CHANGES IN BOLZANO'S DEFINITION OF MATHEMATICS

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ABSTRACT. The paper will discuss some changes in Bolzano's definition of mathematics by comparing some passages from the *Beyträge*, *Wissenschaftslehre* and *Grössenlehre*: is mathematics a theory of forms or a theory of quantities? Relevant changes between 1810 and 1848 will be analyzed with respect to the distinction between analytic and synthetic propositions and to Kant's influence. It will be argued that the change in Bolzano's views can be better explained if one interprets Bolzano's philosophical stance in the *Beyträge* as deeply influenced by Kant's philosophy: notwithstanding the harsh criticism to be found in the Appendix, Bolzano is still somehow Kantian, or at least more than he thinks. It is only in the *Wissenschaftslehre* and in the *Grössenlehre*, where Bolzano develops a new logical theory and better defines some mathematical concepts that he finally distances himself from Kant, and moves from a foundational perspective that is external to mathematics to a philosophical analysis that is related to the solution of certain specific mathematical problems.

1. INTRODUCTION

It is well known that Bolzano, after having criticized in the *Beyträge* the traditional[1] definition of mathematics as a theory of quantities,

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[1]For a discussion of the origin and development of the so-called 'traditional' definition of mathematics, see [3, chap. 3].

and suggested an alternative characterization as a theory of forms, went back to the traditional definition in the *Grössenlehre*. This paper aims at understanding why Bolzano changed his mind and what effect did this change have on his conception of mathematics. A distinction between some issues that remained substantially unaltered and other issues that radically changed between 1810 and 1848 will be helpful to focus the attention on Bolzano's evolution. The latter will be explained both as an effect of the new logical theory developed by Bolzano in the *Wissenschaftslehre*, and as a consequence of the radicalization of Bolzano's anti-Kantianism.

Firstly, I will claim that Bolzano's return to the definition of mathematics as a science of quantities was not a U-turn (§ 2), because there are several essential issues that were maintained throughout all his writings. Bolzano did not renounce to the idea that mathematics is a truly conceptual science, or to the idea that its domain includes other objects besides numbers and magnitudes. Nor did he change his mind concerning the fact that mathematical treatises include both analytic and synthetic propositions. Besides, he was always convinced that the definition of mathematics should not be based on a domain of objects: he always preferred other definitional criteria, such as a sharp distinction with respect to other forms of knowledge (philosophy, logic, other scientific disciplines), or methodological considerations.

Secondly, I will present some relevant changes that might support the claim that Bolzano's notion of mathematics did nonetheless undergo a significant evolution from 1810 to 1848 (§ 3). The development

of the theory of ideas in themselves induced him to abandon the idea that mathematical objects could be characterized as possible entities:. Besides, he gave a different evaluation of the role of analytic propositions in the *Beyträge* and in the *Wissenschaftslehre*. This can be partly explained as a result of a change in the understanding of the notion of analyticity.

Finally, I will analyze the mentioned elements of continuity and discontinuity between 1810 and 1848 in order to evaluate the coherence or incoherence of Bolzano's evolution. Special attention will be paid to the overcome of Kant's conceptual framework and of the urgency to reverse his definition of mathematics point by point (§ 4). I will discuss the effect of the changes on Bolzano's conception of mathematics. In particular, I will suggest that the evolution is not only the effect of a radicalization of the differences with respect to the Kantian approach, but also a result of the new perspective introduced by the logical theory of ideas and propositions in themselves. The return to the traditional definition is combined with a different understanding of the notion of quantity and should thus be interpreted as Bolzano's attempt to reconcile the doctrine of mathematical numbers and quantities as non actual entities with the notion of a mathesis universalis.

2. From 1810 to 1848: A strong continuity

2.1. **Mathematical objects are not just quantities.** In the first chapter of the *Beyträge*, Bolzano contrasted Euclid's lack of a definition of mathematics and the traditional definition that can be found in contemporary textbooks:

It is well known that the oldest mathematical textbook,
Euclid's Elements—which in some ways is still unsurpassed—
contains no definition of the science with which it is con-
cerned. Whether its immortal author did this out of
a kind of willfulness, or because he thought it was not
worthwhile, or because he did not know any valid defini-
tion to give us, I shall not venture to decide. By contrast,
in all *modern* textbooks of mathematics this definition is
put forward: 'mathematics is *the science of quantity*'.
Kant has already found fault with this definition in his
Kritik der reinen Vernunft (see the 2nd edition, S. 742)
because in it, as he says, '*no essential characteristic* of
mathematics is stated, and the *effect* is also mistaken for
the *cause*.'[2]

Similar criticism of the traditional definition of mathematics was quite
common in 18th century German-speaking philosophers: Kant, but
also Hegel and after him also Grassmann criticized the German def-
inition of mathematics as 'Grössenlehre'. This is partly due to the
fact that Wolff, introducing a German terminology for mathematics
and philosophy in his *Mathematisches Lexicon*, had translated several
distinct notions—moles, volumen, quantitas, magnitudo—by the same
word: 'Grösse'.[3] Bolzano's rejection of the traditional definition begins
by a mention of Kant's earlier critical remarks. This is something more

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[2]Cf. Beyträge § 1 in [8, p. 91].

[3]For a detailed analysis of Wolff's contribution to the unification of the concept
of magnitude and quantity in German terminology, see [4]. In the following the
German word 'Grösse' will be translated into English by 'quantity'.

than an argument *ad auctoritatem.* Bolzano agrees with Kant's objection: a satisfactory definition of mathematics should not be based on its domain—i.e. quantities. While Kant argues that such a definition is inadequate because it mistakes the effect for the cause, and gives an explanation of why mathematics applies to quantities, Bolzano claims on the contrary that the definition is unsatisfactory because the domain of mathematics does not coincide with quantities, and besides the notion of quantity is not univocally determined.

Bolzano's criticism is expressed in § 2 of the *Beyträge,* where he quotes a text published in 1805 by Franz Anton Ritter von Spaun,[4] and argues that the objects of mathematics are neither quantities in the sense of conceivable entities nor quantities in the sense of sensible objects.[5] Contrary to what he will claim in the *Grössenlehre,* Bolzano argues that the objects of mathematics are not "the *objects to which the concept of quantity is especially applicable*" either, because in that

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[4]This text, which contains a criticism of negative quantities, was reviewed in 1807 in the *Jenaische Allgemeine Literatur-Zeitung,* presumably by Bolzano himself. See [5].

[5]"Naturally everything here depends on what is understood by the word '*quantity*'. The anonymous author of the book *Versuch, das Studium der Mathematik durch Erläuterung einiger Grundbegriffe und durch zweckmässigere Methoden zu erleichtern.* Bamberg and Würzburg, 1805 (S. 4), puts forward the following definition of quantity, '*A quantity is something that exists and can be perceived by some sense.*' This definition is always one of two things, either too wide or too narrow, according to whether the author takes the words '*exists*' and '*can be perceived*' in their widest sense when they mean a purely *ideal existence* and a *possibility of being thought,* or in their narrower and proper sense in which they hold only for a *sensible object which actually exists.* In the first case, quantity would be *every conceivable thing without exception* and if we then defined mathematics as the science of quantity we would basically bring all sciences into the domain of this one science. On the other hand, in the second case, only *sensible objects* would be quantities, and the domain of mathematics would obviously then be excessively restricted—because immaterial things, e.g. spirits and spiritual forces, can also become an object of mathematics, and particularly of arithmetic." *Beyträge,* § 2 in [8, p. 91.]

case all sciences, logic included, would have to be considered as mathematical sciences; besides, no consideration of the frequency of application could ever provide a definite distinction between mathematical and non-mathematical sciences.[6]

After having shown that several definitions of mathematics based on its domain of objects as quantities are inadequate, Bolzano turns to the analysis of Kant's suggestion, which is based on the opposition between philosophy and mathematics and on the notion of an a priori intuition that grants the possibility of a construction of mathematical concepts.

Bolzano's criticism of Kant will be further discussed in § 3.3, but it must be noted here that the argument given by Bolzano to reject the definition of mathematics as a science of quantities is at the same time an argument against the Kantian definition. Rejecting the traditional definition because the effect is mistaken for the cause, Kant did not question the idea that mathematics concerns only quantities. On the contrary, he intended to give an explanation of the reason why mathematics concern only quantities: quantities can be represented in space

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[6]"The concept of quantity is *applicable* to all objects, even to *objects of thought*. Therefore if one wanted to consider the mere *applicability of the concept of quantity* to an object a sufficient reason for counting the theory of that object among the mathematical disciplines, all sciences would in fact have to count as mathematics, e.g. even the science in which the proposition is proved that there are only *four* (or as *Platner* more correctly states, only two) syllogistic figures; or the science which states that there are *no more and no less than four sets of three* pure simple concepts of the understanding (categories), etc. Therefore in order to salvage this definition one would have to take into account the *difference between rarer and more frequent applicability*, i.e. count only those objects to be in mathematics to which the concept of quantity can be applied *often and in many ways.* But anyone can see that this would be an extremely vague, and not at all scientific, determination of the boundaries of the domain of mathematics. We must therefore look for a *better* definition." *Beyträge,* § 4 in [8, p. 92.]

and time. So, the form of mathematical knowledge is the reason why it concerns only quantities.[7]

In the *Beyträge*, on the contrary, Bolzano rejects the definition of mathematics as a science of quantities not only because the notion of quantity is too vaguely defined, but also because he believes that, no matter how broad the notion of quantity might be, mathematics would always concern objects that are definitely not quantities. Quantities are in fact considered as magnitudes (extensive or intensive) whose properties can be fully described by numbers. Neither the notion of geometrical point, nor the notion of a permutation can thus be included among quantities.

> If we do not wish to move too far away from the use
> of language (something which we should surely never do
> even in the sciences without necessity), then we must
> understand by quantity, a *whole in so far as it consists*
> *of several equal parts*, or even more generally, *something*
> *which can be determined by numbers.*[8]

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[7]"Die philosophische Erkenntniß ist die Vernunfterkenntniß aus Begriffen, die mathematische aus der Construction der Begriffe. Einen Begriff aber construiren, heißt: die ihm correspondirende Anschauung a priori darstellen." Kant (1787) *KrV*. B742-3. "Alles, was im Raum und in der Zeit vorgestellt wird, hat extensive Größe." Kant (1821), *Vorlesungen über die Metaphysik* (Pölitz), p. 53. "Die Form der mathematischen Erkenntniß ist die Ursache, daß diese lediglich auf Quanta gehen kann." Kant (1787) *KrV*. B742-3). "Da die Größe den Gegenstand der Mathematik ausmacht, und in Betrachtung derselben nur darauf gesehen wird, wie vielmal etwas gesetzt sei, so leuchtet deutlich in die Augen, daß diese Erkenntniß auf wenigen und sehr klaren Grundlehren der allgemeinen Größenlehre (welches eigentlich die allgemeine Arithmetik ist) beruhen müsse." Kant (1864), *Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral*, p. 282.

[8]"The concept of quantity, or of number, does not even appear in many problems of the *theory of combinations* (this very important part of general mathesis). For example, if the question is raised: *which permutations—not how many—of the given things a, b, c, . . . are admissible?* In the particular parts of mathematics,

A similar argumentation is presented in the *Grössenlehre*, where Bolzano goes back to the traditional definition of mathematics. How is this possible? Firstly, Bolzano applies a principle that he had already enunciated in the *Beyträge*, and that constitutes a leitmotiv of his work: the linguistic usage is relevant and should not be modified without necessity.[9] Secondly, Bolzano is still convinced that a science cannot be univocally characterized by its domain of objects—and thus even if mathematics is defined as a science of quantities, this need not mean that it can be applied only to quantities.[10] Secondly, he assumes a different, and more general meaning of the word quantity.[11] So, Bolzano

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chronometry, *geom*etry etc., as the names suggest, some object *other* than the concept of *quantity* (e.g. time, space, etc.) appears everywhere, and the concept of quantity is just *frequently applied* to it. So that in all these disciplines there are several axioms and theorems which do not even contain the concept of quantity. Thus, for example, in chronometry the proposition that *all moments are similar to each other*, and in geometry that *all points are similar to each other*, must be established. Such propositions, which do not contain the concept of a quantity or number at all, could never be established in mathematics if it were merely a *science of quantity*." Cf. B.I.3 in [8, p. 92.].

[9]"The traditional definition of mathematics that has been given up to now does not say anything else than that mathematics is the science of quantities (scientia quantorum). So I hope that my definition will not be blamed for differing too much from the ordinary one." *Grössenlehre*, Einleitung, § 2, Anm. in GA II.A.7 p. 27.

[10]"However I have to explain that I did not keep the same expressions. One should rightly say that mathematics is a theory of quantities, only if the objects that are considered in the different mathematical sciences are quantities altogether and just because of the fact that they are considered in those sciences. But that is not the case. On the contrary, many mathematical sciences rather concern, at least partially, objects that are not quantities, even if the concepts of quantity and number are used to examine them." *Grössenlehre*, Einleitung, § 2, Anm. in GA II.A.7 p. 27.

[11]"I believe that the best thing to do would be to distinguish two meanings of the word 'quantity' [Grösse], a broader and a narrower meaning. In a narrower sense one could understand by quantity what one usually calls continuous quantities; in a broader sense one could take the word to include both the continuous and the so-called non continuous or discrete magnitudes, and nothing else. If mathematics is generally defined as a science of quantities (a definition I am now basically returned to, as we will see soon), one undoubtedly takes the word quantity in its broader sense, for one considers also the theory of numbers as a mathematical discipline, and maybe as one of the most important disciplines. It is in this broader sense, I think,

reverts to the traditional definition of mathematics because he wants to preserve a well-established, almost idiomatic expression, but at the same time he changes its meaning by giving a different explanation of the concept of quantity. The linguistic use, and the mathematical practice are maintained but the usual way of speaking is combined with a conceptual change. Besides, the reference to quantities that is suggested in the definition can be understood on pragmatic grounds: it is useful to underline the propaedeutic function of the general theory of quantities in mathematics.

Sofar, the evolution from the definition of mathematics as a theory of forms to the traditional definition of mathematics as a science of quantities did not appear as a radical change, but we wil see in § 3 what further changes induced Bolzano to abandon the definition of mathematics as a theory of forms.

2.2. **Mathematics without intuitions: a truly conceptual science.** A constant feature of Bolzano's anti-Kantianism is the refusal to grant intuitions any role in mathematics: the latter is a merely conceptual science. In the Appendix to *Beyträge*, where Bolzano makes his critique to Kant, concepts and intuitions are defined as following: "All ideas are either intuitions, i.e. ideas of an individual, or concepts, i.e. ideas of something general."[12] Against Kant, Bolzano argues that there is no such a thing as pure intuition: space and time are forms,

that we call any object a quantity (quantum), when we consider it as belonging to a species of things, between any two of those we can assert one and just one of the following reciprocal relations: either they are equal one to the other or one contains a part that is equal to the other." *Grössenlehre*, Einleitung, § 1, in GA II.A.7 pp. 25-6.

[12]*Beyträge*, Appendix, § 2, in [8, p. 132.]

but they are concepts, just like smell and colour. He also argues that arithmetic does not require the pure intuition of time in order to be universal and necessary knowledge, nor does geometry require the pure intuition of space. Mathematics is a conceptual science, that is to say, a science that contains just concepts. A non conceptual judgment is on the contrary a judgment that contains at least an intuition. Mathematics does not contain any intuition. Its objects are not individual but general.

The harsh criticism of Kant's notion of an a priori intuition is due to Bolzano's strong conviction that mathematics is a conceptual science. In the *Beyträge* he explicitly defines it as a "*a science which deals with the general laws (forms) to which things must conform [sich richten nach] in their existence [Dasein]*", and adds that "mathematics never deals with a single thing as an *individual* but always with whole *genera* [Gattungen]."[13] Having assumed the Kantian distinction between intuitions as ideas of an individual and concepts as ideas of something general, there is no doubt that mathematics, concerning only genera, is a conceptual science.

Bolzano wants to eliminate any appeal to intuition from mathematics. As a mathematician he has to solve the problem of imaginary or ideal quantities, that is to say, objects that do not have a corresponding intuition in mathematics (the infinite line, complicated spatial objects

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[13]*Beyträge*, § 8, in [8, p. 94.]

in stereometry, etc...). Bolzano does not question their possibility, but grounds it on the independence from intuition and imagination.[14]

2.3. **Mathematics contains analytic and synthetic propositions.**

In the *Beyträge* Bolzano maintained the traditional Kantian definition of analyticity, and he claimed that mathematics contains synthetic propositions (e.g. axioms) and analytic propositions, which play the role of definitions. In the *Wissenschaftslehre* Bolzano introduced a different definition of analyticity, but mathematics still contains both analytic and synthetic propositions. Only their role is antithetic to the role they played in Kant's theory.

The distinction between analytic and synthetic judgments is introduced in the *Beyträge* with respect to judgments of the form $<S$ is a kind of $P>$, which means that S is a species of the genus P. If I can find a differentia specifica D so that S is decomposable into P cum D, then the judgment is analytic: $<P$ cum D is a kind of $P>$. If I cannot find D, then I can only decompose S into P cum S, which is but an apparent definition, and S is a simple concept. Bolzano mentions the following examples: a) "A point is a spatial object" is a synthetic judgment, because it can only be decomposed into "A spatial object which is a point is a spatial object"; b) "Human are animals" on the contrary

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[14]"The proposition that every straight line can be extended to infinity has no intuition behind it: the lines which our imagination can picture are not infinitely long. In stereometry we are often concerned with such complicated spatial objects, that even the most lively imagination is no longer able to imagine them clearly; but we none the less continue to calculate with our concepts and find truth." *Beyträge*, Appendix, § 9, in [8, p. 136.]

is analytic, because it can be decomposed into: "An animal who is rational is an animal".[15] The Kantian distinction between analytic and synthetic judgments is thus maintained: a judgment is called analytic if the predicate is contained directly, or indirectly, in the definition of the subject; every other judgment is synthetic.[16]

The distinction between analytic and synthetic judgments is relevant to understand their respective roles in mathematics. Analytic judgments cannot be considered as axioms, for they are composite and thus provable. They are actually linguistic propositions rather than judgments, because they inform us on different designations used to denote the objects rather than on the objects themselves. Like Kant, Bolzano considers them as not ampliative, and therefore as not properly deserving a place in a scientific system, if not as linguistic conventions.[17]

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[15]"A classification of judgements quite different from those considered so far, which has, since Kant, become particularly important, is the classification into analytic and synthetic judgements. In our so-called necessity judgements, the subject appears as a species whose genus is the predicate. But this relation of species to genus can be of two kinds: either there is a characteristic which can be thought of and stated in itself, which is added in thought as a differentia specifica to the genus (predicate P) to produce the species (subject S) or not. In the first case the judgement is called analytic; in every other case, which may be any of the classes mentioned in § 15, it is called synthetic." *Beyträge*, II §17 in [8, p. 115.]

[16]"In other words an analytic judgement is such that the predicate is contained directly, or indirectly, in the definition of the subject, and every other one is synthetic." *Beyträge*, II §17 in [8, p. 115.]

[17]"From this definition it now follows immediately that analytic judgements can never be considered as axioms, indeed, in my opinion they do not even deserve the name of judgements, but only that of propositions, they teach us something new only as propositions, i. e. insofar as they are expressed in words, but not as judgements. In other words the new [fact] which one can learn from them never concerns concepts and things in themselves but at most only their designations. Therefore, they do not even deserve a place in a scientific system, and if they are used, it is only to recall the concept designated by a certain word, just as with conventions. In any case, it is decided even according to the usual view, that analytic judgements are not axioms, for their truth is not recognized from them themselves, but from the definition of the subject." *Beyträge*, II § 18 in [8, p. 115.]

Synthetic judgments are axioms for two reasons. On the one hand, because "there are true definitions only for concepts which are composite, and therefore also decomposable again,"[18] that is to say only analytic propositions might play the role of definitions. On the other hand, because axioms have to be unprovable and all analytic propositions are provable.[19] So, after having proved that there are some synthetic judgments (e.g. judgments whose subject is a simple concept),[20] Bolzano argues that mathematics has axioms, because there are simple concepts that belong properly to mathematics.[21] Provided that one does not consider synthetic judgments as based on any intuition, then one might agree with Kant tat the majority of propositions of arithmetic are synthetic.[22]

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[18]*Beyträge*, II § 18 in [8, p. 104.]

[19]"I believe I have found out that all judgements whose subject or predicate are composite concepts must be provable judgements." *Beyträge*, II § 16 in [8, p. 114.] "Hence it now follows that the really unprovable propositions, or axioms, are only to be sought in the class of those judgements in which both subject and predicate are completely simple concepts." *Beyträge*, II § 20 in [8, p. 117.] "If therefore the word 'axiom' is to be taken in an objective sense we must understand by it a truth which we not only do not know how to prove but which is in itself unprovable." *Beyträge*, II § 11 in [8, p. 110.]

[20]"Therefore if all our judgements were analytic there could also be no unprovable judgements, i. e. axioms at all. [...] we want to try and demonstrate, in a way independent of § 15, that there actually are synthetic judgements. All judgements whose subject is a simple concept are thereby already synthetic." *Beyträge*, II § 19 in [8, p. 116.]

[21]"If the foregoing is correct, the question can now be answered, 'whether mathematics also has axioms?'. Of course, if all mathematical concepts were definable concepts, then there could be no axioms in the mathematical discipline. But since there are simple concept which belong properly to mathematics (§8), one certainly has to acknowledge actual axioms in it. The domain of the axioms stretches as far as that of the purely simple concepts: where the latter end and the *definitions* begin, there also the axioms come to an end and the *theorems* begin." *Beyträge*, II § 22 in [8, p. 119.]

[22]"The principle of sufficient reason, ant the majority of propositions of arithmetic are, according to Kant's correct observation, synthetic propositions." *Beyträge*, Appendix § 8 in [8, p. 135.]Cf. B.A.8.

A new definition of analytic and synthetic propositions is presented
by Bolzano in the *Wissenschaftslehre* (especially § 148), where it is
grounded on a new theory of truth. Bolzano distinguishes between an-
alyticity and logical analyticity: both are associated to a proposition in
itself, the objective counterpart of a sentence or of a judgment. Each
proposition in itself has a degree of validity (Gültigkeit) with respect
to some of its parts (representations in itself) that might be considered
as variables. The degree of validity of a proposition with respect to
one or more of its parts is expressed by the ratio of the number of true
propositions and the number of all (objectual) propositions that are
obtained by variation of that part or those parts of the proposition. A
proposition is universally valid or invalid with respect to those parts
when the ratio is 1 or zero respectively. A proposition is analytic if it
is universally valid or invalid with respect to at least one of its variable
parts. A proposition is logically analytic if it is universally valid or in-
valid with respect to all its non logical parts. A proposition is synthetic
if it is not analytic.[23]

Mathematical truths are partly analytic partly synthetic. There are
universal formulations of mathematical truth that are synthetic but
if one instantiates them, one obtains an analytic truth. For example,
"The angles of this triangle are together equal to two right angles" is

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[23]"But suppose there is just a single [auch nur ein einziger] idea in it [in a propo-
sition] which can be arbitrarily varied without disturbing its truth or falsity, i. e.
if all the propositions produced by substituting for this idea any other idea we
pleased are either true altogether or false altogether [...] I permit myself, then, to
call propositions of this kind, borrowing an expression from Kant, analytic. All the
rest, however, i. e. in which there is not a single idea that can be arbitrarily varied
without affecting their truth or falsity, I call synthetic propositions." *Wissenschaft-
slehre*, § 148. For the criticism of Kant see also § 315.

analytic with respect to 'this'. Similarly, "The angles of an equilateral triangle are together equal to two right angles" is analytic with respect to 'equilateral'. On the contrary, "The angles of any triangle are together equal to two right angles" is synthetic, and the previously mentioned truths are provable from this truth.[24] This example looks paradoxical: a universal proposition is considered to be synthetic, whereas if one instantiates it, one obtains an analytic proposition. De Jong [6] has suggested an interesting interpretation: a Kantian example of a synthetic a priori proposition that was considered to be grounded on intuition is transformed into an analytic proposition. Bolzano's criticism of Kant's recourse to intuitions is here more radical, and it explains why analytic truths deserve a place in a scientific system: given the fact that they are not self-evident, they often deserve not only a place but also a proof in the system, e.g. a derivation from a general synthetic principle.[25]

In short, Bolzano does not change his mind concerning the fact that mathematical treatises include both analytic and synthetic propositions, even if, as we mentioned, the role of such propositions might vary according to a variation of their respective definitions. Nor does

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[24] Cf. *Wissenschaftslehre*, § 197.

[25] "From these examples one can already derive that not every analytic proposition expresses a self-evident truth, and thus the aim of teaching analytic propositions to somebody is not fully superfluous; on the contrary it's quite clear that even pure analytic propositions are sometimes so remarkable that they not only deserve a place in a textbook, but they make us feel obliged to guarantee their truth with a proof. And indeed it cannot be denied that such analytic propositions, whose truth is not directly evident, can easily be known as true, after one has learned a synthetic truth from which they follow." *Wissenschaftslehre*, § 447.

he renounce to the idea that mathematics is a truly conceptual science, or that its domain includes other objects besides quantities.

3. Yet something changes

3.1. **A sharp distinction between mathematics and other sciences.** In the *Grössenlehre* the sharp distinction between mathematics and other sciences that Bolzano is considered as unattainable, or at least as incompatible with the usual linguistic practice of mathematicians. And the latter 'mathematical need' has now become more important than the 'philosophical' need for a foundation of science that might account for a sharp disciplinary distinction.[26]

As we will discuss further in § 3.3 this change is related to the overcome of the urgency to give a definition that is specular to the one given by Kant. By the way it is to be noted that these criteria are not only typical features of the Kantian conception of mathematics, but also necessary conditions for a satisfactory definition of a scientific discipline according to the standards that became widespread after the flourishing of encyclopedias in the 18th century: a science is defined by

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[26]"[...] a science deserves to be called mathematics if a considerable part of its theory contains determinations of quantity whose correctness might be understood only on the basis of considerations on the nature of magnitudes, considerations that require a proper introduction. This addition will probably appear to many as objectionable, and so it appeared to me – I make no secret of it – since it reduces the difference between mathematical sciences and non mathematical sciences to a 'more or less' question. I do not deny this might be a mischief, but I do not see how one could avoid it, without determining the concept of mathematics in a way that differs radically from the dominant linguistic use and that will originate more confusion than advantages. [...] Anyway, this seems to me to have been the rule followed by mathematicians, as they increased the number of the mathematical sciences [...]" *Grössenlehre*, Einleitung, §2, Anmerkung in GA.II.7, p. 29.

its position in the tree of knowledge. Besides, the opposition to philosophy was a standard way to account for the difference in certainty, rigor and intersubjective agreement between the two disciplines in the "geometric century".[27]

The definition of mathematics as a theory of forms explains the opposition between philosophy and mathematics: "the former concerns itself with the question, how must things be made in order that they should be possible? The latter raises the question, which things are real—and indeed (because it is to be answered *a priori*)—*necessarily real*? Or still more briefly, *mathematics* would *deal* with *hypothetical necessity, metaphysics with absolute necessity.*"[28]

This issue has definitively become less urgent in the *Grössenlehre*, because Bolzano is more and more interested in giving an appropriate role to logic rather than in giving a symmetric and opposite definition of philosophy and mathematics.

3.2. **From possible entities to ideas in themselves.** In the *Beyträge* mathematics concerns the question "how must things be made in order that they should be possible". Mathematics is an hypothetical science: it concerns the conditions of the possible existence of objects—not only real objects, but also objects that exist only in the imagination.[29] But

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[27]For a characterization of the era between Spinoza and Kant as a "geometric century" see [1].

[28]*Beyträge*, I. § 9, in [8, pp. 94–95.]

[29]"I therefore think that mathematics should best be defined as a science which deals with the general laws (forms) to which things must conform in their existence. By the word 'things' I understand here not merely those which possess an objective existence independent of our consciousness, but also those which simply exist in our imagination, either as individuals (i.e. intuitions) or simply as general concepts, in other words, everything which can in general be an object of our capacity for

from the *Wissenschaftslehre* onwards Bolzano developed a new theory of ideas and propositions in themselves, and mathematics might concerns such entities as well.

> More than thirty years ago I believed I could trace a more precise boundary between mathematics and the other sciences, as I attributed to mathematics all those truths that do not concern the real existence (Daseyn) but only the conditions for the possibility of existence. But I abandoned this thought as soon as I realized that mathematical theories do not refer only to things that are real or that might become real (i.e. to things that are possible).[30]

Bolzano explicitly admits that he has changed his mind. In the *Beiträge* mathematical objects were possible. In the *Wissenschaftslehre* mathematical objects in general—and not only Gedankendinge—are not actual and cannot become actual, so they are not possible.

3.3. **A progressive distantiation from Kant.** It is well known that Kantian definition of mathematics as the "knowledge gained by reason from the construction of concepts" has been extremely influential in the early 19th century, and that Bolzano was among its harsher critics. As a matter of fact, Bolzano's criticism of Kant is one of the most coherent features of his epistemology: if the *Betrachtungen* apparently

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representation. Furthermore if I say that mathematics deals with the laws to which these things conform in their existence, this indicates that our science is concerned not with the proofs of the existence of these things but only with the condition of their possibility." Cf. Beyträge, I, § 8, in [8, p. 94.]

[30]*Grössenlehre*, Einleitung, § 2, Anmerkung in GA.II.7, p. 30..

accept several Kantian remarks on geometry, from the *Beyträge* on-
wards Bolzano never stops criticizing Kant's notion of a priori intu-
ition, Kant's claim that mathematics contains propositions based on
an a priori intuition of space and time, and the kind of distinction he
introduces between philosophy and mathematics. These issues, briefly
mentioned in the *Beyträge* (§ 9), in the *Grössenlehre* and in the *Para-
doxes of the Infinite*—where Kant's "idea of space as a (subjective) form
of intuition" is referred to as an unfortunate idea[31]—are extensively dis-
cussed in the Appendix to the *Beyträge*,[32] in the *Wissenschaftslehre,*
in *Athanasia*, and of course in Prihonsky's *Neuer Anti-Kant*.[33]   But
Bolzano's anti-Kantianism underwent a significant evolution.

In the *Beyträge* the need to criticize Kant dictated Bolzano's agenda.
His main aim is apparently that of deconstructing Kant's definition
point by point. Where Kant speaks of intuition, Bolzano speaks of pure
concepts; where Kant speaks of quantities, Bolzano speaks of forms; if
Kant ignores logic, Bolzano develops it in details as part of an analysis

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[31]"until finally *Kant* got the unfortunate idea, still repeated by many today, of
considering space as well as time not to be something objective, but to be a mere
(subjective) *form of our intuition.*" Cf. [8, p. 646.]

[32]"The critical philosophy seems to promise us one [a better definition] [...] mathe-
matics is a science of the construction of concepts (A712). [...] For my part I wish
to admit openly that I have not yet been able to convince myself of the truth of
many doctrines in the critical philosophy, and especially of the correctness of the
Kantian claims about pure intuitions and about the construction of concepts using
them. I still believe that in the concept of a pure (i.e. a priori) intuition there
already lies an intrinsic contradiction. Much less can I persuade myself that the
concept of number must necessarily be constructed in time and that consequently
the intuition of time belongs essentially to arithmetic." *Beyträge*, Appendix, §§ 5-6
in [8, p. 93.]

[33]Concerning the distinction between philosophy and mathematics see in particular
pp. 216-17, repr. in [7, pp. 172-173.]

of the notion of a mathematical proof. Besides, Bolzano's terminology is Kantian, because Bolzano adopts Kant's definition of intuition, concept, analytic, synthetic, etc. Thirdly, Bolzano himself makes a connection between the need of a new definition and Kant's criticism quoting a critical review of a work by G.U.A. Vieth, possibly written by himself.[34] Finally, like Kant, Bolzano's characterization of mathematics is developed by opposition to philosophy, and in order to explain why mathematics applies to a certain domain of objects, and thus how it can be sharply distinguished from other sciences.

From the *Wissenschfatslehre* onwards a new logical theory and new definitions of the main concepts are introduced. From a certain point of view one could argue that this corresponds to a radicalization of the opposition to Kant, or at least to a form of opposition that is no more internal but rather external to the Kantian framework. In a certain sense, the return to the traditional definition of mathematics as a science of quantities was the only further anti-Kantian assumption that he could make.

To summarize, the changes in the conception of mathematics are due to changes in Bolzano's logic: the new theory of truth involves a different definitions of analytic and synthetic judgments, and the introduction of ideas in themselves involves a different conception of mathematical entities. But they can also be related to a gradual overcome of Kant's terminology (e.g the definition of analyticity, but also the renounce to define mathematics by opposition to philosophy), and to a

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[34]*Beyträge*, I, § 7, in [8, p. 93.] For further details on this review, see [5].

radicalization of Bolzano's anti-Kantianism (e.g. the different interpretation of the role of analytic and synthetic truths in mathematics, but also the return to the traditional definition that Kant had explicitly criticized).

4. Conclusion

The analysis of Bolzano's changes in the definition of mathematics has shown that they are less radical than one might think. Several issues are common to the three works: 1) the definition of mathematics does not depend on its objects; 2) mathematics is exclusively conceptual; 3) its propositions are partly analytic and partly synthetic; 4) its objects are not just numbers, concrete continuous magnitudes, or magnitudes that can be measured. Notwithstanding some similarities, there are relevant issues that change in the three works: 1) mathematical objects are possible entities in 1810 but in 1837 and in 1848 they are conceived also as ideas in themselves, which cannot be possible, because they can never become actual; 2) mathematics is defined as a theory of forms in the *Beyträge* and as a science of quantities in the *Grössenlehre*; 3) mathematics is defined by opposition to philosophy and to other scientific disciplines in th *Beyträge* but not in the *Grössenlehre*; 4) in the *Beyträge* Bolzano remarks that analytic truths should not occur in a scientific treatment, given that they are only conventions used to recall the concept designated by a certain word, but in the *Wissenschaftslehre* analytic truths are considered as relevant enough to deserve a place in a scientific system, and even a proof; 5) Bolzano's anti-Kantianism evolves from 1810 to 1848.

Given the elements of continuity and the elements of discontinuity that we have described in the previous paragraphs, it is now possible to evaluate the coherence of Bolzano's evolution. Bolzano clearly renounced to an aim that he praised in 1810—the idea of a sharp distinction between mathematics and other sciences—in favor of a more fluid and pragmatic classification of knowledge, but he always believed that mathematics cannot be adequately described by the indication of its domain of objects, and that mathematics does not concern only quantities, even if it is based on a quantitative approach to objects. Most differences concern minor points, or are perfectly understandable in the light of new definitions of the concepts involved. But what about the fundamental difference between forms that are possible and ideas or propositions in themselves that are not?

The difference is the result of Bolzano's continuous effort to defend the possibility of applying mathematical principles and theorems to the largest possible domain of objects. In *Grössenlehre* Bolzano considers arithmetic and combinatorics as general sciences that can be applied to almost all kinds of things, and thus also to non actual objects. This is true of arithmetic, because numbers are not real objects, but just determinations of objects that can be used to determine all other quantities.[35] And it is true of combinatorics, because the latter can be applied to any object that can be part of a whole.[36] Bolzano's

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[35]"Eine besondere Gattung von Grössen, welche sich unsrer Betrachtung vor andern darbieten, deren wir uns eben desshalb u. ihrer Einfachheit wegen zur genauern Bestimmung aller übrigen Grössen bedienen, sind — die Zahlen." *Grössenlehre*, Einleitung, § 3.5, in GA.II.7, p. 34.

[36]"Denn sicher ist doch das Gebiet der Gegenstände, auf welche sich diese Gegenstände in irgend einem Betrachte als Theile vorstellte, die in ein eigenes Ganze

conception of mathematics as a general mathesis—first presented in 1810—has not disappeared in 1848. It is exactly for the sake of generality that Bolzano abandons the previous definition of mathematics and modifies the meaning of 'Grösse' from quantity to quantity in general, as it was common in the tradition of the mathesis universalis.[37]

Bolzano's change in his definition of mathematics is thus a consequence of a deeper understanding of the meaning and the nature of the general mathesis. Bolzano's evolution is thus coherent. Rather than a change in his conception of mathematics, it is the result of a change in his logical and epistemological theory, and can be better understood in the light of an increased distance from the Kantian framework.

While Bolzano was still pervasively influenced by Kant in the *Beyträge*, so that Bolzano felt the need to present a definition of mathematics that should have the same advantages and characteristics of the Kantian definition, even if by opposition, his anti-Kantianism becomes much more radical, as a consequence of the introduction of a new logical perspective. At this point, he does not need to produce a point by point counter-argumentation of Kant's theses: in particular, there is no more need for a definition of mathematics that might guarantee a sharp distinction between mathematics and other sciences, or explain the certainty of mathematics by opposition to philosophy. Bolzano's adhesion to the project of a mathesis universalis and his resistance to any bipartition of mathematics in theory of numbers and theory of

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vereinigt werden sollen, und dass diese Vereinigung einem Gesetze von der Art unterliege, dass es bei einer endlichen Menge von ihnen nur eine endliche Menge Verbindungen gibt." *Grössenlehre*, Einleitung, § 3.4, in GA.II.7, p. 36.

[37]On the characterization of quantities in the *Grössenlehre* see especially [9, p. 342.]

magnitudes—numbers are themselves a kind of quantity—are coherent with his anti-Kantian conception of mathematics.

The idea of mathematics as being the science of quantities in general is the continuity element in Bolzano's evolution, which can be appreciated only if one develops a deep analysis of Bolzano's notion of quantity and of the changes it underwent with time. According to this interpretation, the conception of mathematics remains substantially unaltered, but Bolzano's investigation on the foundations of mathematics moves from a philosophical perspective that is still profoundly indebted to Kant's terminology and to Kant's definitional criteria towards a mathematical and algebraic conception which is much more concerned with a precise clarification of the primitive concepts of the discipline. It is exactly for this reason that Bolzano goes back to the traditional definition of mathematics: the linguistic use can be preserved, provided that one preliminarily clarifies the meaning of the concepts involved.

According to this interpretation, Bolzano's change in the definition of mathematics depends on a change in the logic (the introduction of ideas and propositions in themselves), which is, as I have shown elsewhere,[38] strictly connected to the solution of certain mathematical needs (e.g. the need to explain how mathematics might contain meaningful propositions concerning objectless ideas). The belief in the importance of logic for mathematics is probably the main reason for the radicalization of Bolzano's anti-Kantianism, and is coherently reflected in the

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[38]Cf. [2].

evolution of his definition of mathematics. Bolzano's return to the traditional definition is the best compromise he could find in order to keep together the new logical theory with the idea of a mathesis universalis.

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