Longitudinal Dispersion Coefficients within Turbulent and Transitional Pipe Flow

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Abstract The longitudinal dispersion coefficient is used to describe the change in characteristics of a solute cloud, as it travels along the longitudinal axis of a pipe. Taylor [1954] proposed a now classical expression to predict the longitudinal dispersion coefficient within turbulent pipe flow. However, experimental work has shown significant deviation from his prediction for $Re < 20000$. This paper presents experimental results from tracer studies conducted within the range $2000 < Re < 50000$, from which longitudinal dispersion coefficients have been determined. Initial results are also presented for a numerical model that aims to predict the longitudinal dispersion coefficient over the same range of Reynolds numbers.

1 Introduction

Longitudinal dispersion can be defined as the spreading of a solute along the flow's longitudinal axis. This process leads to a change in characteristics of a contamination cloud from an initial state of high concentration and low spatial variance, to a downstream state of lower concentration and higher spatial variance.

Within potable water networks it is important to quantify the changing characteristics of solutes as they travel through the network.

Current water quality models for distribution networks assume steady, highly turbulent flow [Tzatchkov et al. 2009]. These assumptions are valid for the majority of the flow conditions experienced in the main network. However, one part of the network for which these assumptions are not valid is the network's periphery, where water leaves the main network and travels to the point of consumption. Here, in the so called 'dead end' regions of the network, discharge is contingent upon the

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intermittent demand of the consumer, hence the flow is unsteady, and can assume any flow rate from the relatively high main network rate, through to zero in times of no demand [Lee 2004]. In addition, pressure transients caused by any change in the flow conditions (closed valve, leaking pipe, network maintenance etc) can result in periods of both unsteady and low flow in the main network. This scenario is of particular interest in water quality modeling, as negative pressure created in such pressure transients can lead to contaminant intrusion into the network through any leaks in the system [LeChevallier et al. 2003]. Thus, contaminant can be released into a low and unsteady flow.

Taylor [1953, 1954] developed two equations to predict the longitudinal dispersion coefficient within steady laminar and turbulent pipe flow respectively. His equations are still widely used. However, experimental data has shown a significant divergence between predictions made by Taylor's equation for turbulent flow, and experimentally determined longitudinal dispersion coefficients within turbulent flow for $Re < 20000$.

This paper presents experimentally determined longitudinal dispersion coefficients for steady pipe flow in the range of Reynolds Numbers $2000 < Re < 50000$, a range set to highlight conditions under which Taylor's model does not describe experimental data. Furthermore, a simple numerical model is developed for the same range of Reynolds Numbers.

2 Background and Previous Work

Longitudinal dispersion is primarily caused by differential advection associated with the flow's longitudinal velocity profile. When fluid flows through a pipe, the velocity varies with radial position from the maximum velocity obtained at the pipe's centreline, to zero at the pipe's boundary. When a cross-sectionally well-mixed tracer is introduced across a pipe, tracer will be advected in accordance with the velocity at its corresponding radial position. Hence, tracer at the centre of the pipe will travel further in a given period of time than tracer at the boundary of the pipe, and thus the tracer disperses. The tracer is further spread in all directions by the effects of molecular and turbulent diffusion. The degree to which these diffusion mechanisms act to spread the tracer directly in the longitudinal direction is negligible when compared to the effects of the differential advection. However, the two diffusion mechanisms are significant with regard to longitudinal dispersion because of their ability to spread the tracer radially. As radial diffusion increases, each particle of tracer experiences a larger number of radial positions and corresponding velocities, thus reducing the effects of the differential advection. Hence, there is an inverse relationship between molecular and turbulent diffusion and longitudinal dispersion.

Taylor [1953,1954] showed that, after some initial development period, the spatial distribution of the cross sectional mean concentration of a solute is Gaussian, with a variance that increases linearly with distance. Through this, Taylor showed that the cross-sectional average concentration distribution can be described by Fick's second law of diffusion, such that:

$$\frac{\partial c}{\partial t} = D_{xx}\frac{\partial^2 c}{\partial x^2} - \bar{u}\frac{\partial c}{\partial x} \tag{1}$$

where c is the cross-sectional mean concentration, t is time, D_{xx} is the longitudinal dispersion coefficient, x is the distance along the longitudinal axis and $\bar{u}$ is the cross-sectional mean velocity.

For turbulent flow, Taylor [1954] considered the radial distribution of a concentration in terms of the following partial differential equation:

$$\frac{\partial}{\partial r}\left(D_r(r)r\frac{\partial c(x,r)}{\partial r}\right) = r\left(u(r)\frac{\partial c(x,r)}{\partial x} + \frac{\partial c(x,r)}{\partial t}\right) \tag{2}$$

where r is the radial position from the centreline, $c(x,r)$ is the concentration at position (x,r), and $D_r(r)$ and $u(r)$ are the radial diffusion coefficient and velocity at position r, respectively.

Taylor assumed the radial diffusion coefficient was equivalent to the turbulent diffusion coefficient D_t, such that $D_r = D_t$, and defined the turbulent diffusion coefficient by considering it in terms of Reynolds analogy, i.e. the assumption that the transfer of matter, heat and momentum are analogous, such that:

$$D_r(r) = D_t(r) = \frac{\tau_t(r)}{\rho(\partial u(r)/\partial r)} \tag{3}$$

where $\tau_t(r)$ is the turbulent stress at position r, $\tau_t(r) = \tau \cdot p$, where τ is the wall shear stress and p is dimensionless position $p = r/a$, where a is the pipe's radius.

Taylor assumed a 'universal' velocity distribution, of the form:

$$\frac{u_c - u(r)}{u_*} = f(p) \tag{4}$$

where u_c is the maximum velocity, u_* is the frictional velocity $u_* = \bar{u}\sqrt{f/8}$, where f is the friction factor. $f(p)$ is a geometric relationship for the velocity distribution as a function of dimensionless position p. Taylor derived the values of $f(p)$ as the mean value of the data of Stanton and Pannell [1914] and Nikuradse [1932]. In addition, Taylor proposed an empirical relationship for the maximum velocity, $u_c = \bar{u} + (4.25u_*)$.

Taylor used this definition of the radial diffusion coefficient (Equation 3) and velocity profile (Equation 4) to solve Equation 2, which gave the following expression for the longitudinal dispersion coefficient within turbulent pipe flow:

$$D_{xx} = 10.1au_* \tag{5}$$

The Reynolds number, $Re = \bar{u}d/\nu$, where d is the pipe diameter and ν is kinematic viscosity, effectively quantifies how turbulent a flow is. Figure 1 shows the results of previous experimental investigations into the relationship between the longitudinal dispersion coefficient and Reynolds number for the range $2000 < Re < 50000$, compared to Taylor's theory (Equation 5). From Figure 1 it can be seen that for

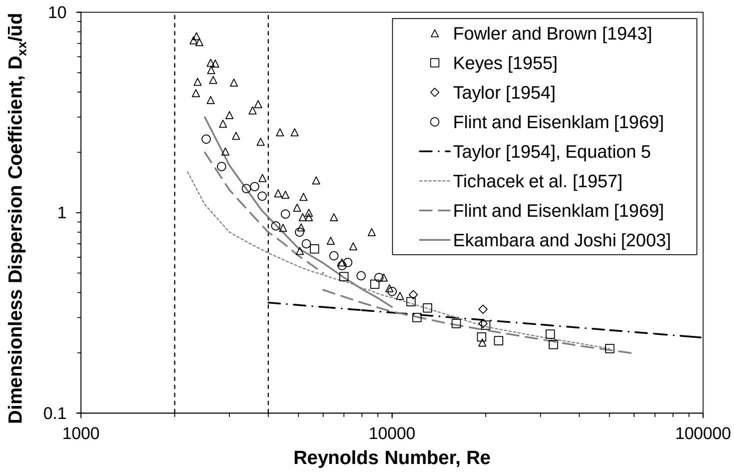


Fig. 1 *Comparison between the experimental data of Fowler and Brown [1943], as presented by Levenspiel [1958], Taylor [1954], Keyes [1955] and Flint and Eisenklam [1969] and the theorys of Taylor [1954] (Equation 5), Tichacek et al. [1957], Flint and Eisenklam [1969] and Ekambara and Joshi [2003].*

$Re > 20000$, the longitudinal dispersion coefficient is relatively small and independent of Reynolds number. For $Re < 20000$, it increases significantly to a point that at $Re \approx 2000$, its value is approximately 25 times larger at than the value for $Re > 20000$. Furthermore, at approximately the same point that the longitudinal dispersion coefficient begins to increase, it also begins to diverge from Taylor's prediction. Some insight into this phenomenon can be gained by considering the relationship between the Reynolds number and the velocity profile throughout this range.

When the flow is laminar, at $Re \approx 2000$, the velocity profile is parabolic and thus there is a high degree of spreading due to differential advection. When the flow is turbulent, at $Re > 4000$, the velocity is more uniform than the laminar profile leading to a decrease in differential advection. From $2000 < Re < 4000$ the flow is transitional, and thus the profile transitions from the parabolic profile at $Re \approx 2000$, to the more uniform profile at $Re \approx 4000$. Figure 2 shows a comparison between an analytically predicted laminar velocity profile and two theoretical turbulent velocity profiles [Nikuradse 1932]. From Figure 2 it can also be seen that the is a significant difference in the velocity distribution between $2000 < Re < 4000$, whereas there is only slight difference between $4000 < Re < 100000$.

For laminar flow the velocity profile can be determined analytically as [White 2008]:

$$\frac{u(p)}{u_c} = 1 - p^2 \tag{6}$$

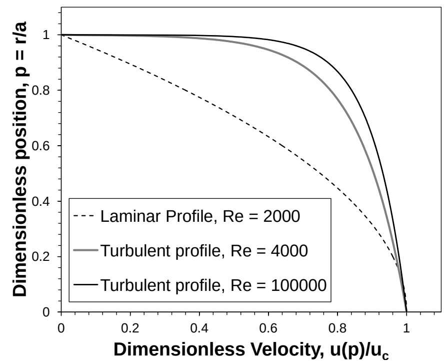


Fig. 2 *Comparison between a laminar velocity profile at* $Re = 2000$*, (Equation 6), and the theoretical turbulent velocity profile of Nikuradse [1932], at* $Re = 4000$ *and* $Re = 100000$*.*

where $u(p)$ is the velocity at position p.

For turbulent flow, the velocity profile cannot be determined analytically, and thus turbulent velocity profiles are generally proposed as empirical expressions. Conventionally turbulent velocity profiles are defined in terms of the dimensionless velocity and distance terms $u^+ = u(p)/u_*$ and $y^+ = u_* y/\nu$, where y is the actual distance from the wall.

Figure 3 shows the experimentally obtained velocity profile of Durst et al. [1995] for a turbulent flow at $Re = 7442$. From Figure 3 it can be seen that there are three

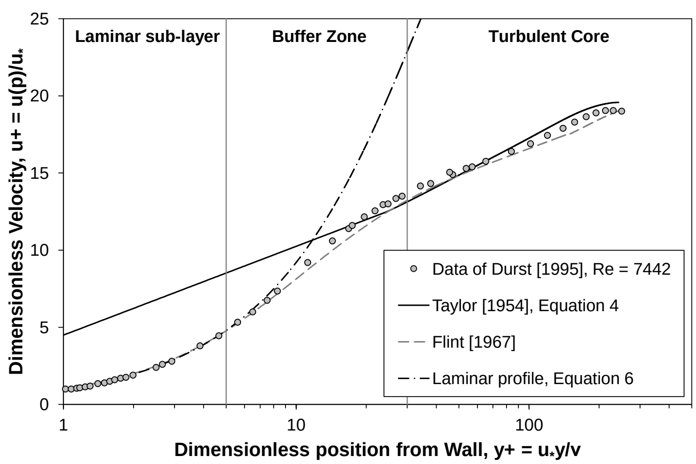


Fig. 3 *Comparison between the experimental data Durst et al. [1995], a laminar velocity profile (Equation 6), Taylor's turbulent velocity profile (Equation 4) and turbulent velocity profile of Flint [1967].*

parts to a turbulent velocity profile. The main part of the profile is the 'turbulent core', which occurs at $y^+ > 30$. In this part of the flow, the profile is fully turbulent. Here, the profile is logarithmic besides a small portion near the centreline where the profile plateaus, deemed the 'wake' region. For $y^+ < 5$, even in fully developed turbulent flow, the flow remains laminar. This portion of the flow is deemed the 'laminar sub-layer'. From $5 < y^+ < 30$, the flow transitions from being laminar to fully turbulent, a portion of the flow deemed the 'buffer zone'. The region of the flow corresponding to the non-turbulent portion, i.e. the length corresponding to $y^+ < 30$, grows as Reynolds number decreases. At $Re > 20000$, the laminar sub-layer and buffer zone constitute less that 5% of the flow whereas at $Re = 4000$, the laminar sub-layer and buffer zone constitute around 20% of the flow. As Reynolds number reduces from $Re = 4000$, the laminar sub-layer grows further to the point that the entire flow is laminar, at around $Re = 2000$.

Figure 3 shows a comparison between Taylor's velocity profile (Equation 4), and the data of Durst et al. [1995]. Taylor's profile was derived from highly turbulent data, and thus neglects a laminar sub-layer and buffer zone. Due to this, Taylor's equation for the longitudinal dispersion coefficient is only valid for $Re > 20000$, the portion of the flow where the size of the laminar sub-layer and buffer zone are small enough to be considered negligible with regards to longitudinal dispersion.

Several authors have used improved velocity profiles to build upon Taylors original analysis for the longitudinal dispersion coefficient.

Tichacek et al. [1957] solved Equation 2 using experimental velocity profiles to produce a model for the longitudinal dispersion coefficient for $2200 < Re < 50000$.

Flint [1967] proposed an expression for the velocity profile for turbulent flow which included a laminar sub-layer and buffer zone. Figure 3 shows a comparison between Flints turbulent velocity profile and the data of Durst et al. [1995]. In addition, Flint [1967] proposed a further expression for the velocity profile within transitional and low turbulent flow, covering $2500 < Re < 6000$. Flint and Eisenklam [1969] solved Equation 2 using the theoretical velocity profile of Flint [1967] to produce a model for the longitudinal dispersion coefficient for $2500 < Re < 100000$.

Ekambara and Joshi [2003] solved equation Equation 2 using a low Reynolds number $\kappa - \varepsilon$ CFD code to provide a prediction for the longitudinal dispersion coefficient for $2500 < Re < 10000$.

Figure 1 shows a comparison between these models for the longitudinal dispersion coefficient and experimental data.

None of the models discussed cover the whole range $2000 < Re < 20000$, the range over which Taylors equation do not describe experimental data. Furthermore, all of the models discussed involve solving the problems governing differential equation, and thus are mathematically complex.

The aim of the present work is to experimentally determine the longitudinal dispersion coefficient over the range $2000 < Re < 50000$, and proposes a relatively simple numerical model for the same range, with particular emphasis on $2000 < Re < 3000$, the range not previously fully described.

A suitable model for this objective is the 'zonal' model of Chikwendu [1986], which provides an analytical solution for the longitudinal dispersion coefficient for a given velocity profile and radial diffusion coefficient by dividing the flow into N number of zones (See Section 5 for more detailed explanation of model).

Thus, the numerical model proposed in this paper will consider the most suitable definition of the velocity profile for $2000 < Re < 50000$, in conjunction with the model of Chikwendu [1986].

3 Experimental Setup and Method

A series of experiments was conducted to determine the longitudinal dispersion coefficient for a range of flow rates corresponding to the range of Reynolds numbers $2000 < Re < 50000$. The tests were conducted using a re-circulating system where the main test pipe was 16.6 metres long, with an internal diameter of 24 mm.

The flow rate was obtained by measuring a volume of water collected over a set period of time. For each set of tests at a fixed flow rate, the flow rate was measured three times both before and after each set of injections. Thus, the flow rate for each run was the mean value of six repeats.

Dye injections of Rhodamine WT were made using a computer controlled peristaltic pump. For each injection, Rhodamine WT at a concentration ranging from 700 - 1500 ppb was injected for a one second period. Injections were made at a distance 3.5 metres downstream from the start of the test section, a length sufficient to allow for the flow to become fully developed [White 2008].

The response of the dye to the flow was recorded as cross-sectional average concentration vs. time profiles using two Turner Designs series 10 fluorometers, which were 6 metres apart and 7.1 and 13.1 metres downstream of the injection point respectively. The instruments were calibrated before and after the full series of dye injections to confirm the calibration relationship held throughout the tests.

For each flow rate, three injections were made. Thus the longitudinal dispersion coefficients discussed in the Section 4 represent the mean value of 3 repeats.

4 Experimental Results and Analysis

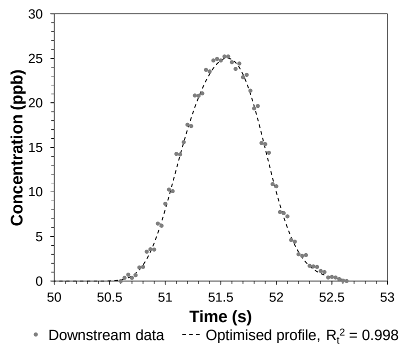
An initial estimate of the longitudinal dispersion coefficient was made through the ‘method of moments’ [Rutherford 1994].

These estimates were optimised through the following routing procedure [Rutherford 1994]:

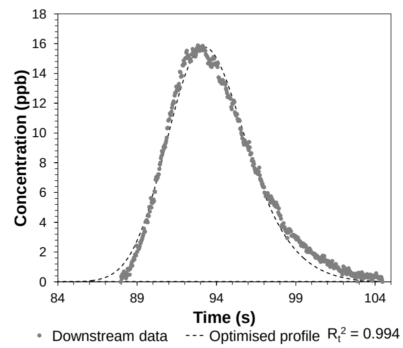
$$c(x_2,t) = \int_{-\infty}^{\infty} \frac{c(x_1,\gamma)\bar{u}}{\sqrt{4\pi D_{xx}\bar{T}}} \exp\left[-\frac{\bar{u}^2(\bar{T}-t+\gamma)^2}{4D_{xx}\bar{T}}\right] d\gamma \tag{7}$$

where $\bar{T}$ is the travel time, the difference between the centroid of two profiles, and γ is an integration variable, or pseudo time.

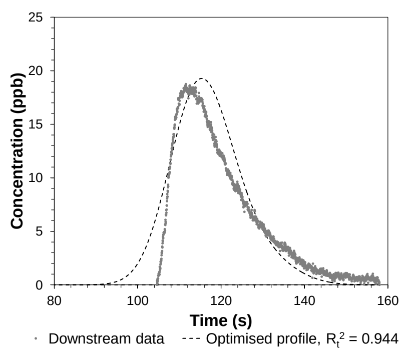
A routing procedure, such as Equation 7, takes the experimentally obtained upstream profile, routes it to a downstream position on the basis of the travel time, and spreads it on the basis of the longitudinal dispersion coefficient. Thus, initially the upstream data was routed onto the downstream data using the travel time and longitudinal dispersion coefficient obtained through the method of moments. The routed downstream profile could then be compared to the downstream profile through some criteria of fit, namely R_t^2 [Young et al. 1980]. The longitudinal dispersion coefficient and travel time were then optimised to give the best fit to the downstream data on the basis of the value of R_t^2. Figure 4 shows a sample of the results for the downstream concentration profiles, compared to the optimised profiles from Equation 7 for several representative flow rates covering approximately $2000 < Re < 50000$.



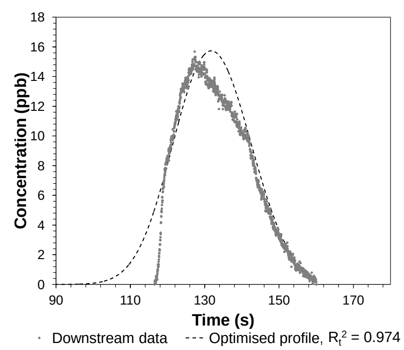
(a) *Re*=50892



(b) *Re*=5994



(c) *Re*=2670



(d) *Re*=2185

Fig. 4 *Comparison between experimental downstream concentration profiles, and concentration profiles routed from upstream data and optimised to best fit downstream data through Equation 7.*

Table 1 summaries the experimental results for each flow rate considered. From Figure 4 and Table 1 it can be seen that for fully turbulent flow, for $4000 < Re < 50000$, the routing procedure fits the data well. For transitional flow, for $2000 < Re < 4000$, as the profiles begin to deviate from the Gaussian assumption in a non-trivial manner, the goodness of the fit of the model decreases.

From Table 1 it can also be seen that for the majority of the tests for fully turbulent flow, for $4000 < Re < 50000$, mass balance is around 100 %. However, for transitional flow, for $2000 < Re < 4000$, mass balance drops to around 90 %. These results could be either a due to tracer being caught in the laminar sub-layer, or experimental error at low flow rates due to the tracer not being cross-sectionally well mixed. Figure 5 shows a comparison between the longitudinal dispersion coefficient determined experimentally in the present work and previous experimental data, from which it can be seen that the results of the present work fall favorably within previous experimental investigation.

Table 1 Summary of experimental results. Each value represent the mean value of three repeats.

Re	Mass balance [%]	$D_{xx}/\bar{u}d$	R_t^2
50892 ± 814	102.3 ± 0.7	0.326 ± 0.042	0.998 ± 0.000
32363 ± 359	101.0 ± 1.2	0.351 ± 0.020	0.998 ± 0.000
20380 ± 168	101.0 ± 1.2	0.347 ± 0.020	0.999 ± 0.000
14815 ± 96	101.6 ± 1.7	0.396 ± 0.003	0.998 ± 0.000
10365 ± 57	102.6 ± 0.4	0.494 ± 0.005	0.996 ± 0.000
5994 ± 66	99.037 ± 3.4	0.626 ± 0.009	0.994 ± 0.001
5148 ± 291	91.4 ± 1.2	0.828 ± 0.033	0.993 ± 0.001
3784 ± 29	104.7 ± 8.1	1.525 ± 0.076	0.989 ± 0.002
2670 ± 13	89.3 ± 1.9	2.682 ± 0.596	0.949 ± 0.008
2185 ± 21	88.4 ± 5.1	4.603 ± 0.383	0.946 ± 0.029

± Represents 1 Stand Deviation.

5 Proposed Numerical Model

The model of Chikwendu [1986] provides a longitudinal dispersion coefficient on the basis of a velocity profile and a radial diffusion coefficient, such that:

$$D(N)_{xx} = \sum_{j=1}^{N-1} \frac{a^2 p_j^4 (1-p_j^2)^2 [u_{f,1\to j} - u_{s,j\to N}]^2}{4D_{r,j,(j+1)}} (W_j + W_{j+1}) + \sum_{j=1}^{N} q_j D_{x,j} \quad (8)$$

where D_x is diffusion in the longitudinal direction, $W_j = p_j - p_{j-1}$ and $u_{f,1\to j}$ and $u_{s,j\to N}$ are the fast and slow zone velocities:

$$u_{f,1\to j} = \frac{1}{p_j^2} \sum_{k=1}^{j} q_k u_k \quad (9)$$

$$u_{s,j\to N} = \frac{1}{1-p_j^2} \sum_{k=j+1}^{N} q_k u_k \quad (10)$$

where $q_j = (r_j^2 - r_{j-1}^2)/a^2$.

Chikwendu's model provides an analytical solution for the longitudinal dispersion coefficient by dividing the flow into a discrete number of zones N. Each zone has its own mean velocity and radial exchange between adjacent zones. Differential advection is accounted for by considering the mean velocity of every zone above each point j, the 'fast zone', less the mean velocity of every zone below each point j, the 'slow zone'. Through this, a dispersion term is calculated for each zone, which is summed to give the longitudinal dispersion coefficient.

Initially, Chikwendu's model was used to reproduce the Taylor's result for turbulent flow, to demonstrate the applicability of the model and to determine an approximate value for the number of zones required. Taylor's velocity profile (Equation 4), proposes the velocity profiles as a function of the geometric function $f(p)$. Taylor gives an expression for the function $f(p)$ for $0.9 < p < 1$, but between $0 < p < 0.9$ only gives 14 experimentally derived values for the function. Thus, in order to use his profile at a higher resolution, an expression was fit to Taylor's values for $0 < p < 0.9$. The radial diffusion coefficient, friction factor and the maximum velocity were used as defined by Taylor [1954]. In addition, the diffusion term in the longitudinal direction, D_x, was also neglected in accordance with Taylor's analysis. At N=3000, the model reproduced Taylor's results to within ~ 0.5 % over the whole range.

The laminar sub-layer and buffer zone of the turbulent profile of Flint [1967] was added to Taylor's velocity profile for $y^+ < 30$, to better describe experimental data for $4000 < Re < 50000$.

For transitional flow, the transitional velocity profile of Flint [1967] was used for $3000 < Re < 4000$, a range over which it conforms to the experimental data of Senecal and Rothfus [1953]. For $2000 < Re < 3000$, Flint's expression diverges from the experimental data of Senecal and Rothfus [1953], and fails to converge upon the analytical prediction of the velocity profile at $Re = 2000$. Within this range, Flint's expression predicts a distribution that appears more turbulent than the majority of the data, whereas the analytical laminar profile predicts a distribution which appears more laminar than the majority of the data. Therefore, to predict the velocity profile within this range, an expression was suggested that postulates a velocity profile as a combination of the distribution of Flint's profile at $Re = 3000$, and the analytical laminar profile at $Re = 2000$. The relative proportion of each distribution used is governed through a transition factor α, such that:

$$f(p)_T = \alpha f(p)_{L(2000)} + (1-\alpha) f(p)_{F(3000)} \tag{11}$$

where $f(p)$ is a dimensionless velocity distribution, $f(p) = u(p)/u_c$, $f(p)_T$ is the transitional dimensionless velocity distribution between $2000 < Re < 3000$, $f(p)_{L(2000)}$ is the dimensionless velocity distribution from the analytically laminar profile at $Re = 2000$ (Equation 6), and $f(p)_{F(3000)}$ is the dimensionless velocity distribution from Flint's profile at $Re = 3000$.

Values for α were obtained by fitting Equation 11 to the 8 profiles of Senecal and Rothfus [1953] between $2000 < Re < 3000$. The trend for α was found to conform to an 'S' trend, which could be described through a sigmoidal function of the form $\alpha = 0.306/(0.2981 + e^{9.747\gamma})$, where $\gamma = (Re - 2000/1000) - 0.5$.

For transitional flow, values for the maximum velocity were obtained by fitting to the data of Senecal and Rothfus [1953], and the friction factor was assumed to vary linearly from the turbulent value at $Re = 3000$, to the laminar value at $Re = 2000$.

Figure 5 shows the results using Chikwendus model, with the velocity profiles described in the present work for the range $2000 < Re < 100000$.

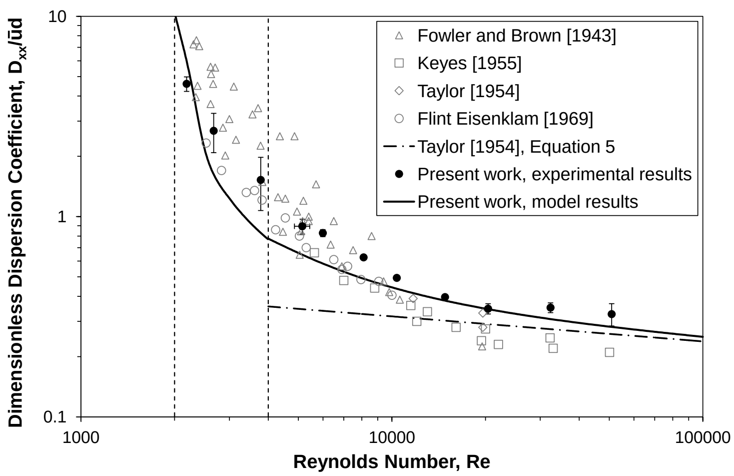


Fig. 5 *Comparison between the the optimised experimental data of the present work, the experimental data of Fowler and Brown [1943], Taylor [1954], Keyes [1955] and Flint and Eisenklam [1969], Taylor's theory (Equation 5), and the model of the present work.*

6 Conclusions

Experimental data has been presented by optimising the longitudinal dispersion coefficient through a Fickian type model. Fits to downstream data are good for turbulent flow for $4000 < Re < 50000$, but it fails to fully predict the downstream distributions for transitional flow for $2000 < Re < 4000$. The results for the longitudinal dispersion coefficient compare favorably with previous experimental work and show a deviation from Taylor's result at $Re < 20000$.

A simple numerical model is proposed that builds upon Taylor's prediction for turbulent flow by adding a laminar sub-layer and buffer zone to Taylor's velocity profile. The use of this velocity profile within Chikwendu's model predicts the general trend in the data for the longitudinal dispersion coefficient of increasing from Taylor's prediction for $4000 < Re < 20000$. The model presented here was extended to transitional flow by considering two further velocity profiles, that of Flint [1967] for $3000 < Re < 4000$, and the profile derived in the present work on the basis of the data of Senecal and Rothfus [1953] for $2000 < Re < 3000$. The use of these expressions within Chikwendu's model predicts the large increases in the longitudinal dispersion coefficient within the transitional region in a manner reasonably consistent with experimental data.

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