The Shortest Identities for Max-Plus Automata with Two States

Laure Daviaud
University of Warsaw

With Marianne Johnson, University of Manchester

MFCS 2017
A natural and fundamental question:

Which pairs of inputs can be distinguished by a given computational model?
Given a class of computational models:
Given a class of computational models:

→ Are all the pairs of distinct inputs distinguishable by an instance of the class?
Given a class of computational models:

→ Are all the pairs of distinct inputs distinguishable by an instance of the class?

→ Is there an instance which can distinguish all the inputs?
Given a class of computational models:

→ Are all the pairs of distinct inputs distinguishable by an instance of the class?

→ Is there an instance which can distinguish all the inputs?

→ What is the minimal size of an instance distinguishing two given inputs?
Are all the pairs of distinct inputs distinguishable by an instance of the class? YES

Is there an instance which can distinguish all the inputs? NO

What is the minimal size of an instance distinguishing two given inputs? yields to the profinite theory Automata.
Are all the pairs of distinct inputs distinguishable by an instance of the class? YES
Are all the pairs of distinct inputs distinguishable by an instance of the class? **YES**

Is there an instance which can distinguish all the inputs? **NO**
Are all the pairs of distinct inputs distinguishable by an instance of the class? **YES**

Is there an instance which can distinguish all the inputs? **NO**

What is the minimal size of an instance distinguishing two given inputs? **yields to the profinite theory**
Quantitative extensions of automata

Weighted automata [Schützenberger]
Weighted automata \cite{Schützenberger}

Quantitative extensions of automata

Semiring \((S, \oplus, \otimes)\): transitions are weighted by elements of \(S\)
Quantitative extensions of automata

Semiring \((S, \oplus, \otimes)\): transitions are weighted by elements of \(S\)

Paths: \(\otimes\)  
Non-determinism: \(\oplus\)
Weighted automata \([\text{Schützenberger}]\)

Quantitative extensions of automata

Semiring \((S, \oplus, \otimes)\): transitions are weighted by elements of \(S\)

Paths: \(\otimes\)  
Non-determinism: \(\oplus\)

\[
[A] : w \mapsto \bigoplus_{\rho \text{ accepting path labelled by } w} \left( \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{|w|} \right)
\]
An example in \((\mathbb{R}, +, \times)\)

\[ A = \{0, 1\} \]
An example in \((\mathbb{R}, +, \times)\)

\[ A = \{0, 1\} \]

\[ 100101 \mapsto 2^0 + 0 + 0 + 2^3 + 0 + 2^5 \]
An example in $\mathbb{R}, +, \times$

\[ A = \{0, 1\} \]

\[ 100101 \mapsto 2^0 + 0 + 0 + 2^3 + 0 + 2^5 \]

\[ \rightarrow \text{Are all the pairs of distinct inputs distinguishable by an instance of the class? YES} \]
An example in \((\mathbb{R}, +, \times)\)

\[ A = \{0, 1\} \]

\[ 100101 \mapsto 2^0 + 0 + 0 + 2^3 + 0 + 2^5 \]

\[ \text{\(\rightarrow\)} \text{ Are all the pairs of distinct inputs distinguishable by an instance of the class? } \text{YES} \]

\[ \text{\(\rightarrow\)} \text{ Is there an instance which can distinguish all the inputs? } \text{YES} \]
An example in \((\mathbb{R}, +, \times)\)

\[ A = \{0, 1\} \]

100101 $\mapsto$ $2^0 + 0 + 0 + 2^3 + 0 + 2^5$

$\rightarrow$ Are all the pairs of distinct inputs distinguishable by an instance of the class? YES

$\rightarrow$ Is there an instance which can distinguish all the inputs? YES

$\rightarrow$ What is the minimal size of an instance distinguishing two given inputs? 1 or 2 states...
Max-plus automata

\[ \mathcal{A} : w \mapsto \max_{\rho \text{ accepting path}} (\rho_1 + \rho_2 + \cdots + \rho_{|w|}) \]
Max-plus automata

\[ [A] : w \mapsto \max_{\rho \text{ accepting path labelled by } w} (\rho_1 + \rho_2 + \cdots + \rho_{|w|}) \]

Are all the pairs of distinct inputs distinguishable by an instance of the class? YES
Max-plus automata

\[ \mathcal{A} : w \mapsto \max_{\rho \text{ accepting path labelled by } w} (\rho_1 + \rho_2 + \cdots + \rho_{|w|}) \]

→ Are all the pairs of distinct inputs distinguishable by an instance of the class? YES

→ Is there an instance which can distinguish all the inputs? NO
Max-plus automata

\[ [A] : w \mapsto \max_{\rho \text{ accepting path labelled by } w} (\rho_1 + \rho_2 + \cdots + \rho_{|w|}) \]

→ Are all the pairs of distinct inputs distinguishable by an instance of the class? YES

→ Is there an instance which can distinguish all the inputs? NO

→ What is the minimal size of an instance distinguishing two given inputs? ???
Given a positive integer $n$, are there $u \neq v$ such that for all max-plus automata $\mathcal{A}$ with at most $n$ states:

$$\lbrack \mathcal{A} \rbrack(u) = \lbrack \mathcal{A} \rbrack(v)$$

?
Some results
Some results

\[ W \mapsto \rho_a |w|_a + \rho_b |w|_b \]
Some results

\[ w \mapsto \rho_a |w|_a + \rho_b |w|_b \]

\[ \rightarrow \] With one state

\[ a : \rho_a \]
\[ b : \rho_b \]

\[ \rightarrow \] With two states

Existence of an identity (length 20) [Izhakian, Margolis]

\[ \rightarrow \] With three states

Existence of identities for all \( n \) [Izhakian]
Some results

→ With one state

$w \mapsto \rho_a|w|_a + \rho_b|w|_b$

→ With two states

Existence of an identity (length 20) [Izhakian, Margolis]

→ With three states

Existence of an identity (length 1795308) [Shitov]
Some results

→ With one state

\[ w \mapsto \rho_a|w|_a + \rho_b|w|_b \]

→ With two states
Existence of an identity (length 20) [Izhakian, Margolis]

→ With three states
Existence of an identity (length 1795308) [Shitov]

→ Triangular
Existence of identities for all \( n \) [Izhakian]
Max-plus automata with two states

Theorem [D., Johnson]

There are two identities of minimal length which hold in the class of max-plus automata with two states:

\[ a^2 b^3 a^3 babab^3 a^2 = a^2 b^3 ababa^3 b^3 a^2 \]

and

\[ ab^3 a^4 baba^2 b^3 a = ab^3 a^2 baba^4 b^3 a \]

→ counter-example to a conjecture of Izhakian
An interesting list of criteria

→ list of criteria to eliminate all the shorter identities
An interesting list of criteria

→ list of criteria to eliminate all the shorter identities
An interesting list of criteria

→ list of criteria to eliminate all the shorter identities

- block-permutation
- triangular
- weights restricted to \( \{0, 1\} \)
Ultimate (far away) goal:
Characterize all the identities holding for the class of max-plus automata with at most $n$ states, for all $n$...

...still quite far: Does there exist an identity for all $n$?

...again quite far: Characterize the set of all the identities for $n = 2$.

Is there a strict subset of max-plus automata containing all their computational power?

Perspectives
Perspectives

Ultimate (far away) goal:
Characterize all the identities holding for the class of max-plus automata with at most $n$ states, for all $n$...
Ultimate (far away) goal:
Characterize all the identities holding for the class of max-plus automata with at most $n$ states, for all $n$...

still quite far: Does there exist an identity for all $n$?
Ultimate (far away) goal:
Characterize all the identities holding for the class of max-plus automata with at most $n$ states, for all $n$...

- still quite far: Does there exist an identity for all $n$?

- again quite far: Characterize the set of all the identities for $n = 2$
Ultimate (far away) goal:
Characterize all the identities holding for the class of max-plus automata with at most $n$ states, for all $n$...

- still quite far: Does there exist an identity for all $n$?
- again quite far: Characterize the set of all the identities for $n = 2$
- Is there a strict subset of max-plus automata containing all their computational power?