On the Conception of the *Beyträge*: the Start of Bolzano's Philosophical Thinking

Johan Blok

University of Groningen

It is with the *Beyträge* that Bolzano took his first public step on the field of philosophy of mathematics and logic. It was the result of several years of study of many texts on logic, philosophy and mathematics. Publication of Bolzano's notes allows not only insight in these studies, but more importantly how they inspired Bolzano to develop his own idea's. Investigation of Bolzano's notes and his intellectual context promises to reveal more about Bolzano's start as a philosopher and logician.

The motivating topic for his first philosophical publication is the state of mathematics. The question rises whether Bolzano was the first to complain about this in the eighteenth century, since generally mathematics was treated as the paradigmatic example of apodictic knowledge. Investigation of Bolzano's notes, among them the recently published notebooks of 1803-1810, shows that Bolzano had detailed knowledge of authors who also worried about the state of mathematics. The most important author to whom Bolzano refers is J.A.C. Michelsen, a reputable professor in mathematics and physics. Michelsen wrote a book entitled *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern* that was published in 1789. The title on its own already makes it exceptional for the eighteenth century in which mathematics was generally regarded as the paradigmatic example of apodictic knowledge. The first section of my paper shows how this text not only contributed to Bolzano's complaints about the state of mathematics, but also how it inspired Bolzano to focus on the order within mathematics.

In order to improve the state of mathematics, Bolzano offers a novel conception of logic in which simple concepts are combined into principles which function as the objective ground of other judgements. In the second section I investigate Bolzano's notion of complex concepts, as well as, how he characterizes principles in relation to the distinction between analytic an synthetic judgements.

1 The Problematic State of Mathematics

Contrary to Wolff and Kant who regarded mathematics as the paradigmatic example of apodictic knowledge, Bolzano emphasizes that mathematics ask for many improvements. Although mathematics is one of the most perfect sciences, even its elementary parts have many deficiencies. In terms of the metaphor of a building: the foundations are not secure.[1] In arithmetic and algebra the main problems are opposite, irrational and imaginary numbers, as well as, infinitesimals. Despite the many issues in arithmetic and algebra, in geometry it is much worse. Bolzano for example complains about the absence of proper definitions of crucial concepts, like that of a line.[2] More importantly, many geometrical demonstrations use an intermediate concept that does not belong there such as for example the notion of movement.[3] Bolzano's criticism is such fundamental as to criticize not only the order and organisation of Euclid's elements, but even reject Euclid's method of proof.[4] Although the mathematical developments in the eighteenth century press the problems further, Bolzano's most important criticism concerns classical geometry. Maybe the new developments contribute to bring to the fore that the Euclidean method of proof is fundamentally flawed, but they are not the origin of the main concern of Bolzano.[5] The main problem is much more than just incorporating new mathematical developments into the existing framework. The framework itself must be turned upside down.

1.1 Order within Mathematics

Several of the deficiencies mentioned by Bolzano where already at the center of the debate among mostly Kantian authors on mathematics and its foundations in publications of the last decade of the eighteenth century. Bolzano's notes indicate that he studied the details of a work by

[1] Beytrage, Vorrede

[2] Cf. BGA 2B2/1 p. 71. []

[3] Bolzano, *Betrachtungen \"uber einige Gegenst\"ande der Elementar geometrie (1804)*, p ? Schulz also aims to present a geometry free of 'fremdartigen' concepts such as the concept of motion (Schulz, *Anfangsgründe der reinen Mathesis*, p. a2; §30, p. 16). One cannot takes the proofs for algebra from geometry. One must be able to understand algebra higher arithmetic complete and thoroughly without any knowledge of geometry (ibid., p. 11).

[4] *Beyträge,* [todo].

[5] Compared to Behboud it must be emphasized that Bolzano's reform of mathematics was much more fundamental than the problems addressed by the Berlin academy in their question after a strict theory about the infinite.Behboud, *Bolzanos Beiträge zur Mathematik und ihrer Philosophie*, p. 2, 5 For Bolzano the problems did not start with that.

Michelsen entitled *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern* that was published in 1789.[6] In this work Michelsen carefully raises fundamental criticism on the foundations of mathematics. As such, its topic alone makes the work an exception among the mathematical philosophical works of the eighteenth century. Accordingly, Michelsen's very first sentence in the preface almost sounds as an excuse for focusing more on the imperfections of mathematics than on the enormous advantages compared to other disciplines.[7] Detailed criticism of some issues raised by Michelsen show that Bolzano read the work by Michelsen carefully.

Surprisingly, Michelsen explicitly asks attention for the disorder in mathematics, both between the parts of mathematics and between the propositions within a part of mathematics. Michelsen acknowledges that this idea is not new, but that it is actually not worked out in practice.[8] Even Kästner complains about the distinctness of the many textbooks on geometry. Contrary to previous authors, Michelsen extend the problem of disorder to elementary geometry and the other parts of mathematics. According to Michelsen, the disorganised state of mathematics asks for improvement:

> Und doch kann nicht nur, es muß selbst, wenn die Mathematik Vernunftwissenschaft aus der Construction der Begriffe seyn soll, in der Mathematik eine so durchaus bestimmte, Ordnug gebe, daß dieselbe, wenn bloß auf die Sache gesehen wird, nur eine ist[.][9]

As an example Michelsen discusses the different places of the Pythagorean theorem and refers to a collection of 23 proofs of this theorem. He even mentions a new proof based on differential and integral calculus. Michelsen notices that the theorem can have many places an that its place has a huge influence on the length and comprehension of its proof. Yet, he hesitates to give precise rules to determine the most suitable place. The only advise Michelsen offers is a vague indication that mathematics must be ordered according to something like its 'natural' order. It seems that 'natural' here must be interpreted as the order in which we invent mathematical truths.[10]

In addition to his vague notion of a natural order Michelsen asks for a more complete method of mathematics in order to achieve that nothing in the organisation of mathematical objects,

[6] In the *Beyträge* Bolzano also repeatedly refers to work by Michelsen.

[7] Michelsen, *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern*, p. 4.

[8] ibid., p. 49.

[9] ibid., p. 50.

[10] [Schulz also mentions in his preface that the material must be presented in natural order. [ref]]

definitions, theorems and proof remains arbitrary.[11] As a result of an improved method of mathematics all elements stand in relation to each other:

> Auf diese Art gliche also ie ganze Mathematik Einen Kette, und wenn gleich die Glieder derselben auf vielfache und mannifatige Art in einander griffen, so herrschte doch durchaus die regelmäßige Ordnung, und es müßte leicht sey, von jedem Glieder in ununterbrochener Folge zu jedem andern zu kommen.[12]

Michelsen's notion of order is such that starting from a single point, all elements become interconnected:

> Man findet indeß nicht alle Erklärungen, Forderungen und Grundsätze vor allen Augaben und Lehrsätze, sondern man entdeckt dieses alles in der Ordnung, daß die Erklärungen ud Grundsätze zerstreut zwischen den Augaben und Lehrsätzen zu stehen kommen. Auch bilden die Säße der Mathematik, auf die gedachte Art gesucht, keine Kette aus lauter einfachen Gliedern, sondern, es greift jedes Glied, un zwar je weiter es vom Anfange entfernt ist desto mehr, in eine Menge von Gliedern ein, und die ganze Kette geht von einem sehr geringen Anfange ohne Ende fort.[13]

Such an almost holistic view accounts for the many proofs of the Pythagorean theorem.

Michelsen's focus on the order within mathematics must have inspired Bolzano.[14] For example when he critcizes Euclid in the *Beyträge.* [extend] Here the notion of order starts to be quite general as it applies to mathematical concepts as well as other entities. When Bolzano discusses his notion of proof, he is much more specific about the nature of objective cohesion. It consists of a chain of grounds and consequences in one direction. More importantly it starts from a first principle. As we will see later, Bolzano does not one to regard all logically valid inferences as an objective connection of ground and consequence (*Folge*). Correspondingly, Bolzano defends the view that for each theorem there is only one objective proof. Michelsen's example of the Pytagorean theorem with its many proofs explains why Bolzano treats this issue in the *Beyträge.*[15] Bolzano's strict notion of an objective connection of ground and consequence does not allow the

[11] Michelsen, *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern*, p. 59.
[12] ibid., p. 216.
[13] ibid., p. 207-208.
[14] Generally, Bolzano is quite positive about Michelsen although Bolzano critizes Michelsen for some alleged flaws in mathematics that are easy to correct (GA 2B2/2 p. 105). Schulz in Anfangsgrunden der reinen Mathesis is also concerned with the organisation of mathematics. p. 11;
[15] Bolzano treats the same topic in the later introduction to the *Grössenlehre*, entitled *On Mathematical Method.* []

degree of interconnectedness of Michelsen's notion of order. Accordingly Bolzano does not allow more than one objective proof of one and the same theorem. [todo]

1.2 The Organisation of Mathematics

Apart from the order of the elements of mathematics in terms of chains, Michelsen asks for an overview over the whole of mathematics. This requires to organize mathematics into disciplines. With regard to the division of mathematics into domains he notices that one must first possess knowledge of the propositions or content of mathematics. It is only afterward that one can draw boarders between disciplines. These boarders can then be drawn by distinguishing between the different kinds of quantities (see diagram 1).[16]

Michelsen is quite cautious not to take such boarders very strict:

> Allein dagegen sind ach die Grenzen, die wir ziehen, warlich keine Linien; [...] und auf der andern Seite trennt man vielleicht, was durch ein natürliches Band verknüpft ist.[17]

In the work of Michelsen the organisation into the disciplines is a kind of order that differs from the order in the chain of theorems. Moreover, the former is quite a relative one with a more practical purpose, namely to obtain an overview. The *Beyträge* also contains two notions of order: the objective order of ground and consequence between judgements and the organisation into disciplines by means of a further specification of *Ding überhaupt*.

In a work on mathematics published one year later, Michelsen distinguishes between general and elementary mathematics. In the Kantian spirit, both parts of mathematics involve the construction of mathematical objects. According to Bolzano's paraphrase of the distinction in his notes, elementary mathematics not only use the construction as a means, but also as the object of investigation.[18] General mathematics on the other hand uses construction to investigate the mathematical concepts themselves. Roughly, Michelsen's notion of general mathematics is that part of mathematics that requires the help of philosophy because it investigates a general concept, namely that of quantity in general. Michelsen differs from the traditional division one finds in eighteenth century German textbooks on mathematics as for example Wolff and

[16] Michelsen, *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern*, p. 208. The term 'gemeine' in Michelsen's text is interpreted as 'niedere' and BuchstabenRechekunst as Algebra.
[17] ibid., p. 53.
[18] ibid., p. 117.

Kästner. Traditionally one starts by dividing the concept of quantity in to discrete and continuous quantities. The former consists of arithmetic and algebra, whereas the latter consists of geometry. Given the traditional definition of mathematics as the science of quantities no place is left for a general part of mathematics, although it is claimed that both parts can use each other.

In his notes on Michelsen's distinction Bolzano is quite positive about his division of general mathematics into three parts, namely lower general mathematics, higher general mathematics, and transcendental mathematics.[19] The term transcendental here refers to the infinite small quantities (infinitesimals) as they play a role in differential and integral calculus. This distinction is different from the diagram which is based on a division in terms of kinds of quantities. It seems that this division picks out lower algebra as lower general mathematics; higher mathematics as higher general mathematics [?], and higher algebra as transcendental mathematics. Of interest is that Michelsen provides place for a general mathematics which applies to any kind of quantity. Bolzano's general mathematics plays an analogous role in that it applies to any object. Bolzano's division of mathematics into parts in the Beytrage involves an organisation according to a further specification of whether it is restricted to specific kinds of objects.

Among all authors that worried about the state of mathematics Michelsen seems to have had quite an influence on the early Bolzano. Bolzano's focus on the order of mathematics stems from his detailed readings of the work of Michelsen.

2 From a Logic of Analysis to a Logic of Synthesis

Bolzano's mathematical considerations lead to the urge for a modified epistemology. The standard treatment of the method of mathematics in the dominating textbooks of for example Kästner was still in the tradition of Leibniz and Wolff. In order to account for modern mathematical objects and a strict objective order between propositions Bolzano developed a new approach to concepts and principles.

2.1 Simple and Complex Concepts

In the eighteenth century German speaking tradition the first chapter of a textbook on mathematics is usually devoted to the method of mathematics. In the tradition of Leibniz and Wolff this boils down to a very short introduction to logic in which the examples and the terminology are adapted to mathematics. Part of this is a treatment of how new mathematical concepts come

[19] 2B2/2, p. 117.

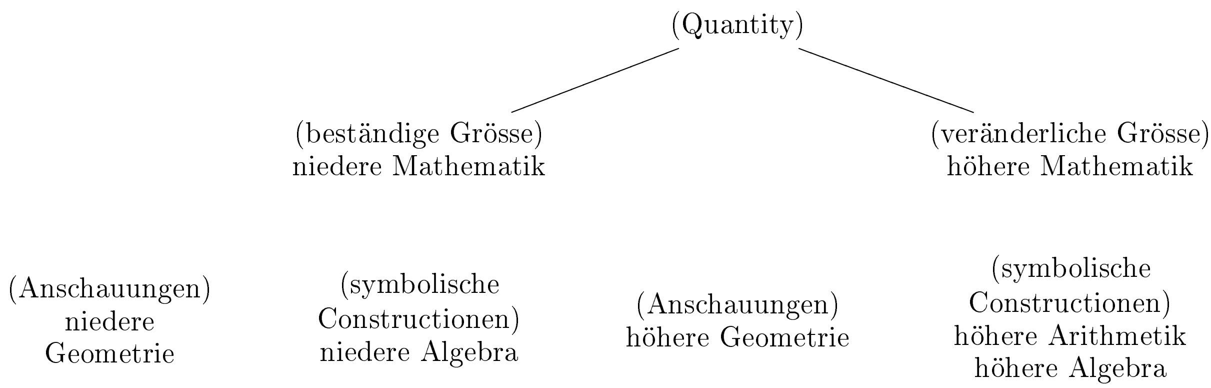

Fig. 1: Organisation of mathematics according to Michelsen in his *Gedanken*.

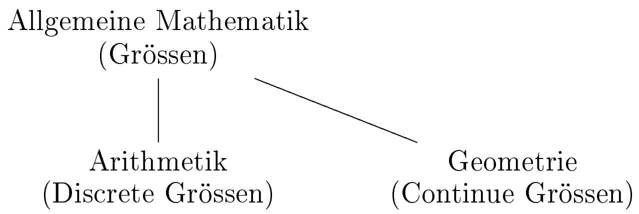

Fig. 2: Traditional organisation of mathematics (Wolff/Kästner).

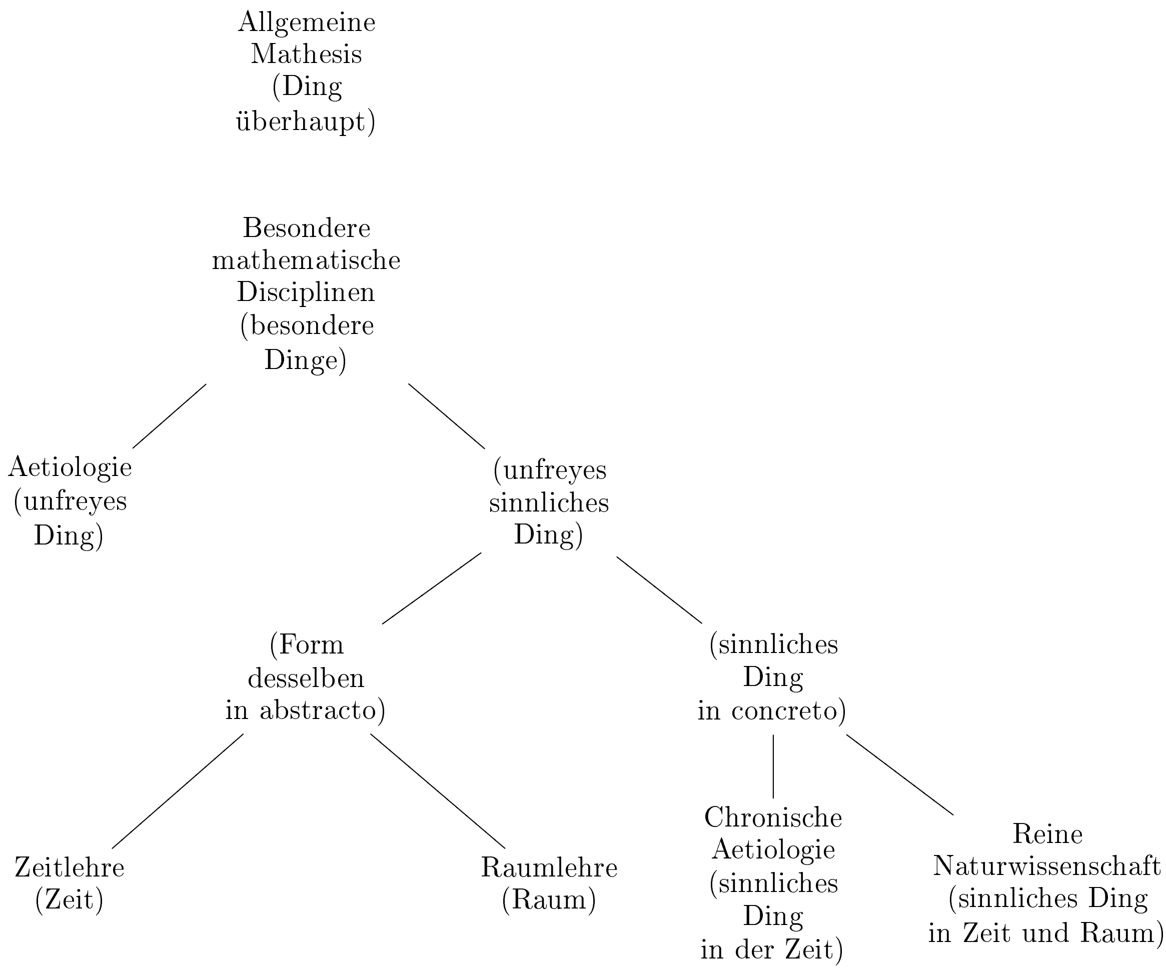

Fig. 3: Organisation of mathematics according to the early Bolzano.

into being. The *Beyträge* does not contain such a part. Bolzano only offers some (heuristic?) rules to determine whether a concept is simple or complex. Without further explanation, Bolzano claims that the fabrication of complex concepts out of simple ones has to obey the laws of possibility of concepts.[20] In the *Beyträge* Bolzano does not explain what these laws are. Fortunately, Bolzano treats the Zusammensetzung of representations in his unpublished continuation of the Beytrage.[21] In his treatment of *Allgemeinen Mathesis,* Bolzano devotes a section to this topic under the title *Von dem Begriffe der Zusammendenkbarkeit als er ersten allgemeinen Eigenschaft der Dinge.*[22]

Bolzano distinguishes several manners of connecting concepts. They boil down to a crucial distinction between A *et* B and *A quod* B. An example of the first connection is 'house and wooden', whereas an example of the latter is 'wooden house'.[23] As the example shows, the latter connection alters the concept. Therefore, the *quod* connection is an objective connection, whereas the *et* connection is a subjective one which only takes place in our mind.[24] For this reason a *quod* connection is called a real, whereas the *et* connection is called ideal. Since according to Bolzano the object general mathematics is *Ding uberhaupt*, real connections do not belong to general mathematics.[25] For a real connection always picks out only a particular kind of objects. Thus, the concepts of general mathematics are ideal connections of simple concepts. Since the connection only has to exist in our mind it does not restrict possible mathematical objects to those that have a corresponding object. This allows Bolzano to account for the problematic modern mathematical concepts such as infinitesimals and imaginary numbers.

2.2 On the Nature of Principles

During the first decade of the nineteenth century Bolzano did not yet make the distinction between proposition (*Satz*) and judgement (*Urtheil*) of the *Wissenschaftslehre* of 1837. Similar to Wolff and Kant he uses these two terms almost interchangeably in his early work.[26] Consequently,

[20] *Beyträge*, II, §6.
[21] [almost within mathematics itself] 2A5 p. 33 §30]
[22] The title indicates that it really is a part of general mathematics, although it traditionally belonged to logic.
[23] 2A5, §33-34, p. 34-35.
[24] 2A5 §37, p. 35. Michelsen might have influenced Bolzano in this respect when Michelsen claims that mathematical concepts are creatures of our mind (Michelsen, *Gedanken über den gegenwärtigen Zustand der Mathematik und die Art die Vollkommenheit und Brauchbarkeit derselben zu vergrößern*, p. 117 (?), p. 152). The laws of thought, that is, logic enables us to attain knowledge of these creatures. However, it is completely unsure whether there are objects in reality that corresponds to these mathematical concepts (ibid., p. 99). According to Michelsen this is the task of applied mathematics.
[25] 2A5, §38, p.36.
[26] As we will see, at some places he reserves the term proposition for analytic judgements because he does not regard them to be proper judgements [todo refs].

principles (*Grundsätze*) are judgements, although of a specific kind. It must be noted that the early Bolzano, in accordance with the tradition, defines judgements as connections between concepts such that something is stated.[27] Traditionally one can find two main characterisations of principles (*Grundsätze*), namely self-evident and unprovable. These characterisations are often combined. Within a Leibnizian-Wolffian context the principles are self-evident because they directly stem from definition and they are unprovable because they are judgements that do not follow from another judgement. Principles are unprovable judgements because they stand at the very beginning of a chain of syllogisms.

In the *Beyträge,* Bolzano rejects the first characterisation because it is a dubious ground to strictly distinguish between principles (*Grundsätze*) and theorems (*Lehrsätze*).[28] Evidence not only comes in degrees, but also depends on many subjective and arbitrary circumstances. Thus, Bolzano is left with the characterisation of principles as unprovable. As he does in many other occasions, Bolzano carefully Bolzano distinguishes between a subjective and objective notion of unprovability.[29] In the case of the former one is not able to provide a proof although it is theoretically possible to find a proof. If a judgement is objectively unprovable it cannot be proved by the very nature of the judgement itself. An objectively unprovable judgement functions as a the final ground from which provable judgements follow:

> Grundsätze sind daher Sätze, welche in objektiver Hinsicht nur immer als Grund und nie als Folge betrachtet werden können.[30]

As such the decision whether a judgement is a principle greatly depends on the inventiveness of the logician. In fact, the nature of logical inference (syllogisms) is such that a conclusion combined with one of the premises gives the other premise as its logical conclusion. Bolzano seems to partly recognise this when he mentions an example of a kind of inference in which two conclusions follow from one premise (see diagram 2.2).[31] The Leibniz-Wolffian tradition appeals to the direct relation to definitions to defend which judgement is the real premise and thereby functions as a principle. Bolzano's rejection of the self-evidence of principle as well as his rejection of definitions as the starting point of scientific enterprise asks for a new epistemological criterion to identify certain judgements as principles.[32] In notes written before the publication

[27] Cf. GA 2A5 p. 33, p. 146. [TODO: remark about Beyträge II §14.
[28] *Beyträge*, II §10 (p. 40).
[29] *Beyträge*, II §11-12 (p. 41-42). As in the later *Wissenschaftslehre*, the objective version is indicated with the attribute in itself (*an sich*).
[30] *Beyträge*, II §12.
[31] *Beyträge*, II §12 (p. 44).
[32] In the *Beyträge*, Bolzano explicitly asks this question. II §14

A ist B cum C.

A ist B, A ist C $\therefore$

Fig. 4: Example of two conclusions on the basis of one premise.

of the *Beyträge,* Bolzano attempts to characterize principles by means of the distinction between simple and complex concepts. His idea is that in the case of principles at least the subject is a simple concept. In some early notes Bolzano repeatedly attempts to prove the thesis that every true a priori judgement about a complex concept is provable, and hence, not a principle. The manuscript that contains these notes most likely stems from 1809 and hence is very close to the conception of the *Beyträge* published in 1810.[33] The amount of attempts show that the thesis was of crucial importance to Bolzano.

Bolzano's first attempt starts with the traditional observation that the ground of an a priori judgement is to be found in its subject.[34] In the case of a complex subject this is ground is to be found in its parts. Therefore one must investigate its parts and judge about them to find the ground of the a priori judgement. This means that the judgement relies on other judgement(s); hence, it is provable. For example 'a European is mortal'.[35] According to its definition the complex subject 'European' has the following parts 'human being who lives in Europe'. The predicate of mortality applies to one of these parts, namely 'human being'. In a slightly different version Bolzano argues that the complex subject in the end is composed of simple concepts. Accordingly, the judgement depends on judgements about these simple concepts. Evidently, such a dependence makes the judgement provable.

It seems Bolzano whether the ground of a judgement indeed is only to be found in one of the parts of its complex subject. In subsequent attempts Bolzano distinguishes several cases. In the first case the ground indeed is to be found in one of the parts of a complex subject: Bolzano's first attempt is satisfactory in this case. In the second case Bolzano considers that the ground is not to be found in one of the parts of the complex subject, but only in the whole of it. He seems to think of a ground that does not apply to an individual part, but only to a certain combination of parts. Bolzano's first argument that also in this case the judgement is provable is formulated

[33] Cf. GA 2B14, p. 12.
[34] GA 2B15, p. 193. [Refer to Wolff/Meier (Leiden paper)]
[35] 2B15, p. 195.

A ist B cum C.

A ist B, A ist C ∴

Fig. 5: Composition as a proper ground/consequence relation.

so short it is hardly comprehensible. The other two cases are judgements of such a form that it is claimed that the complex subject is possible. These cases are crossed out by Bolzano and a remark is added that he is not yet certain whether there proofs are correct.

After this failed attempt Bolzano changes his claim to apply to the predicate: if a predicate is complex the true a priori judgement is provable. Bolzano argues as follows: from such a judgement 'M is (X+Y)' it follows (by syllogism) that 'M is X' and 'M is Y'. Indeed, this is exactly the case we encountered before in diagram 2.2 By means of composition, according to Bolzano to be carefully distinguished from syllogisms, one can argue vice versa (see diagram 2.2). The question now is which of the judgements are unprovable. In the *Beyträge* Bolzano discusses exactly this case and claims:

> Ich kann wohl subjektiv aus der wie immererkannten Wahrheit des ersten dieser drei Sätze die Wahrheit der beiden anderen erkennen, aber ich kann den ersten nicht objektiv als Grund von den zwei anderen ansehen.[36]

In the *Beyträge* Bolzano does not really give an argument. Fortunately, such an argument is contained in the manuscript:

> Diese sind aber einfacher als jener. Mithin kann man sagen: er werde durch sie erwiesen. Da fordert es nun die gute Ordnung offenbar, daß man erst jene aufstelle und dann dieses aus ihnen herleite.[37]

Thus, the judgements 'M is ' and 'M is Y' are the objective ground for the judgement 'M is (X+Y)' because they are less complex. Hence, a judgement with a complex predicate is provable.

Bolzano's new attempt to prove that a judgement about a complex subject is provable uses the claim about the judgement with a complex predicate. Bolzano distinguishes three cases. In the first case, the predicate is complex. This case is already proved, so we only have to discuss

[36] *Beyträge*, II §12.
[37] GA 2B15 p. 196-197.

the cases where the predicate is simple. The second case is when the simple predicate can be attributed to a part of the complex subject. That the judgement in this case is provable is already shown in the very first attempts. In the third case, the predicate only applies to the subject as a whole and not to an individual part. At this point Bolzano raises the question whether there are indeed judgements of this kind. Bolzano gives an example: 'the spacial thing that is shared by two distinct straight lines, is a point' and claims that the judgement in fact should be: is only one point. Even further transformed in to a negative sentence: That what is shared by two distinct straight lines are not two points.] Not clear what his argument is here: Bolzano ends with a description of the case: a simple predicate that only adheres to the 'Vereinigung der Begriffe A, B, C' and not to an of the constituents on their own. Since at this points, his notes break of it is not completely certain what the example is intended to show. Yet, the example suggests that there actually do not exist judgements of the kind of the third case. It is suggested that judgements that seem to be of this kind, can be transformed such that it is revealed that the predicate is complex.

In the *Beyträge* Bolzano starts with the Kantian distinction between analytic and synthetic judgements in which the predicate of analytic judgements is mediately or immediately contained in the definition of the subject.[38] According to Bolzano this definition entails that an analytic judgement can never function as a principle. For an analytic judgement is provable by means of the definition of the subject. This argument confirms that Bolzano denied the existence of judgements of the third case. According to Bolzano it also immediately follows from the definition of the distinction between analytic and synthetic judgements that judgements with a simple subject are synthetic.[39] Since Bolzano claimed that an analytic judgement cannot be a principle, the conclusion (via the step that principles are synthetic judgements) that principles have a simple subject already seems inevitable. Nevertheless, Bolzano adds the arguments developed on his notes in the *Beyträge* to argue that either a complex subject or a complex predicate makes a judgement provable.[40] As conclusion Bolzano's criterion or characteristic for principles is that they both have a simple subject and a simple predicate:

> [D]ie eigentlich unerweislichen Sätze oder Grundsätze sind unter der Klasse bloß jener Urteile zu suchen seie, in welches beides, Subjekt und Prädikat, ganz einfache Begriffe

[38] *Beyträge*, II §17.
[39] *Beyträge*, II §19.
[40] *Beyträge*, II §20.

sind.[41]

It must be noted that Bolzano's formulation is quite explicit in leaving space for judgements with simple subjects and predicate that are nevertheless not principles. Thus, contrary to the tradition, principles do not stem immediately from the definition of complex concepts, but instead consist of simple concepts.

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[41] *Beyträge*, II §20.