(b)
\]

\[
I = (0 \ 0 \ 0 \ \infty)
\]

\[
F = \begin{pmatrix}
0 \\
\infty \\
0 \\
0
\end{pmatrix}
\]
In fact, these are matrices...

\[ \mu(a_1 \cdots a_n) = \mu(a_1) \cdots \mu(a_n) \]

\[ l = (0 \ 0 \ 0 \ \infty) \quad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix} \]

\[ \mu(w)_{i,j} = \text{max of the weights of the runs from } i \text{ to } j \text{ labelled by } w \]

\[ f(w) = l \mu(w) F \]
Questions?
Questions?

- **Decidability:**
  does there exist an algorithm that...
  ?


Questions?

- **Decidability:**
  does there exist an algorithm that... ?

- **Simplification:**
  minimise the number of states = minimise the dimension
determinisation = find an equivalent set of matrices that...
Questions?

- **Decidability:**
  does there exist an algorithm that... ?

- **Simplification:**
  minimise the number of states = minimise the dimension
determinisation = find an equivalent set of matrices that...

- **Comparison**
Questions?

- **Decidability:**
  does there exist an algorithm that... ?

- **Simplification:**
  minimise the number of states = minimise the dimension
determinisation = find an equivalent set of matrices that...

- **Comparison**

*Going both ways...*
Joint spectral radius
Joint spectral radius

\( \Gamma \): finite set of matrices of size \( d \times d \)

Joint spectral radius of \( \Gamma \):

\[
\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \| M_1 \cdots M_\ell \|_\infty \left| M_1, \ldots, M_\ell \in \Gamma \right. \right\}
\]
Joint spectral radius

\( \Gamma \): finite set of matrices of size \( d \times d \)

Joint spectral radius of \( \Gamma \):

\[
\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \|M_1 \cdots M_\ell\|_\infty \right\} \quad \text{for} \quad M_1, \ldots, M_\ell \in \Gamma
\]

**Problem:** Is there an algorithm that, given \( \Gamma \), computes \( \rho(\Gamma) \)?

Best previous result: NP-hardness to compute and to approximate [Blondel, Gaubert, Tsitsiklis]
Joint spectral radius

\[ \Gamma: \text{ finite set of matrices of size } d \times d \]

Joint spectral radius of \( \Gamma \):

\[ \rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \| M_1 \cdots M_\ell \|_\infty \mid M_1, \ldots, M_\ell \in \Gamma \right\} \]

**Problem:** Is there an algorithm that, given \( \Gamma \), computes \( \rho(\Gamma) \)?

Best previous result: NP-hardness to compute and to approximate [Blondel, Gaubert, Tsitsiklis]

**Answer:** No!
Joint spectral radius

\( \Gamma: \) finite set of matrices of size \( d \times d \)

Joint spectral radius of \( \Gamma \):

\[
\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \| M_1 \cdots M_\ell \|_\infty \bigg| M_1, \ldots, M_\ell \in \Gamma \right\}
\]

**Problem:** Is there an algorithm that, given \( \Gamma \), computes \( \rho(\Gamma) \)?

Best previous result: NP-hardness to compute and to approximate

[Blondel, Gaubert, Tsitsiklis]

**Answer:** No!

**Problem:** Is there an algorithm that, given \( \Gamma \) and \( \epsilon > 0 \), computes \( \rho(\Gamma) \) up to \( \epsilon \)?
Joint spectral radius

Γ: finite set of matrices of size $d \times d$

Joint spectral radius of Γ:

$$\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \|M_1 \cdots M_\ell\|_\infty \mid M_1, \ldots, M_\ell \in \Gamma \right\}$$

**Problem:** Is there an algorithm that, given Γ, computes $\rho(\Gamma)$?

Best previous result: NP-hardness to compute and to approximate
[Blondel, Gaubert, Tsitsiklis]

**Answer:** No!

**Problem:** Is there an algorithm that, given Γ and $\varepsilon > 0$, computes $\rho(\Gamma)$ up to $\varepsilon$?

**Answer:** Yes!
Comparison problem: Given a max-plus automaton computing a function $f$, do we have for all words $w$, $f(w) \geq |w|$?

Theorem [Krob, 92] — The comparison problem is undecidable.
Comparison problem: Given a max-plus automaton computing a function $f$, do we have for all words $w$, $f(w) \geq |w|$?

Theorem [Krob, 92]
The comparison problem is undecidable.

Theorem [D., Guillon, Merlet]
The comparison problem for automata with all states initial and final is undecidable.
**Comparison problem**: Given a max-plus automaton computing a function \( f \), do we have for all words \( w \), \( f(w) \geq |w| \)?

---

**Theorem [Krob, 92]**

The comparison problem is undecidable.

---

**Theorem [D., Guillon, Merlet]**

The comparison problem for automata with all states initial and final is undecidable.

Reduction to the problem of computing the joint spectral radius.
Comparison problem: Given a max-plus automaton computing a function $f$, do we have for all words $w$, $f(w) \geq |w|$?

Theorem [Krob, 92]
The comparison problem is undecidable.

Theorem [D., Guillon, Merlet]
The comparison problem for automata with all states initial and final is undecidable.

Reduction to the problem of computing the joint spectral radius $\rightarrow$ even with fixed weights/number of states
Approximation problem:
Input: a max-plus automaton computing a function $f$ and $\varepsilon > 0$
Output: a rational $r$ such that $r - \varepsilon \leq \inf \left\{ \frac{f(w)}{|w|} \mid w \in A^* \right\} \leq r + \varepsilon$

Theorem [Colcombet, D.]
There is an algorithm that solves the approximation problem.

Consequence: There is an algorithm that approximates the joint spectral radius.
Some open problems…
Some open problems…

- Is the comparison problem decidable for automata with 2, 3... states?
  - Is the joint spectral radius decidable for matrices of dimension 2?
Some open problems…

- Is the comparison problem decidable for automata with 2, 3… states?
  = Is the joint spectral radius decidable for matrices of dimension 2?

- Separation of words: an automaton separates two words if it computes different values on these two words
Some open problems...

- Is the comparison problem decidable for automata with 2, 3... states?
  = Is the joint spectral radius decidable for matrices of dimension 2?

- Separation of words: an automaton separates two words if it computes different values on these two words
  - Two words are always separable
  - There is no max-plus automaton that separates every pair of words
Some open problems...

- Is the comparison problem decidable for automata with 2, 3... states?
  = Is the joint spectral radius decidable for matrices of dimension 2?

- Separation of words: an automaton separates two words if it computes different values on these two words
  - Two words are always separable
  - There is no max-plus automaton that separates every pair of words

Fix a number of states \( d \), are there two words that are not separable by a max-plus automaton with at most \( d \) states?
Some open problems...

- Is the comparison problem decidable for automata with 2, 3... states?
  = Is the joint spectral radius decidable for matrices of dimension 2?

- Separation of words: an automaton separates two words if it computes different values on these two words
  - Two words are always separable
  - There is no max-plus automaton that separates every pair of words

Fix a number of states \( d \), are there two words that are not separable by a max-plus automaton with at most \( d \) states?
  = Is there an identity on finitely generated semigroup of matrices of dimension \( d \)?

\( d = 2 \) [Izhakian, Margolis], \( d = 3 \) [Shitov], triangular [Izhakian]
Problem:
Input: a \( \mathbb{Z} \)-max-plus automaton computing a function \( f \geq 0 \)
Output: “yes” if \( f \) is computable by a \( \mathbb{N} \)-max-plus automaton, “no” otherwise

Is this problem decidable?
Problem:
Input: a $\mathbb{Z}$-max-plus automaton computing a function $f \geq 0$
Output: “yes” if $f$ is computable by a $\mathbb{N}$-max-plus automaton, “no” otherwise

Is this problem decidable?

Translation:
Input: a finite set of matrices $\Gamma$ with coefficients in $\mathbb{Z} \cup \{-\infty\}$ such that all $(M_1 \cdots M_k)_{1,2} \geq 0$

Question: is there a finite set $\Gamma'$ of matrices with coefficients in $\mathbb{N} \cup \{-\infty\}$ and $\mu$ a bijection $\Gamma \to \Gamma'$ such that for all $M_1, \ldots, M_k \in \Gamma$:

$$(M_1 \cdots M_k)_{1,2} = (\mu(M_1) \cdots \mu(M_k))_{1,2}?$$
Problem:
Input: a max-plus automaton computing a function \( f \)
Output: “yes” if \( f \) is computable by a deterministic max-plus automaton, “no” otherwise

Is this problem decidable?
Determinisation

Problem:
Input: a max-plus automaton computing a function $f$
Output: “yes” if $f$ is computable by a deterministic max-plus automaton, “no” otherwise

Is this problem decidable?

Translation:
Input: a finite set of matrices $\Gamma$
Question: is there a finite set $\Gamma'$ of matrices with at most one finite coefficient per row and $\mu$ a bijection $\Gamma \to \Gamma'$ such that for all $M_1, \ldots, M_k \in \Gamma$:

\[
(M_1 \cdots M_k)_{1,2} = (\mu(M_1) \cdots \mu(M_k))_{1,2} \ ?
\]
There are a lot of things to gain by using this connection and taking advantage of the best of both worlds.