Solutions of the Boltzmann Equation corresponding to an Invariant Manifold of Molecular Dynamics

by

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Contents

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Declarations

I take no credit for the ideas around which the material is based; the spatial simplification was devised by R.D. James and the methods to study the linearised collision operator are C. Cercignani's. Except as above, I declare that, to the best of my knowledge, the material contained in this dissertation is original and my own work, except where otherwise indicated, cited, or commonly known.

The material in this dissertation is submitted to the University of Warwick for the degree of Master of Science, and has not been submitted to any other university or for any other degree.

Abstract

A method of Objective Molecular Dynamics is considered for a system of gaseous particles dilutely and periodically distributed in $\mathbb{R}^d$, to find the existence or non-existence of solutions to the Boltzmann equation in the case of motion on an evolving Torus.

A reference configuration of the d dimensional torus is used to describe this system, where motion is given by the Boltzmann equation with a forcing term. The balance laws of mass, momentum and energy are analysed, to find a time dependence of the form $\xi(t)G(\eta(t)w)$ for the system. This then immediately gives the solution in scaled velocity space for uniform dilation as a Maxwellian

In the case of simple shearing, a linearisation of the collision operator and Maxwellian molecular interactions are used, and this results in the expectation of non-existence of scaled solutions of this form.

Chapter 1

Introduction

In 1872, Boltzmann published [Bol72], a result deriving an equation, later called the Boltzmann equation, that described the evolution of the velocities of particles in a dilute gas, within a region of space. He proceeded to prove a result we now call Boltzmann's H theorem, which proves that the entropy of a gas in a fixed box with energy and mass conserved increases. This is the only known proof of the second law of thermodynamics. This proof also gives, for this situation, the existence of a density, called a Maxwellian, that is a time independent solution of the equation. It can also be shown that the density of the gas converges to this as $t \to \infty$ in a suitable sense.

For many years hence, people have attempted to prove similar versions to this for other spaces. This work is interested in a situation where one has a spatially periodic distribution of one species of gaseous atoms in $\mathbb{R}^d$. One has an initial deformation given by a matrix B and a deformation over time given by the matrix A. This situation falls into the wider category of Objective Molecular Dynamics, introduced in [DJ07] with further work in [DJ10] and [DJ12].

As such, this system can be viewed as the evolution of a gas, given by the Boltzmann equation, in the time dependent torus $\mathcal{U}(t) = \mathbb{R}^d/(B+tA)$. However, this poses difficulties with regards to the moving boundary. Thus instead, the situation is related to the evolution of a gas, given by the Boltzmann equation together with a forcing term, on the fixed torus $\mathbb{T}^d = \mathbb{R}^d/Z^d$. This method of considering the evolution of the system in a reference configuration should be compared with the Lagrangian method of describing motion, although this is more for intuition rather than for correct mathematical thoughts. This system contains the special cases of shearing, uniform dilation or contraction and a combination of the two. However, time independent solutions only make sense in the case of shearing, although energy is not conserved here so a renormalisation is needed. One possible renormalisation is given in section 2.6.

The equation for the density on the torus is

$$\frac{\partial g}{\partial t} + \nabla_x \cdot \left[B_t^{-1} w g\right] - \nabla_w \cdot \left[A B_t^{-1} w g\right] = Q[g, g]$$

which is derived in section 2.1. Observe that $A = 0$ corresponds to unforced motion on the torus which is well understood, and the stationary density is a Maxwellian in this case.

The main aim is to prove the existence or non-existence of a stationary density for the system, when it makes sense to do so. This is attempted with the following simplifications.

1. spatial homogeneity to remove the ∇_x term from the equation, although where this is not necessary, inhomogeneous equations are derived where the derivation is the same for both cases.

2. a Maxwellian interaction, so the interaction does not depend on the relative velocities is used. This simplifies the collision kernel to be solely a function of the angle between the colliding particles.

3. a time dependence of the form

   $$g(t, w) = \xi(t) G(\eta(t) w)$$

   for some real valued functions ξ, η, is used to derive a simpler equation. This is introduced in 2.5. The role of η is to renormalise due to the lack of conservation of energy, shown in section 2.4 and the ξ is so that it has total mass 1.

These assumptions result in a simpler equation of the form

$$\nabla_v \cdot [G(Cv)] = Q[G, G](v)$$

for some time dependent matrix C, and this is derived in section 2.6 for the motion on the torus. The case of simple shearing and uniform dilation are particularly useful here as the matrix C does not depend on time. Much of this work follows with a certain closeness [MJ]. It is shown that in the case of uniform dilation, the function G is a Maxwellian.

Turning to the study of the simple shearing case, the collision operator is linearised in section 3.1 and properties of this linearised operator are studied in order to prove the non existence of a solution to the simpler problem of

$$L(h) = \nabla_v \cdot [Cv\mu(v)]$$

where μ is the Maxwellian about which L is linearised.

Suggestions are then made as to why this equation has no solutions in the case of shearing, the expectation being that the rescaling is not suitable for shearing, since in shearing one has an energy lump moving in one direction, whereas the scaling is uniform.

Chapter 2

A Derivation of a Boltzmann Equation in Objective Molecular Dynamics

Firstly, in sections 2.1 and 2.2, a derivation of a general Boltzmann equation is given where one has a periodic gas in $\mathbb{R}^d$, where one relates this to a forced gas lying on the torus $\mathbb{T}^d$ to simplify the periodicity relation.

One then considers the Lyapunov function $\mathcal{H}$ and its use in Boltzmann's H Theorem, in section 2.3, though this is only useful in the case of $A = 0$, as is seen in section 2.4 on balance laws, where mass is shown to be conserved, but energy in general is not.

The molecular interaction type introduced by Maxwell, where one assumes a repulsion of the form $Cr^{-(2d-1)-1}$ is assumed and motivated, in section 2.5, and this is then used to simplify the collision operator to write a simpler equation using a time dependence of the form

$$\xi(t)G(\eta(t)w)$$

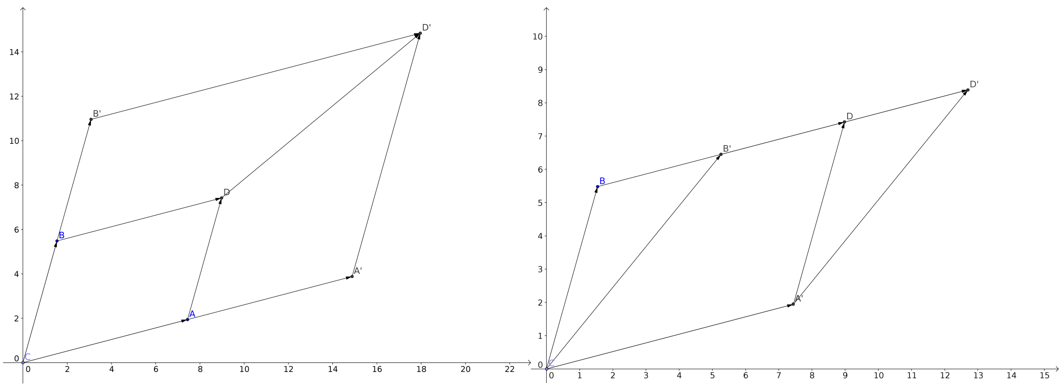
in section 2.6, to get an equation which describes a parametric dependence of G upon the deformation $A \in M^{d\times d}(\mathbb{R})$ and the initial periodicity $B \in M^{d\times d}_{\geq 0}(\mathbb{R})$.

The function G is shown to be a Maxwellian in the case of uniform dilation.

2.1 Motion of Particles

One considers an initial configuration of $\mathbb{R}^d$ given by a matrix $B \in M^{d\times d}(\mathbb{R})$ with positive determinant, which gives a right handed basis of $\mathbb{R}^d$. This partitions $\mathbb{R}^d$ into integral translates of the cell $\mathcal{U}(0) = \mathbb{R}^d/B$. M particles are then distributed in this cell from a density, with some initial velocities and the motion is that of Newtonian dynamics. and this distributes particles in all $\mathbb{R}^d$ with the relation that if a particle is at point $y_0 \in \mathcal{U}(0)$ then there is a particle at $y_0 + Bk$ where $k \in \mathbb{Z}^d$. This distributes particles periodically throughout $\mathbb{R}^d$.

Then a linear deformation of this space given by a matrix $A \in M^{d\times d}(\mathbb{R})$ is considered. The deformation A is in some way related to the initial configuration B, and as such A is generally considered as a multiple of B or a tensor product of several columns of B, though this is by no means an exhaustive list of the possibilities of A. Figure 2.1 shows two situations that can arise within this type of periodicity, those of uniform dilation in 2.1a and simple shearing in 2.1b.



(a) The case of uniform dilation **(b)** The case of simple shearing

Figure 2.1: Examples of situations described by periodicity in $\mathbb{R}^d$

This linear deformation A deforms the point y to the point tAy in time t and so the deformation from some fixed configuration is given by the mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$ by

$$y \mapsto (B + tA)y =: B_t y \tag{2.1}$$

Observe that this motion partitions $\mathbb{R}^d$ into translations of the cell $\mathcal{U}(t) = \mathbb{R}^d/(B_t\mathbb{Z}^d)$, where one has the relation

$$f(t, y, v) = f(t, y + B_t k, v) \qquad \forall k \in \mathbb{Z}^d$$

between the cells. Thus it is enough to consider this system as the motion of M particles in the moving periodic cell $\mathcal{U}(t)$, which henceforth is done.

Observing that the map B_t as given in equation (2.1) maps $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ to $\mathcal{U}(t)$ one can consider the dynamics on the unit torus by performing the map B_t^{-1} from $\mathcal{U}(t)$. This gives rise to two different situations. The first is the **reference** configuration as given by the dynamics on $\mathbb{T}^d$ and the latter is the **actual** configuration as given by the dynamics in the cell $\mathcal{U}(t)$. Henceforth if one uses the bold words above then we refer to that situation.

The velocities on the torus are modified by subtracting the velocity of the point y in the actual cell, which is $AB_t^{-1}y$. This comes about due to the comparison of the Lagrangian description, with the moving boundary, and the Eulerian description with the fixed boundary.

O.M.D. is now introduced to justify from a physical point of view why a linear deformation is considered, and other deformations need not be considered.

2.1.1 Objective Molecular Dynamics

In O.M.D., introduced in [DJ07], the mappings $B_t = B + tA$ are replaced by time dependent isometries of $\mathbb{R}^d$. By this is meant that one considers a collection G of time dependent isometries of $\mathbb{R}^d$ that characterises the periodicity of the particles. For each instance of time, the collection G is a group of isometries with the group operation given by composition of functions.

One now considers M atoms $y_k(t)$ for $k = 1, \dots, M$, called **simulated atoms**, and these are translated by every element in the collection G, where the cardinality of G is given by N. As such, one then considers elements

$$y_{p,k}(t) := \iota_p(y_k(t), t) \qquad k = 1, \dots, M \qquad \iota_p \in G$$

In the situation above, the ι_p are given by $\iota_p(y) = y + B_t p$ for $p \in \mathbb{Z}^d$.

Two assumptions are made upon the particular motion. If $F_{p,k} : \mathbb{R}^{dMN} \to \mathbb{R}^d$ is the force on atom p, k, then for all $Q \in O(d)$ and $b \in \mathbb{R}^d$ one has

$$QF_{p,k}(\dots, y_{i_1,1}, \dots, y_{i_1,M}, \dots) = F_{p,k}(\dots, Qy_{i_1,1} + b, \dots, Qy_{i_1,M} + b, \dots) \quad (2.2)$$

and one also has permutation indifference, meaning if σ is a species preserving permutation, then,

$$F_{\sigma(p,k)}(\dots, y_{i_1,1}, \dots, y_{i_1,M}, \dots) = F_{p,k}(\dots, y_{\sigma(i_1,1)}, \dots, y_{\sigma(i_1,M)}, \dots) \quad (2.3)$$

where preserving species means that the atomic mass and number of the atom is unchanged under this permutation. In the gaseous situation in question, one has only one species of atom for simplicity.

A final assumption is made on the time dependence of the elements of G. This is as follows. If $\iota_p(x,t) = Q(t)x + c(t) \in G$ then one has

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y_{p,k}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \iota_p(y_k(t),t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} y_k(t) \tag{2.4}$$

and this then implies that $c(t)$ must be an affine function of t, and that $Q(t)$ must be constant. This is why it suffices for the form of the deformation B_t to be linear, since the system should satisfy Galilean invariance.

2.2 Boltzmann Equation

This was introduced in 1872 by Boltzmann in his paper [Bol72] and gave lectures on the subject, which have been translated in [Bol64].

While it is not considered here, much work has been done to try to establish the relationship between the molecular dynamical representation of a gas, and the statistical physics method of the Boltzmann equation. We here just use the Boltzmann representation with no justification as to its approximation of a molecular system.

The Boltzmann equation describes the evolution of a molecular density function f which takes as arguments time in $[0,\infty)$, position in a subset of $\mathbb{R}^d$, $d \geq 2$, denoted by X, and velocity in $\mathbb{R}^d$, and gives the distribution of particle velocities. As such, it is a function in $C^1([0,\infty), L^1(X \times \mathbb{R}^d))$, so that at each time $t \in [0,\infty)$ a density function $f(t,\cdot,\cdot)$ is given. Then the evolution of the density is governed by the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_y f = Q[f,f] \tag{2.5}$$

where we have

$$\begin{aligned} Q[f,f] = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \Big(& f(t,y,v_\star - ((v_\star - v)\cdot\nu)\nu) f(t,y,v + ((v_\star - v)\cdot\nu)\nu) \\ & - f(t,y,v_\star) f(t,y,v) \Big) \mathcal{S}(\nu, v_\star - v) \mathrm{d}\nu \mathrm{d}v_\star \end{aligned} \tag{2.6}$$

where ν is an impact parameter in $\mathbb{S}^{d-1}$, and $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. The terms in Q correspond to first a gain at sites (y,v) and at $(y,v_\star)$ and the second term corresponds to loss at (y,v) and $(y,v_\star)$. The gain and loss characterise a collision.

When two particles collide, the pre-collision velocities are changed to post-collision velocities, dependent on the angle of collision, and on the function $\mathcal{S}$.

Certain properties of the collision operator Q are derived in section 2.3.2, and these are used in the conservation laws in section 2.4 and to prove the H theorem. A linearisation of the collision operator is also considered in section 3.1.

For simplicity of notation, one introduces the following:

$$\begin{aligned} f'_\star(t,y,v,v_\star,\nu) &= f(t,y,v_\star - ((v_\star - v)\cdot\nu)\nu) \\ f'(t,y,v,v_\star,\nu) &= f(t,y,v + ((v_\star - v)\cdot\nu)\nu) \\ f_\star(t,y,v_\star) &= f(t,y,v_\star) \end{aligned} \tag{2.7}$$

although the variables on the left hand side will normally be suppressed, and so the Q term becomes

$$Q[f,f](t,y,v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big(f'_\star f' - f_\star f\Big)\mathcal{S}(\nu, v_\star - v)\mathrm{d}\nu\mathrm{d}v_\star$$

The term $\mathcal{S}$ in the collision operator shall be called the collision kernel. This characterises how the particles interact. For example, it could be of the form $|v_\star - v|^\gamma\beta(\nu)$. The form used throughout this work will be $\mathcal{S} = \beta(\nu)$ where one assumes that there is no dependence on the relative velocities of the particles that are colliding. It should also be noted that it is a positive function.

It is very important to note here that the results one obtains are in general specific to one type of collision kernel. Different kernels give fundamentally different results on the density and on the motion itself. Thus results discussed later requiring the interaction to be Maxwellian do not hold for other types of collision kernel.

2.2.1 Relation to Above Dynamics

The dynamics of the system introduced at the start of the section can be characterised by the Boltzmann equation in two different ways. One can consider the space X to be $\mathbb{T}^d$ or alternatively $\mathcal{U}(t)$. On the torus one has more sophisticated dynamics, as it has to consider the changing of space.

We assume that the density in $\mathcal{U}(t)$ is denoted by $f(t,y,v)$ and the density in the torus is $g(t,x,w)$, so $g : [0,\infty) \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$. The relation between the two is the following:

$$g(t,x,w) = \det(B_t)\, f(t, B_t x, w + Ax)$$

or alternatively

$$\boxed{f(t,y,v) = \det\left(B_t^{-1}\right) g(t, B_t^{-1}y, v - AB_t^{-1}y)} \tag{2.8}$$

Now the aim is to derive an equation for the density g. This will require the following result on the derivative of the determinant, taken from [GS08].

Lemma 2.1 *Suppose that C is a $d \times d$ matrix with time dependent entries and which is invertible for all time, then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det C) = \det(C)\,\mathrm{tr}\left(C^{-1}\frac{\mathrm{d}C}{\mathrm{d}t}\right)$$

Proof Let $\psi(C) = \det C$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(C) = \frac{\mathrm{d}}{\mathrm{d}t}\psi(C_{11},\ldots,C_{dd})$$

and by the chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(C) = \frac{\partial\psi}{\partial C_{11}}(C)\frac{\mathrm{d}C_{11}}{\mathrm{d}t} + \cdots + \frac{\partial\psi}{\partial C_{dd}}(C)\frac{\mathrm{d}C_{dd}}{\mathrm{d}t} = \mathrm{tr}\left((\nabla\psi(C))^T\frac{\mathrm{d}C}{\mathrm{d}t}\right)$$

and since $\nabla\psi(C) = \det(C)C^{-T}$ one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(C) = \det(C)\mathrm{tr}\left(C^{-1}\frac{\mathrm{d}C}{\mathrm{d}t}\right)$$

as required. $\square$

Furthermore one needs

Lemma 2.2 *Suppose that the matrix M contains entries that depend on time, and that for all t, $M(t)$ is invertible. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M(t)^{-1}\right) = -M(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(M(t))\,M(t)^{-1}$$

Proof We have that

$$M(t)^{-1}M(t) = I$$

and then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M(t)^{-1}M(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}I = 0$$

and then the product rule and rearranging gives the result. $\square$

Now calculating the derivatives of f in equation 2.5 and expressing these in terms of g give:

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,y,v) &= \frac{\partial}{\partial t}\left[\det\left(B_t^{-1}\right) g(t, B_t^{-1}y, v - AB_t^{-1}y)\right] \\
&= \frac{\partial}{\partial t}\left(\det\left(B_t^{-1}\right)\right) g(t, B_t^{-1}y, v - AB_t^{-1}y) \\
&\qquad + \det\left(B_t^{-1}\right) \frac{\partial}{\partial t}\left(g(t, B_t^{-1}y, v - AB_t^{-1}y)\right) \\
&= \det\left(B_t^{-1}\right)\Bigg[-\mathrm{tr}\left(B_t^{-1}A\right) g(t,x,w) + \frac{\partial g}{\partial t}(t,x,w) + \frac{\partial}{\partial t}\left(B_t^{-1}y\right)\cdot\nabla_x g + \\
&\qquad + \frac{\partial}{\partial t}\left(v - AB_t^{-1}y\right)\cdot\nabla_w g\Bigg] \\
&= \det\left(B_t^{-1}\right)\Bigg[-\mathrm{tr}\left(B_t^{-1}A\right) g(t,x,w) + \frac{\partial g}{\partial t}(t,x,w) - \left(B_t^{-1}Ax\right)\cdot\nabla_x g + \\
&\qquad + \left(AB_t^{-1}Ax\right)\cdot\nabla_w g\Bigg]
\end{aligned}$$

where in the third line one has used Lemma 2.1 and to get the final line one uses Lemma 2.2, and one denotes $x = B_t^{-1}y$ and $w = v - AB_t^{-1}y$. Furthermore,

$$\begin{aligned}
\nabla_y f(t,y,v) &= \nabla_y\left[\det\left(B_t^{-1}\right) g(t, B_t^{-1}y, v - AB_t^{-1}y)\right] \\
&= \det\left(B_t^{-1}\right)\left[\left(\nabla_y\left(B_t^{-1}y\right)\right)\nabla_x g + \left(\nabla_y\left(v - AB_t^{-1}y\right)\right)\nabla_w g\right] \\
&= \det\left(B_t^{-1}\right)\left[B_t^{-T}\nabla_x g - (AB_t^{-1})^T\nabla_w g\right]
\end{aligned}$$

which is more easily seen if one considers the coordinates of this derivative, and so

$$v\cdot\nabla_y f(t,y,v) = \det\left(B_t^{-1}\right)\left[B_t^{-1}v\cdot\nabla_x g - AB_t^{-1}v\cdot\nabla_w g\right]$$

and the Q term becomes:

$$\begin{aligned}
Q[f,f](t,y,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Big(f(t,y,v_\star - ((v_\star - v)\cdot\nu)\nu) f(t,y,v + ((v_\star - v)\cdot\nu)\nu) \\
&\qquad - f(t,y,v_\star) f(t,y,v)\Big)\mathcal{S}(\nu, v_\star - v)\mathrm{d}\nu\mathrm{d}v_\star \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \det\left(B_t^{-1}\right)\Big(g(t,x,w_\star - ((w_\star - w)\cdot\nu)\nu)\times \\
&\qquad \times g(t,x,w + ((w_\star - w)\cdot\nu)\nu) \\
&\qquad - g(t,x,w_\star) g(t,x,w)\Big)\mathcal{S}(\nu, w_\star - w)\mathrm{d}\nu\mathrm{d}w_\star \\
&= \det\left(B_t^{-1}\right) Q[g,g](t,x,w)
\end{aligned}$$

where one again denotes $x = B_t^{-1}y$ and $w = v - AB_t^{-1}y$ and one obtains this by a change of variables $w_\star = v_\star - AB_t^{-1}y$.

Then observing that by combining the ∇_x and ∇_w terms together, and using the relation $w = v - AB_t^{-1}y$, one obtains an equation of the form

$$\boxed{-\mathrm{tr}\left(B_t^{-1}A\right)g + \frac{\partial g}{\partial t} + B_t^{-1}w \cdot \nabla_x g - AB_t^{-1}w \cdot \nabla_w g = Q[g,g]} \tag{2.9}$$

for the density g. One can rewrite this using several product rules to get

$$\frac{\partial g}{\partial t} + \nabla_x \cdot \left[B_t^{-1}wg\right] - \nabla_w \cdot \left[AB_t^{-1}wg\right] = Q[g,g] \tag{2.10}$$

Looking at this equation, the terms $\mathrm{tr}(AB_t)$, B_t^{-1} and AB_t^{-1} will in general be dependent on time, and thus there will be no hope in deriving a time independent solution of this equation. Thus several cases of the equation are considered where the algebra for consideration is much simpler.

Thus the next goal is to analyse the blow up of energy to find the right rescaling of the velocity to annul this problem.

2.2.2 Special Cases of Deformation

One is motivated by the physical setting to consider the following types of deformation matrix A, which are introduced with suggestive nomenclature, solely to give some physical justification.

2.2.3 Shearing

The first case is where one has B arbitrary, and $A = a \otimes n$ with $a \cdot n = 0$, with a one of the columns of B and n determining the hyper-surface along which the deformation moves. Then it is clear that $\mathrm{tr}A = a \cdot n = 0$ and it is also the case that

$$AB_t^{-1} = A(B + tA)^{-1} = a \otimes n = A$$

and so this has $\mathrm{tr}\left(AB_t^{-1}\right) = \mathrm{tr}\left((a \otimes n)\right) = 0$ and so using equation (2.9) one has the form

$$\frac{\partial g}{\partial t} + B_t^{-1}w \cdot \nabla_x g - (a \otimes n)w \cdot \nabla_w g = Q[g,g]$$

for the Boltzmann equation. If the density is spatially homogeneous then this simplifies to

$$\boxed{\frac{\partial g}{\partial t} - (a \otimes n)w \cdot \nabla_w g = Q[g,g]} \tag{2.11}$$

and in particular note that the only term here that depends on time is the density g. Thus in this spatially homogeneous shearing case it makes sense to consider time independent states of the system.

Analysing whether a Maxwellian can be a time independent solution of this equation, one observes that a time independent solution solves

$$-(a \otimes n)w \cdot \nabla_w g = Q[g, g]$$

and one checks whether a Maxwellian can satisfy this. A Maxwellian, of the form $Ae^{-\beta|w-u|^2}$, for $\beta > 0, A \in \mathbb{R} \setminus \{0\}$ and $u \in \mathbb{R}^d$, has $Q\left[Ae^{-\beta|w-u|^2}, Ae^{-\beta|w-u|^2}\right] = 0$ and

$$\nabla_w \left(Ae^{-\beta|w-u|^2}\right) = -2\beta Ae^{-\beta|w-u|^2}(w - u)$$

and so one has

$$\left(-2\beta Ae^{-\beta|w-u|^2}\right)(a \otimes n)w \cdot w = \left(-2\beta Ae^{-\beta|w-u|^2}\right)(a \otimes n)w \cdot u$$

and the value of β and A does not affect the equation. One has here that the left hand side is a quadratic in w and the right hand side is linear in w. Since this equation should hold for arbitrary w, a Maxwellian cannot be the stationary solution in this case.

This is not surprising, since physically one expects energy to increase due to the work being done on the system to shear it. Thus one where energy is constant would not be physical.

2.2.4 Dilation/Contraction

Here $A = \alpha B$ with B an arbitrary change of basis matrix, for some $\alpha \neq 0$. For $\alpha > 0$ one considers this dilation for all time. However, for $\alpha < 0$ in the contraction case one considers this only for times in $t \in [0, -1/\alpha)$. In both cases, one has $B_t = (1 + \alpha t)B$ and thus the inverse to this is

$$B_t^{-1} = \frac{1}{1 + \alpha t} B^{-1}$$

where B^{-1} is well defined as the assumption that $\det B > 0$ was made. Then

$$AB_t^{-1} = \frac{\alpha}{1 + \alpha t} I$$

and so $\operatorname{tr}\left(AB_t^{-1}\right) = \operatorname{tr}\left(\frac{\alpha}{1+\alpha t} I\right) = \frac{\alpha d}{1+\alpha t}$

Then using (2.10) and the assumption of spatial homogeneity, the Boltzmann equation in this case is

$$\boxed{\frac{\partial g}{\partial t} - \frac{\alpha}{1+\alpha t}\nabla_w \cdot [wg] = Q[g,g]} \tag{2.12}$$

As one can see, in the case of dilation, it does not make sense to consider time independent solutions of the equation, as the divergence in velocity space depends on time.

2.3 Boltzmann H-Theorem

Returning to the general equation (2.9), considerations are made as to whether there is a Lyapunov function for the equation. First some properties of these proposed Lyapunov functions, Boltzmann's H function, are discussed, as well as some properties of the collision operator Q are shown, which are used later. This is the consideration of the next part, although the use is limited as in section 2.4 it is shown that energy is not conserved. The method for showing the H theorem is similar to [Cer75] and in [CIP94]. The proof, as done by Boltzmann, is in [Bol64], which is a translation from the German.

2.3.1 H function

For the solution g, one defines $\mathcal{H}\colon [0,\infty)\times\mathbb{T}^d \to \mathbb{R}$ to be

$$\boxed{\mathcal{H}(t,x) = \int_{\mathbb{R}^d} g(t,x,w)\log g(t,x,w)\mathrm{d}w} \tag{2.13}$$

This $\mathcal{H}$ is used for systems which are spatially homogeneous and so $\mathcal{H}(t,x) = \mathcal{H}(t)$ here, and the following quantity H is used for systems where one does not have spatial homogeneity. Furthermore, one can define $H : [0,\infty) \to \mathbb{R}$ by

$$\boxed{H(t) = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} g(t,x,w)\log g(t,x,w)\mathrm{d}w\mathrm{d}x}$$

It is of interest to discuss the sign of the time derivative of both these quantities, as then this may give stability of the system near a stationary solution. As such, properties of the collision operator are considered with the aim of analysing the sign of the time derivative. This property of stability should remind the reader of Lyapunov functions.

These quantities in general will not be positive, and positivity is a desirable property. However, the following lemma ensures that they are bounded below, and so one can add a constant to make both positive.

Lemma 2.3 *If $g(t,\cdot,\cdot) \in L^1(\mathbb{R}^d)$ for each $t > 0$ and*

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} g \mathrm{d}w \mathrm{d}x = 1$$

and g is such that $g(t,x,w) > 0$ for all t,x,w then $\mathcal{H}$ is bounded below.

Note that if one instead takes $g \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ then one has that H is bounded below. **Proof** We work only with $\mathcal{H}$. By definition one has

$$\begin{aligned}
\mathcal{H} &= \int_{\mathbb{R}^d} g \log g \mathrm{d}w \\
&= \int_{\mathbb{R}^d} \mathbb{1}(\{g \geq 1\}) g \log g \mathrm{d}w + \int_{\mathbb{R}^d} \mathbb{1}(\{g < 1\}) g \log g \mathrm{d}w \\
&\geq -\int_{\mathbb{R}^d} \mathbb{1}(\{g < 1\}) g(-\log g) \mathrm{d}w
\end{aligned} \tag{2.14}$$

and for $g \in L^1(\mathbb{R}^d)$ one must have, outside of some ball of radius R, $B(0,R)$ that g decays like $\frac{1}{\|w\|^{d+\varepsilon}}$ for $\varepsilon > 0$. Then one has

$$\begin{aligned}
\int_{\mathbb{R}^d \cap \{g<1\}} g(-\log g) \mathrm{d}w &= C + \int_{(\mathbb{R}^d \setminus B(0,R)) \cap \{g<1\}} \frac{1}{\|w\|^{d+\varepsilon}} \left(-\log \left(\frac{1}{\|w\|^{d+\varepsilon}} \right) \right) \mathrm{d}w \\
&= C + K \int_R^\infty r^{-1-\varepsilon} (d+\varepsilon) \log(r) \mathrm{d}r \\
&< \infty
\end{aligned} \tag{2.15}$$

where $C = \int_{(\mathbb{R}^d \cap B(0,R)) \cap \{g<1\}} g\,(-\log g)\,\mathrm{d}w < \infty$ and K contains integrals over angles. This gives a lower bound on $\mathcal{H}$ as required. The bound for H is proved similarly. □

Thus I use H and $\mathcal{H}$ somewhat liberally to refer to the values as defined above, or to the values above with a constant added, to ensure that the terms are positive for simplicity.

Before analysing $\mathcal{H}$ and H further, properties of the collision operator Q are looked at as these are necessary tools to analyse the H functions.

2.3.2 Collision Operator

The following properties can be found in [Cer75] and in [CIP94]. One defines a bilinear quantity $Q\colon L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ by

$$Q[f,g]=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\left(f'_\star g'+f'g'_\star-fg_\star-f_\star g\right)\mathcal{S}(\nu,v_\star-v)\mathrm{d}\nu\mathrm{d}v_\star \tag{2.16}$$

and clearly this is symmetric in its arguments. Note that one has $Q[f,f]$ to be the right hand side of the Boltzmann equation in (2.5).

Lemma 2.4 *For $\phi\colon\mathbb{R}^d\to\mathbb{R}$ such that $\phi\in L^2(\mathbb{R}^d)$ one has*

$$\int_{\mathbb{R}^d}Q[f,g]\phi(w)\mathrm{d}w=\frac{1}{8}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\left(f'_\star g'+f'g'_\star-fg_\star-f_\star g\right)\times \\ \times\left(\phi+\phi_\star-\phi'-\phi'_\star\right)\mathcal{S}(\nu,v_\star-v)\mathrm{d}\nu\mathrm{d}v_\star\mathrm{d}w$$

Proof This is a simple consequence of the fact that the formula for Q is symmetric in the following changes of coordinates.

$$\begin{aligned}&w\mapsto w_\star\\&w\mapsto w'=w+((w_\star-w)\cdot nu)\nu\\&w\mapsto w'_\star=w_\star-((w_\star-w)\cdot\nu)\nu\end{aligned}$$

□

Observe that the equation in this lemma could be used to define the collision operator in a weak sense as an operator on L^2.

As a consequence of this representation, one notes that

Corollary 2.5 *For $a\in\mathbb{R}$,*

$$\int_{\mathbb{R}^d}aQ[f,g]\mathrm{d}w=0$$

For some $b\in\mathbb{R}^d$, one has

$$\int_{\mathbb{R}^d}b\cdot wQ[f,g]\mathrm{d}w=0$$

$$\int_{\mathbb{R}^d}|w|^2Q[f,g]\mathrm{d}w=0$$

Furthermore, if $h:\mathbb{R}^d\to\mathbb{R}$ satisfies

$$\int_{\mathbb{R}^d}h(w)Q[f,g]\mathrm{d}w=0$$

then h is a linear combination of the above three terms.

Proof Simple applications of lemma 2.4 give the first three results, whereby one notes that the integral is zero if $\phi + \phi_\star - \phi' - \phi'_\star = 0$. The last part is proved in [CIP94, Theorem 3.1.1] and is somewhat involved so is omitted. □

These functions, the constant, scalar product and norm, are called the **collision invariants** for this operator. They provide a key rôle in determining the Maxwellian as the stationary state of the Boltzmann equation, in certain cases, discussed in [CIP94, p.51].

To prove the Boltzmann H theorem, the following inequality is also needed.

Lemma 2.6 (Boltzmann Inequality) *Suppose that $f \in L^2(\mathbb{R}^d)$ and $f > 0$. Then*

$$\int_{\mathbb{R}^d} \log(f) Q[f,f] \mathrm{d}w \leq 0$$

Proof Using Lemma 2.4 one has

$$\begin{aligned}\int_{\mathbb{R}^d} \log(f) Q[f,f] \mathrm{d}w &= \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left(f'_\star f' - f f_\star\right) \log\left(\frac{f f_\star}{f' f'_\star}\right) \times \\ &\qquad \times \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star \mathrm{d}w \\ &= \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f' f'_\star \left(1 - \frac{f f_\star}{f' f'_\star}\right) \log\left(\frac{f f_\star}{f' f'_\star}\right) \times \\ &\qquad \times \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star \mathrm{d}w\end{aligned}$$

and note that $f' f'_\star > 0$, ν is a positive measure on the sphere, and $\mathcal{S}$ is positive as well, and for any $\lambda \geq 0$ one has that

$$(1 - \lambda) \log \lambda \leq 0$$

to conclude. □

2.3.3 H time derivative sign

The following calculations are somewhat formal, since in section 2.4 one shows that in general there is no conservation of energy. Furthermore one needs to assume enough regularity of the functions so that one can exchange integrals and derivatives where one needs to.

The aim is to get an inequality for $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$. This is done by noting that this term arises by multiplication of the Boltzmann equation with $\log g$ and then integration over w, with a product rule for this term $\frac{\partial g}{\partial t} \log g$.

We perform the calculation for general A. One takes equation (2.9) which is

$$-\mathrm{tr}\left(B_t^{-1}A\right)g + \frac{\partial g}{\partial t} + B_t^{-1}w\cdot\nabla_x g - AB_t^{-1}w\cdot\nabla_w g = Q[g,g]$$

and multiplies by $\log(g)$ and integrates over $\mathbb{R}^d$. The trace term becomes

$$-\int_{\mathbb{R}^d}\mathrm{tr}\left(B_t^{-1}A\right)g\log(g)\mathrm{d}w = -\mathrm{tr}\left(B_t^{-1}A\right)\mathcal{H}$$

and the second and third terms are simplified using a product rule and the divergence theorem, similar to the final term, where one gets

$$\begin{aligned}\int_{\mathbb{R}^d}AB_t^{-1}w\cdot\nabla_w g\log(g)\mathrm{d}w &= \int_{\mathbb{R}^d}\nabla_w\cdot\left[AB_t^{-1}wg\log g\right]\mathrm{d}w \\ &\qquad - \int_{\mathbb{R}^d}\mathrm{tr}\left(AB_t^{-1}\right)g\log(g)\mathrm{d}w \\ &\qquad\qquad - \int_{\mathbb{R}^d}AB_t^{-1}w\cdot\nabla_w g\mathrm{d}w \\ &= -\mathrm{tr}\left(AB_t^{-1}\right)\mathcal{H} - \int_{\mathbb{R}^d}AB_t^{-1}w\cdot\nabla_w g\mathrm{d}w\end{aligned}$$

where the last line follows from an application of the divergence theorem, with the fact that $g\to 0$ as $w\to\infty$ means that the first term here is zero. Thus one has an equation for $\mathcal{H}$ of the form

$$\begin{aligned}-\mathrm{tr}\left(B_t^{-1}A\right)\mathcal{H} + \int_{\mathbb{R}^d}\mathrm{tr}\left(AB_t^{-1}\right)g\mathrm{d}w + \frac{\partial\mathcal{H}}{\partial t} + \int_{\mathbb{R}^d}B_t^{-1}w\cdot\nabla_x\cdot(g\log(g))\,\mathrm{d}w \\ + \mathrm{tr}\left(AB_t^{-1}\right)\mathcal{H} = \int_{\mathbb{R}^d}Q[g,g][\log(g)+1]\mathrm{d}w\end{aligned}$$

where it has been noted that the terms $\int\frac{\partial g}{\partial t}$, $\int B_t^{-1}w\cdot\nabla_x g$ and $\int AB_t^{-1}w\cdot\nabla_w g$ combine to give $\int Q[g,g]-\int\mathrm{tr}(AB_t^{-1})g\mathrm{d}w$. Then using Lemma 2.6, the Boltzmann inequality, one gets

$$\frac{\partial\mathcal{H}}{\partial t}\leq -\int_{\mathbb{R}^d}\mathrm{tr}\left(AB_t\right)g\mathrm{d}w - \int_{\mathbb{R}^d}B_t^{-1}w\cdot\nabla_x\left(g\log(g)\right)\mathrm{d}w$$

Introducing $H=\int_{\mathbb{T}^d}\mathcal{H}\mathrm{d}x$ one obtains from the above that

$$\frac{\mathrm{d}H}{\mathrm{d}t}\leq -\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\mathrm{tr}\left(AB_t^{-1}\right)g\mathrm{d}w\mathrm{d}x - \underbrace{\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}B_t^{-1}w\cdot\nabla_x\left(g\log(g)\right)\mathrm{d}w\mathrm{d}x}_{=0}$$

where the second term is zero by the divergence theorem, and, since one can assume that $g \in L^1\left(\mathbb{T}^d \times \mathbb{R}^d\right)$, this becomes

$$\boxed{\frac{\mathrm{d}H}{\mathrm{d}t} \leq -\mathrm{tr}\left(AB_t^{-1}\right) M^g(t)} \tag{2.17}$$

where M^g is the total mass, as defined in definition 2.11, and is always positive.

If g is space homogeneous then one uses $\mathcal{H}$ and one obtains

$$\boxed{\frac{\partial \mathcal{H}}{\partial t} \leq -\mathrm{tr}\left(AB_t^{-1}\right) \rho^g(t)} \tag{2.18}$$

due to the fact that g is a density with norm ρ^g, called the mass density (defined in Definition 2.8).

Since $\mathcal{H}$ is bounded below, one can add a constant to ensure that $\mathcal{H}$ is positive without changing any of the above calculations, except for adding a positive constant to the above. Then one just needs to consider the sign of $\mathrm{tr}\left(AB_t^{-1}\right)$. Observe that when $A = 0$ then one has that $\frac{\partial \mathcal{H}}{\partial t}$ is clearly decreasing.

2.3.4 Special Cases of Motion

Considerations to the special cases of motion introduced before in section 2.2.2 are now given.

As was shown earlier, in the case of shearing one has $\mathrm{tr}\left(AB_t^{-1}\right) = 0$ and so

$$\frac{\partial \mathcal{H}}{\partial t}, \frac{\mathrm{d}H}{\mathrm{d}t} \leq 0$$

and so $\mathcal{H}$ and H are decreasing with time.

In the case of dilation, one has that

$$\mathrm{tr}\left(AB_t^{-1}\right) = \frac{\alpha d}{1 + \alpha t}$$

and this is positive for $\alpha > 0$ and negative for $\alpha < 0$, and $t \in [0, -1/\alpha)$. Thus one has that, for $\alpha > 0$,

$$\frac{\partial \mathcal{H}}{\partial t}, \frac{\mathrm{d}H}{\mathrm{d}t} \leq 0$$

and so $\mathcal{H}$ and H are decreasing with time.

2.3.5 H Theorem for no deformation

It should come as no surprise, especially from looking at the equations for $\mathcal{H}$ and H, that in a system where one has no deformation, and the particles move unforced on the torus that one has an analogous result to Boltzmann's H theorem.

Theorem 2.7 *Suppose that $A = 0$, and that g has enough regularity to ensure that one can interchange differentiation with respect to t and integration with respect to x and w. Then the following hold:*

(a) Suppose that the gas is spatially homogeneous, i.e. g does not depend on x. Then $\mathcal{H}$ never increases with times and is steady if and only if the distribution function g is Maxwellian.

(b) If the gas is not spatially homogeneous, then H never increases with time. However, a Maxwellian is not a time independent solution of this system.

Proof

(a) If g does not depend on position, then using equation (2.18) above one has

$$\frac{\partial \mathcal{H}}{\partial t} \leq 0$$

which gives the first claim. If now one has $\mathcal{H}$ steady then this furthermore implies that

$$\int_{\mathbb{R}^d} Q[g,g] \log(g) \mathrm{d}w = \frac{\partial \mathcal{H}}{\partial t} = 0$$

then Corollary 2.5 implies that $\log(g)$ must be of the form

$$\log(g) = a - b|w - c|^2$$

for some $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^d$. This this implies that g is a Maxwellian, namely of the form $g = e^{a-b|w-c|^2}$ and this trivially satisfies the Boltzmann equation since both sides are zero.

The only if part follows from g solving the equation and the fact that $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \leq 0$ is a global property.

(b) If one does not have spatial homogeneity, then H satisfies the inequality

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0$$

as required, from equation (2.17).

The equation in this case is

$$\frac{\partial g}{\partial t} + B_t^{-1} w \cdot \nabla_x g = Q[g,g]$$

and if one assumes that g is time independent and a Maxwellian then the first and last terms in the above equation are zero, and so one gets that

$$B_t^{-1} w \cdot \nabla_x g = 0$$

and for g a time independent Maxwellian then one has a left hand side dependent on time and a right hand side that doesn't, and this clearly cannot be the case, unless the Maxwellian is constant, which is unphysical.

□

2.4 Conservation Laws

So far the microscopic properties of the gas have been analysed, and an equation for the microscopic velocities has been derived. This section focuses upon the macroscopic properties of the gas. The properties of mass density, bulk velocity, internal and total energy are considered. Balance laws for the mass density and energy are considered.

These properties are defined from the velocity density g, and the balance laws governing them are derived. The definitions come from [CIP94].

In the situation we have, there are double the quantities of interest. All of the following definitions are for the density g, but all could be repeated with the density f. Thus the following quantities have a superscript to denote they are derived from g. If the superscript is replaced with an f, then they relate to the density f instead.

Definition 2.8 *One defines the* **mass density** *to be the integral of the distribution g, namely*

$$\rho^g(t,x) = \int_{\mathbb{R}^d} g(t,x,w)\mathrm{d}w$$

One defines the **bulk velocity** *to be the average of the distribution g, namely*

$$u^g(t,x) = \frac{1}{\rho^g(t,x)} \int_{\mathbb{R}^d} w g(t,x,w)\mathrm{d}w$$

One defines the **momentum tensor** *to be the second moment of the distribution g, namely*

$$p^g(t,x) = \int_{\mathbb{R}^d} w \otimes w g(t,x,w)\mathrm{d}w$$

and the **stress tensor** *to be the negative of the variance of the distribution g namely*

$$T^g(t,x) = -\frac{1}{2}\int_{\mathbb{R}^d} (w - u^g(t,x)) \otimes (w - u^g(t,x))\, g(t,x,w)\mathrm{d}w$$

In the definition of stress tensor the minus sign is used so that the balance law for internal energy is suggestive of the continuum mechanical description, whereby one has the rate of change of energy is equal to the stress tensor matrix inner product with the gradient of the bulk velocity.

The bulk velocity is the average of the molecular velocities, and as such is the velocity of the wind of the gas. This is what we can directly perceive of the molecular motion at a macroscopic level.

Definition 2.9 *The* **energy density** *is defined to be*

$$J^g(t,x) = \frac{1}{2}\int_{\mathbb{R}^d} |w|^2 g(t,x,w)\mathrm{d}w$$

The **Internal energy** *is defined to be*

$$\mathcal{E}^g(t,x) = \frac{1}{2\rho(t,x)}\int_{\mathbb{R}^d} |w - u^g(t,x)|^2\, g(t,x,w)\mathrm{d}w$$

Observe that the internal energy is the trace of the variance of g and the energy density is the trace of the second moment of g.

From looking at these definitions, one can immediately show

Theorem 2.10

$$p^g(t,x) = \rho^g u^g(t,x) \otimes u^g(t,x) + T^g(t,x)$$

$$J^g(t,x) = \frac{1}{2}\rho^g(t,x)|u^g(t,x)|^2 + \rho^g(t,x)\mathcal{E}^g(t,x)$$

$$2\rho^g\mathcal{E}^g = -\mathrm{tr}T^g \qquad \text{(Ideal Gas Law)}$$

One then uses quantities that are integrals over space, namely

Definition 2.11 *The* **Total Mass**

$$M^g(t) = \int_{\mathbb{T}^d} \rho^g(t,x)\mathrm{d}x$$

and the **Total Energy**

$$E^g(t) = \int_{\mathbb{T}^d} J^g(t,x)\mathrm{d}x$$

The rest of this section aims to derive balance laws for the mass density, momentum and energy density for the system in question.

Recall Corollary 2.5 which gives certain quantities ψ, the collision invariants, such that

$$\int_{\mathbb{R}^d} \psi Q[g,g]\mathrm{d}w = 0$$

and then using this one obtains that

$$-\int_{\mathbb{R}^d} \psi \mathrm{tr}\left(B_t^{-1}A\right) g\mathrm{d}w + \frac{\partial}{\partial t}\int_{\mathbb{R}^d} \psi g\mathrm{d}w + \nabla_x \cdot \int_{\mathbb{R}^d} \psi B_t^{-1} w g \mathrm{d}w - \int_{\mathbb{R}^d} \psi A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w = 0 \quad (2.19)$$

if one assumes enough regularity to do so. Then substituting the different collision invariants ψ one obtains the different balance laws. This is performed explicitly in the following.

2.4.1 Mass Conservation

Taking $\psi = 1$ in the above equation leads to

$$-\int_{\mathbb{R}^d} \mathrm{tr}\left(B_t^{-1}A\right) g\mathrm{d}w + \frac{\partial}{\partial t}\int_{\mathbb{R}^d} g\mathrm{d}w + \nabla_x \cdot \int_{\mathbb{R}^d} B_t^{-1} w g \mathrm{d}w - \int_{\mathbb{R}^d} A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w = 0$$

and so

$$\frac{\partial \rho^g}{\partial t} + \nabla_x \cdot \left(\rho^g B_t^{-1} u\right) = \int_{\mathbb{R}^d} A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w + \mathrm{tr}\left(B_t^{-1}A\right)\rho^g$$

and then the fact that

$$\nabla_w \cdot [A B_t^{-1} w g(t,x,w)] = A B_t^{-1} w \cdot \nabla_w g + \mathrm{tr}(A B_t^{-1}) g$$

gives the density balance equation as

$$\boxed{\frac{\partial \rho^g}{\partial t} + \nabla_x \cdot \left(\rho^g B_t^{-1} u^g\right) = 0} \quad (2.20)$$

Physically this makes sense, since if $A = 0$ then one has the usual density balance, except that the velocities are scaled since space has been scaled as well.

Integrating this equation with respect to x and using the divergence theorem on the ∇_x term gives

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} M^g(t) = 0} \quad (2.21)$$

This makes sense due to the fact that on the torus, the particles are fixed and cannot move, so the density and the mass should be constant.

2.4.2 Momentum Balance Law

Taking equation (2.19) with $\psi = w_i$ for $i \in \{1, \dots, d\}$ gives the equation

$$
\begin{aligned}
-\int_{\mathbb{R}^d} w_i \mathrm{tr}\left(B_t^{-1}A\right) g \mathrm{d}w + \frac{\partial}{\partial t}\int_{\mathbb{R}^d} w_i g \mathrm{d}w + \nabla_x \cdot \int_{\mathbb{R}^d} w_i B_t^{-1} w g \mathrm{d}w \\
- \int_{\mathbb{R}^d} w_i A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w = 0
\end{aligned}
$$

and this then gives, for each $i = 1, \dots, d$, that

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho^g u_i^g) + \nabla_x \cdot \left(\rho^g u_i^g B_t^{-1} u^g + \int_{\mathbb{R}^d} c_i B_t^{-1} c g \mathrm{d}w\right) \\
= \int_{\mathbb{R}^d} w_i A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w + \mathrm{tr}(B_t^{-1}A)\rho^g u_i^g
\end{aligned}
$$

Again this equation makes sense physically. If one takes $A = 0$ then one recovers the usual momentum conservation equation, with scaled velocity, as before. For $A \neq 0$ one has the momentum change modified in a similar manner to before.

Then if one assumes spatial homogeneity, and using the product rule

$$
\nabla_w \cdot \left[w_i A B_t^{-1} w g\right] = w_i A B_t^{-1} w \cdot \nabla_w g + w_i \mathrm{tr}\left(A B_t^{-1}\right) g + (A B_t^{-1} w)_i g
$$

the momentum balance becomes

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^g(t) u^g(t)\right) = -\rho^g(t) A B_t^{-1} u^g(t)} \tag{2.22}
$$

2.4.3 Internal Energy Balance Law

Taking equation (2.19) with $\psi = |w - u^g(t,x)|^2$ gives the equation

$$
\begin{aligned}
-\int_{\mathbb{R}^d} |w - u^g|^2 \mathrm{tr}\left(A B_t^{-1}\right) g \mathrm{d}w + \frac{\partial}{\partial t}\int_{\mathbb{R}^d} |w - u^g|^2 g \mathrm{d}w + \nabla_x \cdot \int_{\mathbb{R}^d} |w - u^g|^2 B_t^{-1} w g \mathrm{d}w \\
- \int_{\mathbb{R}^d} |w - u^g|^2 A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w = 0
\end{aligned}
$$

Using the definition of $\mathcal{E}^g$ one gets this equation as

$$
-2\rho^g \mathcal{E}^g \mathrm{tr}\left(A B_t^{-1}\right) + \frac{\partial}{\partial t}\rho^g \mathcal{E}^g + \nabla_x \cdot \int_{\mathbb{R}^d} |w - u^g|^2 B_t^{-1} w g \mathrm{d}w
$$

$$- \int_{\mathbb{R}^d} |w - u^g|^2 A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w = 0$$

then using the product rule

$$\nabla_w \cdot \left[|w-u|^2 A B_t^{-1} w g \right] = |w-u|^2 A B_t^{-1} w \cdot \nabla_w g + |w-u|^2 g \mathrm{tr}\left(A B_t^{-1} \right) + 2(w-u) \cdot A B_t^{-1} w g$$

one gets

$$\frac{\partial}{\partial t} \rho^g \mathcal{E}^g + \nabla_x \cdot \int_{\mathbb{R}^d} |w-u^g|^2 B_t^{-1} w g \mathrm{d}w - \int_{\mathbb{R}^d} \nabla_w \cdot \left[|w-u^g|^2 A B_t^{-1} w g \right] \mathrm{d}w + \int_{\mathbb{R}^d} 2(w-u^g) \cdot A B_t^{-1} w g \mathrm{d}w = 0$$

and then using the divergence term on the fourth term results in that being zero, as before, and if we assume that g is homogeneous in space then the equation becomes

$$\frac{\partial}{\partial t} \rho^g \mathcal{E}^g + \int_{\mathbb{R}^d} 2(w-u^g) \cdot A B_t^{-1} w g \mathrm{d}w = 0$$

Thus

$$\rho^g \frac{\partial}{\partial t} \mathcal{E}^g + A B_t^{-1} \colon T^g + \rho A B_t^{-1} u^g \cdot u^g - \int_{\mathbb{R}^d} A B_t^{-1} u^g \cdot w g \mathrm{d}w = 0$$

and the last two terms are equal and so we get

$$\boxed{\rho^g \frac{\partial}{\partial t} \mathcal{E}^g(t) = A B_t^{-1} \colon T^g(t)} \tag{2.23}$$

which is the usual equation for the rate of change of energy.

2.4.4 Energy Balance Law

Taking equation (2.19) with $\psi = |w|^2$ gives the equation

$$- \int_{\mathbb{R}^d} |w|^2 \mathrm{tr}\left(B_t^{-1} A \right) g \mathrm{d}w + \frac{\partial}{\partial t} \int_{\mathbb{R}^d} |w|^2 g \mathrm{d}w + \nabla_x \cdot \int_{\mathbb{R}^d} |w|^2 B_t^{-1} w g \mathrm{d}w - \int_{\mathbb{R}^d} |w|^2 A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w = 0$$

and this then gives, using the definitions of J and r,

$$2 \frac{\partial}{\partial t} J^g(t,x) + \nabla_x \cdot r^g(t,x,B_t^{-1}) = \int_{\mathbb{R}^d} |w|^2 A B_t^{-1} w \cdot \nabla_w g \mathrm{d}w + 2\mathrm{tr}(B_t^{-1} A) J^g(t,x)$$

Then assuming spatial homogeneity and using a divergence theorem on the first term on the right hand side gives

$$\frac{\mathrm{d}}{\mathrm{d}t}J^g + \int_{\mathbb{R}^d} w \cdot AB_t^{-1} wg\mathrm{d}w = 0$$

and then observing that $w \cdot Cw = C\colon w \otimes w$ gives

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}J^g(t) = -AB_t^{-1}\colon p^g(t)} \tag{2.24}$$

And if $A = 0$ then one has that the energy is constant, as expected. Furthermore if the system is expanding then energy is not conserved and is decreasing.

2.4.5 Shearing Balance Laws

As was defined earlier, this case is given by $A = a \otimes n$ with $a \cdot n = 0$. Then the calculations before gave that

$$\mathrm{tr}\left(AB_t^{-1}\right) = 0$$

Then one obtains that

$$\frac{\partial \rho^g}{\partial t} + \nabla_x \cdot \left(\rho^g B_t^{-1} u^g\right) = 0$$

and if the gas is spatially homogeneous then this becomes

$$\frac{\partial \rho^g}{\partial t} = 0$$

and furthermore in all cases of homogeneity, the mass is conserved, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}M^g(t) = 0$$

The evolution of the energy is given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^g(t) = -2\int_{\mathbb{R}^d} w \cdot a \otimes nwg\mathrm{d}w$$

2.4.6 Dilation/Contraction Balance Laws

Here $A = \alpha B$ for some $\alpha \neq 0$. Then recall that

$$AB_t^{-1} = \frac{\alpha}{1+\alpha t}I$$

As is always the case, the mass is conserved and the evolution of the energy is given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^g(t) = -\frac{\alpha(d+2)}{1+\alpha t}\mathcal{E}^g(t)$$

2.5 Maxwell Molecules

As was said in the introduction, the assumption of the interaction being Maxwellian is used, and as such, a derivation of why this form of interaction is useful is now given, at least for dimension 3. This section is somewhat self contained and only the part from equation (2.29) is useful for the rest of the work. This simplification of the interaction, where the relative velocity of the two particles in collision does not affect the collision, was first noted by Maxwell in [Max66] and has been reprinted in [Max65].

Definition 2.12 *A collection of* ***Maxwell molecules*** *is a collection of molecules for which the repulsion between any pair of them is of the form* $Cr^{1-(2d-1)}$ *where* r *is the distance between the two molecules, and* C *is some constant.*

2.5.1 Collision term

To fully justify the use of Maxwell molecules, it is useful to derive an equation for the collision term $\mathcal{S}$. This can be found in greater detail in [Cer75]. The following is **explicit for** $d = 3$ **only**, but can be generalised to general dimensions. To do this, one considers the two body problem. Here the potential attributed to each molecule will be assumed to be $U(\rho)$, where ρ is the distance from the particle in consideration. Furthermore one assumes that these potentials have been cut off, namely there exists $\sigma > 0$ such that for $\rho \geq \sigma$ the potential U is constant. Later it will be assumed that $U(\rho) = k\rho^{-(n-1)}$ but for now we perform the calculation for general potentials U.

This collision is considered in a plane which contains the two particles centres of mass and has normal perpendicular to the velocities of the two particles. Figure 2.2 shows such a situation. One furthermore considers this motion as if one of the particles were at rest, and the other had mass

$$\mu = \frac{mm_\star}{m+m_\star}$$

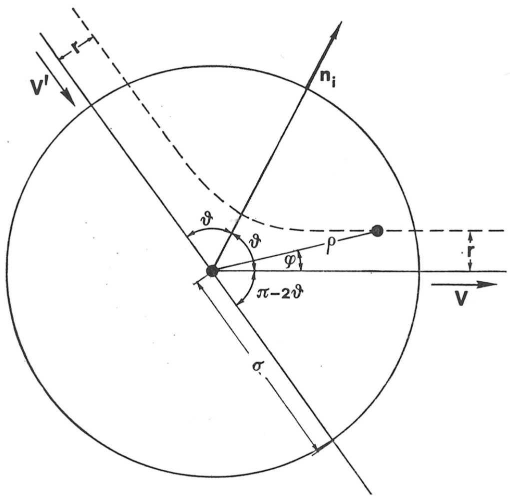


Figure 2.2: Nomenclature for a Two Body interaction. Image copied from [Cer75]

In this plane of motion, η and ϕ will be the radial and angular coordinates respectively.

One supposes that r is the distance of closest approach of the two particles, and θ is the angle between the velocity before, v and after, $v_\star$, the collision. Then one defines

$$\mathcal{S}(\theta, v_\star - v) = \|v_\star - v\| \, r \frac{\partial r}{\partial \theta}$$

Then for this two body problem, conservation of energy and conservation of mass give

$$\begin{aligned} \frac{1}{2}\mu\left(\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 + \rho^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right) + U(\rho) &= \frac{1}{2}\mu\,\|v_\star - v\|^2 + U(\sigma)k \qquad \rho \leq \sigma \\ \rho^2\frac{\mathrm{d}\phi}{\mathrm{d}t} &= r\,\|v_\star - v\| \end{aligned} \tag{2.25}$$

where σ is the cut off of the potential U. Then, using $V = \|v_\star - v\|$, and the fact that the orbit is symmetric in the apse line, and so "the angle can be easily evaluated since it is the angle between V and the apse line", giving that

$$\theta = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} V r \int_{\rho_0}^{\sigma} \rho^{-2}\left[\frac{\mu}{2}V^2\left(1 - \frac{r^2}{\rho^2}\right) - U(\rho) + U(\sigma)\right]^{-\frac{1}{2}} \mathrm{d}\rho + \sin^{-1}\left(\frac{r}{\sigma}\right) \tag{2.26}$$

where ρ_0 is the distance of closest approach which satisfies the equation

$$\frac{\mu}{2}V^2\left(1 - \frac{r^2}{\rho_0^2}\right) = U(\rho_0) - U(\sigma)$$

although clearly $\rho_0 \leq \sigma$ or otherwise there is no collision. Observe that this holds for any potential U. However, if now we assume that

$$U(\rho) = \begin{cases} k\rho^{1-n} & \rho \leq \sigma \\ k\sigma^{1-n} & \rho \geq \sigma \end{cases}$$

then one can easily substitute this potential into equation (2.26) for θ to get

$$\theta = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} Vr \int_{\rho_0}^{\sigma} \rho^{-2} \left[\frac{\mu}{2} V^2 \left(1 - \frac{r^2}{\rho^2}\right) - k\rho^{1-n} + k\sigma^{1-n}\right]^{-\frac{1}{2}} \mathrm{d}\rho + \sin^{-1}\left(\frac{r}{\sigma}\right)$$

Setting

$$\begin{aligned} b &= r\left(\frac{\mu}{2k}V^2 + \frac{k}{\sigma^{n-1}}\right)^{\frac{1}{n-1}} \\ x &= \frac{r}{\rho}\left(1 + \frac{2k}{\mu V^2 \sigma^{n-1}}\right)^{\frac{1}{2}} \\ \lambda &= \frac{r}{\sigma}\left(1 + \frac{2k}{\mu V^2 \sigma^{n-1}}\right)^{\frac{1}{2}} \end{aligned} \tag{2.27}$$

the equation becomes

$$\theta = \int_{\lambda}^{x_0} \frac{\mathrm{d}x}{\sqrt{1 - x^2 - \left(\frac{x}{b}\right)^{n-1}}} + \sin^{-1}\left(\frac{r}{\sigma}\right) \tag{2.28}$$

where x_0 satisfies

$$1 - x_0^2 - (x_0/b)^{n-1} = 0$$

Considering the limiting case, where $\sigma \to \infty$ one obtains

$$\theta = \int_{0}^{x_0} \frac{\mathrm{d}x}{\sqrt{1 - x^2 - \left(\frac{x}{b}\right)^{n-1}}}$$

and

$$b = r\left(\frac{\mu}{2k}\right)^{\frac{1}{n-1}} V^{\frac{2}{n-1}}$$

Thus using these two equations to calculate an equation for r one obtains

$$r = \left(\frac{2k}{\mu}\right)^{\frac{1}{n-1}} V^{-\frac{2}{n-1}} b(\theta)$$

Thus one obtains

$$\mathcal{S}(\theta, \|v_\star - v\|) = \|v_\star - v\|^{\frac{n-5}{n-1}} \left(\frac{2k}{\mu}\right)^{\frac{1}{n-1}} b\frac{\mathrm{d}b}{\mathrm{d}\theta}$$

which is of the form

$$\sin\theta\mathcal{S}(\theta, \|v_\star - v\|) = \beta(\theta)\,\|v_\star - v\|^{\frac{n-5}{n-1}}$$

this coming from [TM80] or from [Cer75, pp 68-71] for the case of $d = 3$.

The general dimension case is mentioned in [Vil02], and the relationship is

$$\boxed{\sin\theta\mathcal{S}(\theta, \|v_\star - v\|) = \beta(\theta)\,\|v_\star - v\|^{\frac{n-(2d-1)}{n-1}}} \tag{2.29}$$

In particular, this implies that for $\lambda > 0$,

$$\mathcal{S}(\theta, \lambda\,\|v_\star - v\|) = \lambda^{\frac{n-(2d-1)}{n-1}}\mathcal{S}(\nu, \|v_\star - v\|)$$

and so for $(2d-1)$th power molecules, where one would have $n = 2d-1$, this is independent of the magnitude of the difference in velocities, so the interaction is of the form $U(\rho) = k\rho^{-(2d-2)}$.

One can furthermore rewrite $Q[g, g]$ as

$$\begin{aligned} Q[g(\lambda\cdot), g(\lambda\cdot)](t, x, w) &= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big(f'_\star f' - f_\star f\Big)\mathcal{S}(\theta, \lambda\,\|w_\star - w\|)\mathrm{d}\nu\mathrm{d}w_\star \\ &= \lambda^{\frac{5-n}{n-1}}\lambda^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big(f'_\star f' - f_\star f\Big)\mathcal{S}(\theta, \|w_\star - w\|)\mathrm{d}\nu\mathrm{d}w_\star \\ &= \lambda^{\frac{5-n}{n-1}-d}Q[g, g](t, x, \lambda w) \end{aligned}$$

Thus for Maxwell molecules, where $n = 5$ in dimension 3, one has this scaling like the dimension of the space.

2.6 Simplified Time Dependence Boltzmann Equation

As has been seen in section 2.4 on balance laws, the energy is in general not conserved. Thus one takes a special simplification of the density g by modifying velocity space to account for this. The simplest such time dependence is

$$\boxed{g(t, w) = \xi(t)G(\eta(t)w)} \tag{2.30}$$

where $\xi, \eta \colon \mathbb{R}_+ \to \mathbb{R}$ and $G \colon \mathbb{R}^d \to \mathbb{R}$, and one also assumes that

$$\xi(0) = 1 = \eta(0)$$

These functions are such that η characterises the renormalisation of velocity space on the torus which decays to zero in such a way that this internal energy remains constant. Then to ensure that the total mass of the system is constant, one must multiply by a factor $\xi(t)$ so that this is the case. This is the justification for this simplification.

This simplification is somewhat arbitrary, but gives a particularly nice equation in the end. This is one of its justifications. Something of the form $g(t, w) = \xi(t)G(t, \eta(t)w)$ may well be useful or more realistic, but the final equation is harder to deal with. Thus as a first simplification, the above is used.

Furthermore for simplicity one assumes that

$$\boxed{u^g(0) = 0}$$

which should be the case, since the bulk velocity is the velocity of the deformation, which in the reference configuration is zero initially.

A physical interpretation of ξ and η would be nice, and as such one considers the density and internal energy definitions to get equations for η and ξ as follows. We henceforth drop the superscripts of the g on the properties for ease of notation.

The definition of density gives

$$\begin{aligned}
\rho(t) &= \int g(t, w)\mathrm{d}w \\
&= \xi(t) \int G(\eta(t)w)\mathrm{d}w \\
&= \frac{\xi(t)}{\eta(t)^d} \int G(z)\mathrm{d}z \\
&= \frac{\xi(t)}{\eta(t)^d}\rho(0)
\end{aligned}$$

and the definition of internal energy gives

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2\rho(t)} \int |w - u^g(t)|^2 g(t, w)\mathrm{d}w \\
&= \frac{1}{2\rho(t)} \int |w|^2 \xi(t) G(\eta(t)w)\mathrm{d}w \\
&= \frac{1}{2\rho(t)\eta(t)^{d+2}} \int |z^2| G(z)\mathrm{d}z
\end{aligned}$$

$$= \frac{\mathcal{E}(0)\rho(0)}{2\rho(t)\eta(t)^{d+2}}$$
$$= \frac{1}{\eta(t)^2\mathcal{E}(0)}$$

and so

$$\eta(t) = \left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)^{-1/2} \qquad \xi(t) = \frac{1}{(\mathcal{E}(t)/\mathcal{E}(0))^{d/2}}$$

where one uses that the density is constant, as shown in section 2.4.1.

Now assuming that one has Maxwell molecules, one has, from section 2.5

$$\boxed{Q[g(\lambda\cdot), g(\lambda\cdot)](w) = \lambda^{-d}Q[g,g](\lambda w)}$$

and defining

$$\overline{\mathcal{E}}(t) = \frac{\mathcal{E}(t)}{\mathcal{E}(0)}$$

one wishes to rewrite the Boltzmann equation (2.9) for this time dependence. Dropping the time dependence of $\overline{\mathcal{E}}$ for ease of notation, and setting $v = \eta(t)w$, then the g term becomes

$$-\mathrm{tr}(AB_t^{-1})g = -\mathrm{tr}(AB_t^{-1})\frac{1}{\overline{\mathcal{E}}^{d/2}}G(\overline{\mathcal{E}}^{-1/2}w) = -\mathrm{tr}(AB_t^{-1})\frac{1}{\overline{\mathcal{E}}^{d/2}}G(v)$$

and the time derivative becomes

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \frac{\partial}{\partial t}\left(\xi(t)G(\eta(t)w)\right) \\
&= \xi'(t)G(v) + \xi(t)\nabla_v G(v)\eta'(t) \\
&= \frac{-\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\mathcal{E}}^{-d/2}\right)}{\overline{\mathcal{E}}^{d}}G(v) - \frac{1}{2\overline{\mathcal{E}}^{d/2}}\overline{\mathcal{E}}^{-3/2}\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{E}}w\cdot\nabla_v G(v) \\
&= -\frac{d\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{E}}}{2\overline{\mathcal{E}}^{\frac{d+2}{2}}}G(v) - \frac{\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{E}}}{2\overline{\mathcal{E}}^{\frac{d+2}{2}}}v\cdot\nabla_v G(v)
\end{aligned}$$

and the velocity derivative becomes

$$\begin{aligned}
\nabla_w g(t,w) &= \nabla_w(\xi(t)G(\eta(t)w)) \\
&= \xi(t)\nabla_w(G\eta(t)w) \\
&= \xi(t)\eta(t)\nabla_v G(v)
\end{aligned}$$

and thus

$$AB_t^{-1}w \cdot \nabla_w g = \frac{1}{\overline{\mathcal{E}}^{d/2}} AB_t^{-1} v \cdot \nabla_v G(v)$$

and so the Boltzmann equation becomes

$$-\left[\mathrm{tr}(AB_t^{-1})\frac{1}{\overline{\mathcal{E}}^{d/2}} + \frac{d\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{E}}}{2\overline{\mathcal{E}}^{\frac{d+2}{2}}}\right] G(v) - \left[\frac{\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{E}}}{2\overline{\mathcal{E}}^{\frac{d+2}{2}}} I + \frac{1}{\overline{\mathcal{E}}^{d/2}} AB_t^{-1}\right] v \cdot \nabla_v G(v) = \frac{1}{\overline{\mathcal{E}}^{d/2}} Q[G,G](v)$$

Now defining

$$\widetilde{T}(0) = \frac{T(0)}{2\rho\mathcal{E}(0)}$$

and using the balance law for $\mathcal{E}$, namely $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{E}} = \frac{1}{\rho\mathcal{E}(0)} AB_t^{-1} \colon T$, this then gives the equation as

$$\frac{1}{\overline{\mathcal{E}}^{d/2}} \left[\left(-\left(I + d\widetilde{T}(0)\right) : AB_t^{-1}\right) G(v) - \nabla_v G(v) \cdot \left(\left(\widetilde{T}(0) \colon AB_t^{-1}\right) I + AB_t^{-1}\right) v\right] = \frac{1}{\overline{\mathcal{E}}^{d/2}} Q[G,G](v)$$

and then since I is a diagonal matrix, one has that

$$\mathrm{tr}\left[\left(\widetilde{T}(0) \colon AB_t^{-1}\right) I + AB_t^{-1}\right] = d\widetilde{T}(0) \colon AB_t^{-1} + \mathrm{tr}AB_t^{-1} = \left(I + d\widetilde{T}(0)\right) : AB_t^{-1}$$

and so the equation becomes

$$\boxed{-\nabla_v \cdot \left[G(v)\left[\left(\widetilde{T}(0) \colon AB_t^{-1}\right) I + AB_t^{-1}\right] v\right] = Q[G,G](v)} \tag{2.31}$$

Then using the Ideal gas law one can show that

$$\mathrm{tr}\left(\widetilde{T}(0)\right) = \frac{1}{\rho\mathcal{E}}\mathrm{tr}\left(\frac{\rho\mathcal{E}T(0)}{2\rho(0)\mathcal{E}(0)}\right) = \frac{2}{\rho\mathcal{E}}\mathrm{tr}T = -1$$

and a dependence of

$$T(t) = \int w \otimes w g(t,w)\mathrm{d}w = \frac{\xi}{\eta^{d+2}} \int z \otimes z G(z)\mathrm{d}z = \overline{\mathcal{E}}(t)T(0)$$

for the stress tensor. The aim now is to analyse the time dependence of AB_t^{-1}. The two special situations of shearing and dilation are now considered as these have time independence in AB_t^{-1}.

2.6.1 Simplified time dependence in the case of Simple Shearing

Suppose that one is in $d = 2$ and that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and that $B = I$. Then the equation for motion with the above time dependence becomes

$$\nabla_v \cdot \left[G(v) \begin{pmatrix} -\frac{T(0)_{12}}{2\rho\mathcal{E}(0)} & -\frac{1}{\rho} \\ 0 & -\frac{T(0)_{12}}{2\rho\mathcal{E}(0)} \end{pmatrix} v \right] = Q[G, G](v)$$

and this is the same as

$$-\frac{T(0)_{12}}{\rho\mathcal{E}(0)} G - \left(\frac{T(0)_{12}}{2\rho\mathcal{E}(0)} v_1 + \frac{1}{\rho} v_2 \right) \partial_{v_1} G(v) - \frac{T(0)_{12}}{2\rho\mathcal{E}(0)} v_2 \partial_{v_2} G(v) = Q[G, G](v)$$

Then one can interpret the functions ξ and η as

$$\xi(t) = e^{2\beta t} \qquad \eta(t) = e^{\beta t}$$

Then one can show the stress tensor to be

$$T(t) = e^{(2-(2+2))\beta t} T(0)$$

With this one can find, using the ideal gas law, that

$$\beta = -\frac{\kappa T(0)_{12}}{3P(0)}$$

where $T(0)$ is the initial stress tensor and $P(0)$ is the initial pressure and $A = \kappa \frac{a}{|a|} \otimes \frac{n}{|n|}$ where $\kappa = |a||n|$.

Furthermore one needs that this $\beta < 0$. Clearly $P(0)$ is positive since it is the product of the initial density and energy, both of which are positive. One thus needs the sign of κ and $(T(0))_{12}$ need to be the same.

In the situation as above, if one takes the equation of motion, multiplies by $\log G$ and integrates over all v, then using the divergence theorem one finds that

$$-\frac{T(0)_{12}}{\rho\mathcal{E}(0)} \int G \log G \mathrm{d}v \leq 0$$

and by adding a constant to ensure that $\int G \log G \mathrm{d}v \geq 0$ one finds that

$$T(0)_{12} \geq 0$$

with equality if and only if G is a collision invariant. Thus $\beta < 0$ since G is not a

Maxwellian, as needed.

Then for $d = 2$ one has

$$T(t) = T(0)e^{\frac{2\kappa T(0)_{12}}{3P(0)}t}$$

and thus

$$\mathcal{E}(t) = -\frac{1}{2\rho}\mathrm{tr}\left(T(0)e^{\frac{2\kappa T(0)_{12}}{3P(0)}t}\right) = \frac{3P(0)}{2\rho}e^{\frac{2\kappa T(0)_{12}}{3P(0)}t}$$

and so

$$\begin{aligned}
\rho\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3P(0)}{2\rho}e^{\frac{2\kappa T(0)_{12}}{3P(0)}t}\right) \\
&= \frac{2\kappa(T(0))_{12}}{3P(0)}\frac{3P(0)}{2\rho}e^{\frac{2\kappa T(0)_{12}}{3P(0)}t} \\
&= \kappa(T)_{12} \\
&= T \colon AB_t^{-1}
\end{aligned}$$

and observe that this is accordance with the conservation law in section 2.4

2.6.2 Simplified time dependence in the case of Dilation

Theorem 2.13 *Suppose that B is the matrix characterising the initial configuration of space, and suppose that $A = \alpha B$ for some $\alpha \in \mathbb{R}$. Then a solution to the equation* (2.31) *in this case is a Maxwellian.*

Proof One has that

$$-AB_t^{-1} = -AB_t^{-1} = \frac{-\alpha}{1+\alpha t}I$$

and one also has that

$$-\widetilde{T}(0) \colon AB_t^{-1} = \frac{-\alpha}{1+\alpha t}\mathrm{tr}\left(\widetilde{T}(0)\right) = \frac{\alpha}{1+\alpha t}$$

and so

$$\left(\widetilde{T}(0) \colon AB_t^{-1}\right)I + AB_t^{-1} = 0$$

thus giving the equation in this case to be

$$Q[G,G](v) = 0$$

resulting in G being a Maxwellian. □

One now turns attention to the case of simple shearing, and that is the focus in the next chapter.

Chapter 3

Applied Functional Analysis to the case of O.M.D.

In this chapter the main aim is to prove the existence or non-existence of a solution to the Boltzmann equation in the case of simple shearing with the special time dependence as given in the previous chapter. This is shown to have the expectation of non-existence.

Firstly a linearisation of the Collision operator is introduced, with the main aim to prove that it is a Fredholm operator, and to analyse its spectrum. These and Fréchet derivatives are introduced in the second section, and the Implicit Function Theorem for Banach spaces is stated, so that one can discount the use of the IFT. The chapter concludes with the use of a result on self adjoint operators to prove the non-existence of a solution to the special time dependence in shearing.

3.1 The Linearised Collision Operator

Since the general collision operator Q is hard to deal with, one linearises this operator in such a manner that one can prove nice properties about the linearisation. From a physical point of view, one can justify this by considering the linearisation about some distribution to be the deviation from some background noise of the system in question.

One supposes that

$$\mu(w) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|w|^2}{2}}$$

and then one linearises the collision operator around this Maxwellian. Observe that for μ, one has

$$\mu' \mu'_\star = \mu \mu_\star$$

from the section on collision invariants.

This linearisation about the Maxwellian is performed as follows. One supposes that the function $g = \mu + \mu^{1/2} h$ and then

$$Q[g, g] = Q[\mu + \mu^{1/2} h, \mu + \mu^{1/2} h] = 2Q[\mu^{1/2} h, \mu] + Q[\mu^{1/2} h, \mu^{1/2} h]$$

and observing that the first term on the right hand side is linear in h, and normalising by multiplying by $\mu^{-1/2}$ one defines:

Definition 3.1 *The* **Linearised collision operator** *is defined to be a map* $L : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *with*

$$L(h) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left(h'(\mu^{1/2})'_\star + (\mu^{1/2})' h'_\star - (\mu^{1/2}) h_\star - h(\mu^{1/2})_\star \right) \times \\ \times (\mu^{1/2})_\star \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star$$

This definition of the linearised collision operator is that used in [CIP94]. This is not the only form though. A similar form, where one takes instead $g = \mu + \mu h$ is used in [Cer88]. This changes the space over which each operator is symmetric. The former, as is soon to be seen, is symmetric on $L^2(\mathbb{R}^d)$ whereas the latter is symmetric on $L^2(\mathbb{R}^d, \mu \mathrm{d}w)$.

Both of these linearisations are used. The former because there is a nice exposition for it in [CIP94] and the latter, namely

$$L^B(h) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left(h' + h'_\star - h_\star - h \right) \mu_\star \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star$$

because the spectrum and eigenfunctions for it have been explicitly calculated in [CdBFU70] and [Cer69] and stated in [Cer88]. The paper by Drange, [Dra75], relates the two different linearisations and uses a decomposition similar to the one derived below. Most of the other literature is on the operator L^B.

Much work has been performed on and using this linearized collision operator. It has been used to prove global classical solutions of the Boltzmann equation near equilibrium in [GS11]. Spectral gap results have also been proved in [Mou06] and [MS07].

The work generally consists of a choice of collision kernel, and the type of cut off employed, for example in [Cer67], or not, as in [Kla77]. The choice of kernel is key, as for some cut off functions, one does not have a self adjoint operator on L^2, and with others, such as the Maxwellian molecule with Grad's cut off, one does.

Lemma 3.2 *On $L^2(\mathbb{R}^d)$, L is a linear, non-positive and self-adjoint operator. Furthermore, the kernel of L is $d+2$ dimensional and is spanned by the set*

$$\{\psi_\alpha\} := \left\{\mu^{1/2}, |w|^2\mu^{1/2}, w_i\mu^{1/2}, i = 1, \ldots, d\right\}$$

The kernel of L^B is spanned by

$$\left\{1, |w|^2, w_i, i = 1, \ldots, d\right\}$$

Proof The linearity is clear from the definitions in equations (2.7).

The self adjointness comes from the fact that

$$\int L(h)g\mathrm{d}w = \int L(g)h\mathrm{d}w$$

and the fact that the maximal domain of this operator is L^2.

We now show non-positivity. One has that, using an equation similar to Lemma 2.4, that

$$\begin{aligned}
\int_{\mathbb{R}^d} hL(h)\mathrm{d}w &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \left(h'(\mu^{1/2})'_\star + (\mu^{1/2})'h'_\star - (\mu^{1/2})h_\star - h(\mu^{1/2})_\star\right) \times \\
&\qquad \times (\mu^{1/2})_\star \left(h + h_\star - h'_\star - h'\right) \mathcal{S}(\nu, w_\star - w)\mathrm{d}\nu\mathrm{d}w_\star\mathrm{d}w \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \left(h'(\mu^{1/2})'_\star + (\mu^{1/2})'h'_\star - (\mu^{1/2})h_\star - h(\mu^{1/2})_\star\right)^2 \times \\
&\qquad \times \mathcal{S}(\nu, w_\star - w)\mathrm{d}\nu\mathrm{d}w_\star\mathrm{d}w \\
&\leq 0
\end{aligned}$$

and is negative since it is minus an integral of a positive, or zero function.

Suppose that $L(h) = 0$. Then one has that

$$h'(\mu^{1/2})'_\star + (\mu^{1/2})' h'_\star - (\mu^{1/2}) h_\star - h(\mu^{1/2})_\star = 0$$

for all $w \in \mathbb{R}^d$. This is the same as

$$\frac{h'}{\left(\mu^{1/2}\right)'} + \frac{h'_\star}{\left(\mu^{1/2}\right)'_\star} - \frac{h_\star}{\left(\mu^{1/2}\right)_\star} - \frac{h}{\mu^{1/2}} = 0$$

at which point $h/\mu^{1/2}$ is a collision invariant, as required. A similar method shows the claim for the kernel of L^B □

The assumption of a Maxwellian interaction, used in the previous section, means that the collision term $\mathcal{S}$ has the form

$$\mathcal{S}(\nu, w_\star - w) = \beta(\nu)$$

As will be seen shortly, it is necessary to use the Grad cut off for the angular collision kernel as was introduced in [Gra58] and discussed in [Cer88].

3.1.1 A Decomposition of the Linearised Collision Operator

One now aims to prove that the linearised collision operator is a sum of a compact operator and a bounded linear operator. To such an end, the following split of the linearised collision operator is performed, similar to [CIP94]. One defines

$$\begin{aligned}
K_1(h) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} h(w_\star) \mu^{1/2}(w) \mu^{1/2}(w_\star) \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star \\
K_2(h) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left(h' \left(\mu^{1/2}\right)'_\star + \left(\mu^{1/2}\right)' h'_\star \right) \mu^{1/2}(w_\star) \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star \qquad (3.1) \\
K_3 &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mu(w_\star) \mathcal{S}(\nu, w_\star - w) \mathrm{d}\nu \mathrm{d}w_\star
\end{aligned}$$

at which point one then has that

$$L(h) = K_2(h) - K_1(h) - K_3 I(h)$$

where I is the identity operator. One has a similar decomposition for L^B.

Observe that K_2 corresponds to the gain part of the collision operator; that part which characterises which velocities are gained after a collision. This is a non local operator and so one expects it to be hard to deal with. The other terms, K_1

and K_3 are the loss parts, so correspond to the velocities which are lost in a collision, and since these are local, they depend only on one velocity, one would expect them to be simple to deal with. The loss term has been split up into two parts because the difference $K_2 - K_1$ will turn out to be a compact operator. Properties of these operators are now discussed, leading up to Theorem 3.6.

It is clear that K_1 can be expressed in the form $\int_{\mathbb{R}^d} h(w_\star)k_1(w, w_\star)\mathrm{d}w_\star$ where

$$k_1(w, w_\star) = e^{-(|w|^2+|w_\star|^2)/4} \int_{\mathbb{S}^{d-1}} \beta(\nu)\mathrm{d}\nu \tag{3.2}$$

Now using the special form of the collision term, and standard formulas for the integration of a Gaussian, one can write K_3 as

$$K_3 = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mu(w_\star)\mathcal{S}(\nu, w_\star - w)\mathrm{d}\nu\mathrm{d}w_\star = 2^d \int_{\mathbb{S}^{d-1}} \beta(\nu)\mathrm{d}\nu \tag{3.3}$$

We now consider K_2, which is somewhat more involved.

$$\begin{aligned} K_2(h) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} &(h(w + ((w_\star - w)\cdot\nu)\nu)\mu^{1/2}(w_\star - ((w_\star - w)\cdot\nu)\nu)+ \\ &+ \mu^{1/2}(w + ((w_\star - w)\cdot\nu)\nu)h(w_\star - ((w_\star - w)\cdot\nu)\nu))\mu^{1/2}(w_\star)\beta(\nu)\mathrm{d}\nu\mathrm{d}w_\star \end{aligned}$$

A change of variables of the following form is now used. Set m to be a unit vector orthogonal to ν in the plane spanned by ν and $w_\star - w$. Then one has that

$$w_\star - w = ((w_\star - w)\cdot\nu)\nu + ((w_\star - w)\cdot m)m$$

which results in $h'\left(\mu^{1/2}\right)'_\star = h'_\star\left(\mu^{1/2}\right)'$ and

$$\beta(\nu) = \beta\left(\frac{w_\star - w}{|w_\star - w|}\cdot\nu\right) = \beta\left(1 - \frac{w_\star - w}{|w_\star - w|}\cdot m\right) =: \tilde{\beta}(m)$$

and defining $\overline{\beta}(\nu) := \beta(\nu) + \tilde{\beta}(m)$ one obtains

$$K_2(h) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} h(w+((w_\star-w)\cdot\nu)\nu)\mu^{1/2}(w_\star-((w_\star-w)\cdot\nu)\nu)\mu^{1/2}(w_\star)\overline{\beta}(\nu)\mathrm{d}\nu\mathrm{d}w_\star$$

A change of variables of the following form is now used to change this equation. For fixed ν, decompose $w_\star - w$ into the parallel and perpendicular parts to the hyperplane with ν as the normal. Do this by setting

$$v = ((w_\star - w)\cdot\nu)\nu$$
$$\widetilde{w} = (w_\star - w) - ((w_\star - w)\cdot\nu)\nu$$

and then, for fixed ν, integration with respect to $w_\star$ is replaced by integration over $|v|$ and $\widetilde{w}$. By varying ν one then varies $v = |v|\nu$ over all $\mathbb{R}^d$ and $\widetilde{w}$ varies over the hyperplane perpendicular to v through the origin. Then due to the reverse polar coordinate change that has been performed, one has that

$$\mathrm{d}\nu\mathrm{d}w_\star = 2|v|^{-(d-1)}\mathrm{d}v\mathrm{d}\widetilde{w}$$

Then observing that the function $\overline{\beta}$ depends only on the angle between the velocity difference and the parameter ν, one has that

$$\overline{\beta}\left(\frac{(w_\star - w)}{|w_\star - w|}\cdot\nu\right) = \overline{\beta}\left(\frac{|v|}{|v+\widetilde{w}|}\right)$$

Then, denoting the hyperplane perpendicular to v through the origin by $\Pi(v)$, one gets

$$K_2(h) = 2\int_{\mathbb{R}^d} h(w+v)\int_{\Pi(v)} \mu^{1/2}(\widetilde{w}+w)\mu^{1/2}(w+v+\widetilde{w})\overline{\beta}\left(\frac{|v|}{|v+\widetilde{w}|}\right)|v|^{-(d-1)}\mathrm{d}\widetilde{w}\mathrm{d}v$$

Then by setting $w + v =: \tilde{\xi}$ one has

$$K_2(h) = 2\int_{\mathbb{R}^d} h(\tilde{\xi})|\tilde{\xi}-w|^{-(d-1)}\int_{\Pi(\tilde{\xi}-w)} \overline{\beta}\left(\frac{|\tilde{\xi}-w|}{|\tilde{\xi}-w+\widetilde{w}|}\right)\mu^{1/2}(\widetilde{w}+w)\times$$
$$\times\,\mu^{1/2}(\widetilde{w}+\tilde{\xi})\mathrm{d}\widetilde{w}\mathrm{d}\tilde{\xi}$$

and from this one can see that K_2 has kernel given by

$$k_2(w,\tilde{\xi}) = |\tilde{\xi}-w|^{-(d-1)}\int_{\Pi(\tilde{\xi}-w)} \overline{\beta}\left(\frac{|\tilde{\xi}-w|}{|\tilde{\xi}-w+\widetilde{w}|}\right)\mu^{1/2}(\widetilde{w}+w)\mu^{1/2}(\widetilde{w}+\tilde{\xi})\mathrm{d}\widetilde{w}$$

This form is not useful though, because the integral over the plane has a dependence upon w and $\tilde{\xi}$. This dependence is now analysed. Observe that

$$\mu^{1/2}(\widetilde{w}+w)\mu^{1/2}(\widetilde{w}+\tilde{\xi}) = \mu^{1/2}\left(\sqrt{2}\widetilde{w} + \frac{1}{\sqrt{2}}(\tilde{\xi}+w)\right)\mu^{1/2}\left(\frac{1}{\sqrt{2}}(\tilde{\xi}-w)\right)$$

and so the kernel becomes

$$k_2(w,\tilde{\xi}) = |\tilde{\xi}-w|^{-(d-1)}\mu^{1/2}\left(\frac{1}{\sqrt{2}}(\tilde{\xi}-w)\right)\int_{\Pi(\tilde{\xi}-w)}\overline{\beta}\left(\frac{|\tilde{\xi}-w|}{|\tilde{\xi}-w+\widetilde{w}|}\right)\times \\ \times\,\mu^{1/2}\left(\sqrt{2}\widetilde{w}+\frac{1}{\sqrt{2}}(\tilde{\xi}+w)\right)\mathrm{d}\widetilde{w}$$

The integral in this expression still depends on w and $\tilde{\xi}$, and to remove this dependence, one notes that the vector $\frac{1}{2}(\tilde{\xi}+w)$ has a part ζ in the plane perpendicular to $\tilde{\xi}-w$. Let $z=\widetilde{w}+\zeta$ and then the part of $\frac{1}{2}(\tilde{\xi}+w)$ perpendicular to the plane is

$$\frac{1}{2}(\tilde{\xi}+w)\cdot\frac{\tilde{\xi}-w}{|\tilde{\xi}-w|}=\frac{1}{2}\frac{|\tilde{\xi}|^2-|w|^2}{|\tilde{\xi}-w|}$$

and so

$$\int_{\Pi(\tilde{\xi}-w)}\overline{\beta}\left(\frac{|\tilde{\xi}-w|}{|\tilde{\xi}-w+\widetilde{w}|}\right)\mu^{1/2}\left(\sqrt{2}\widetilde{w}+\frac{\sqrt{2}}{2}(\tilde{\xi}+w)\right)\mathrm{d}\widetilde{w}= \\ \mu^{1/2}\left(\frac{|\tilde{\xi}|^2-|w|^2}{\sqrt{2}|\tilde{\xi}-w|}\right)\int_{\Pi(\tilde{\xi}-w)}\mu^{1/2}(\sqrt{2}z)\overline{\beta}\left(\frac{|\tilde{\xi}-w|}{|\tilde{\xi}-w|+|z-\zeta|}\right)\mathrm{d}z$$

and so the kernel k_2 is equal to

$$k_2(w,w_\star)=|w_\star-w|^{-(d-1)}\mu^{1/2}\left(\frac{1}{\sqrt{2}}(w_\star-w)\right)\mu^{1/2}\left(\frac{|w_\star|^2-|w|^2}{\sqrt{2}|w_\star-w|}\right)\times \\ \times\int_{\Pi(w_\star-w)}\mu^{1/2}(\sqrt{2}z)\overline{\beta}\left(\frac{|w_\star-w|}{|w_\star-w|+|z-\zeta|}\right)\mathrm{d}z$$

where the integral is the integral of a Gaussian over a plane multiplied by the angular collision kernel.

The point of evaluation of $\overline{\beta}$ corresponds to the angle between the vectors $\frac{\tilde{\xi}-w}{|\tilde{\xi}-w|}$ and $\frac{z-\zeta+\tilde{\xi}-w}{|z-\zeta+\tilde{\xi}-w|}$. Thus one can split up the integration over the plane into integration over the unit sphere in this plane and an integration over $\mathbb{R}$. Then

$$\int_{\Pi(\tilde{\xi}-w)}\mu^{1/2}(\sqrt{2}z)\overline{\beta}\left(\frac{|\tilde{\xi}-w|}{|\tilde{\xi}-w|+|z-\zeta|}\right)\mathrm{d}z= \\ \int_{\mathbb{S}^{d-2}(\tilde{\xi}-w)}\overline{\beta}(\sigma)\mathrm{d}\sigma\int_{\mathbb{R}}\mu^{1/2}(\sqrt{2}z)|z|^{d-2}\mathrm{d}|z|$$

where $\mathbb{S}^{d-2}(w_\star-w)$ means the set of points in $\mathbb{S}^{d-1}$ that are perpendicular to $w_\star-w$.

Then denoting $C(d) := \int_{\mathbb{R}} \mu^{1/2}(\sqrt{2}z)|z|^{d-2}\mathrm{d}|z|$ one obtains that the kernel has the form

$$k_2(w, w_\star) = |w_\star - w|^{-(d-1)}\mu^{1/2}\left(\frac{1}{\sqrt{2}}(w_\star - w)\right)\mu^{1/2}\left(\frac{|w_\star|^2 - |w|^2}{\sqrt{2}|w_\star - w|}\right) \times \\ \times \int_{\mathbb{S}^{d-2}(w_\star - w)} \overline{\beta}(\sigma)\mathrm{d}\sigma C(d) \quad (3.4)$$

which is of a suitable form.

This is summarised in the following lemma:

Lemma 3.3 *Assuming Grad's cut off, namely*

$$\int_{\mathbb{S}^{d-1}} \beta(\nu)\mathrm{d}\nu < \infty$$

The linearised collision operator for Maxwellian interactions can be split up into a sum of the form

$$L = K - K_3 I$$

where K has kernel given by

$$k(w, w_\star) = |w_\star - w|^{-(d-1)}\mu^{1/2}\left(\frac{1}{\sqrt{2}}(w_\star - w)\right)\mu^{1/2}\left(\frac{|w_\star|^2 - |w|^2}{\sqrt{2}|w_\star - w|}\right) \times \\ \times \int_{\mathbb{S}^{d-2}(w_\star - w)} \overline{\beta}(\sigma)\mathrm{d}\sigma C(d) \\ - e^{-(|w|^2+|w_\star|^2)/4} \int_{\mathbb{S}^{d-1}} \beta(\nu)\mathrm{d}\nu$$

and

$$K_3 = 2^d \int_{\mathbb{S}^{d-1}} \beta(\nu)\mathrm{d}\nu$$

3.1.2 Regularity of the Decomposition

Properties of this decomposition are now analysed

Lemma 3.4 *The kernel of the operator $K := K_2 - K_1$ where K_i are defined as in equation* (3.1) *is in $L^1 \cap L^2$ and the integrals are bounded independently of w.*

Proof It should be clear that k_1 is in $L^1 \cap L^2$ and is bounded independently of w if

$$\|\beta\|_{L^1} := \int_{\mathbb{S}^{d-1}} \beta(\nu)\mathrm{d}\nu < \infty$$

which is exactly the assumption of Grad's cut off, since the integral is the integral of a Gaussian in $w_\star$, and it has a Gaussian term in w.

k_2 is somewhat more tricky since it is unbounded as $w_\star \to w$. However, observing that $\mu^{1/2}\left(\frac{|w_\star|^2-|w|^2}{\sqrt{2}|w_\star-w|}\right)$ is bounded above by a constant means that

$$\begin{aligned}\int_{\mathbb{R}^d} k_2(w,w_\star)\mathrm{d}w_\star &\leq C\int_{\mathbb{R}^d} |w_\star - w|^{-(d-1)}\mu^{1/2}\left(\frac{w_\star - w}{\sqrt{2}}(w_\star - w)\right)\times \\ &\qquad\qquad\times \int_{\mathbb{S}^{d-2}(\tilde{\xi}-w)} \overline{\beta}(\sigma)\mathrm{d}\sigma C(d)\mathrm{d}w \\ &\leq CC(d)2\,\|\beta\|_{L^1}\int_{\mathbb{R}^d} |w_\star - w|^{-(d-1)}\mu^{1/2}\left(\frac{w_\star - w}{\sqrt{2}}\right)\mathrm{d}w_\star\end{aligned}$$

where the 2 in front of the integral of the collision kernel comes from the integral of $\tilde{\beta}$ being equal to the integral of β.

Converting to polar coordinates about the point w results in this being a constant times the integral of a negative exponent Gaussian over $\mathbb{R}^d$, which is finite, and can be bounded independent of w.

Now one considers the difference $K = K_2 - K_1$. The kernel of this operator is $k_2 - k_1$ and

$$\int_{\mathbb{R}^d} |k(w,w_\star)|\mathrm{d}w_\star \leq \int_{\mathbb{R}^d} |k_2(w,w_\star)|\mathrm{d}w_\star + \int_{\mathbb{R}^d} |k_1(w,w_\star)|\mathrm{d}w_\star$$

and so $k \in L^1$. To prove that k is in L^2 is essentially the same calculation, except all terms are squared. The bound of the integral independent of w follows from the fact that the integral of both k_1 and k_2 are able to be bounded independently of w.

□

Lemma 3.5 *For any $r \geq 0$ one has*

$$\int_{\mathbb{R}^d} k(w,w_\star)(1+|w_\star|^2)^{-r}\mathrm{d}w_\star \leq \frac{C}{(1+|w|^2)^{r+1/2}}$$

Proof Clearly $\int k_1(w,w_\star)(1+|w_\star|^2)^{-r}\mathrm{d}w_\star$ is bounded in the requisite manner since k_1 has a factor $e^{-|w|^2/4}$ involved, which decays faster than any polynomial. Thus it is left to prove that the term involving k_2 is bounded in this manner. This is performed as follows. One shows that

$$I := C(d)(1+|w|^2)^{r+1/2}\int_{\mathbb{R}^d} k_2(w,w_\star)(1+|w_\star|^{-r})\mathrm{d}w_\star$$

is bounded independently of w.

A change of variables whereby one sets $\gamma = w_\star - w$ is used so that this becomes

$$I = C(d)(1+|w|^2)^{r+1/2} \int_{\mathbb{R}^d} (1+|w+\gamma|^2)^{-r} |\gamma|^{-(d-1)} e^{-\frac{1}{8}\left(\frac{(2w\cdot\gamma+|\gamma|^2)^2}{|\gamma|^2}+|\gamma|^2\right)} \times$$

$$\times \int_{\mathbb{S}^{d-2}(\gamma)} \overline{\beta}(\sigma) \mathrm{d}\sigma C(d) \mathrm{d}\gamma$$

and now splitting this integration into $I = I_1 + I_2$ where I_1 is integration over the set $\{\gamma : |\gamma| > |w|/4\}$ and I_2 is integration over the set $\{\gamma : |\gamma| < |w|/4\}$ one can see that

$$I_1 = C(d)(1+|w|^2)^{r+1/2} \int_{\{|\gamma|>|w|/4\}} (1+|w+\gamma|^2)^{-r} |\gamma|^{-(d-1)} e^{-\frac{1}{8}\left(\frac{(2w\cdot\gamma+|\gamma|^2)^2}{|\gamma|^2}+|\gamma|^2\right)} \times$$

$$\times \int_{\mathbb{S}^{d-2}(\gamma)} \overline{\beta}(\sigma) \mathrm{d}\sigma C(d) \mathrm{d}\gamma$$

$$< CC(d)(1+|w|^2)^{r+1/2} \int_{\{|\gamma|>|w|/4\}} |\gamma|^{-(d-1)} e^{-\frac{1}{8}|\gamma|^2} \mathrm{d}\gamma 2 \|\beta\|_{L^1}$$

since when on this set one has that

$$\frac{e^{-\frac{1}{8}\left(\frac{(2w\cdot\gamma+|\gamma|^2)^2}{|\gamma|^2}+|\gamma|^2\right)}}{(1+|w+\gamma|^2)^{-r}|\gamma|^{-(d-1)}} < C$$

Then this function is bounded independently of w since the integral is of the order of an exponential, decreasing for increasing $|w|$.

Now one considers the integration over the second set $\{\gamma : |\gamma| < |w|/4\}$. This is

$$I_2 := C(d)(1+|w|^2)^{r+1/2} \int_{\{|\gamma|<|w|/4\}} (1+|w+\gamma|^2)^{-r} |\gamma|^{-(d-1)} e^{-\frac{1}{8}\left(\frac{(2w\cdot\gamma+|\gamma|^2)^2}{|\gamma|^2}+|\gamma|^2\right)} \times$$

$$\times \int_{\mathbb{S}^{d-2}(\gamma)} \overline{\beta}(\sigma) \mathrm{d}\sigma \mathrm{d}\gamma$$

One would like to change to polar coordinates, but the term $|w+\gamma|$ is somewhat problematic. However, if $|w| \geq 1$ then one has $1+|w+\gamma|^2 > 1 + \frac{9}{16}|w|^2$ and so, incorporating the angular integration outside of the plane spanned by w and γ into the constant C, the bound becomes by

$$\begin{aligned}
I_2 &< CC(d)2\,\|\beta\|_{L^1}\,\frac{(1+|w|^2)^{r+1/2}}{(1+\frac{9|w|^2}{16})^r}\int_0^\infty\int_0^\pi |\gamma|^{-(d-1)}e^{-\frac{1}{8}\left(\left(2|w|\cos\theta+|\gamma|^2\right)^2+|\gamma|^2\right)}\times\\
&\qquad\qquad\times\sin\theta|\gamma|^{d-1}\mathrm{d}|\gamma|\mathrm{d}\theta\\
&= CC(d)2\,\|\beta\|_{L^1}\,\frac{(1+|w|^2)^{r+1/2}}{(1+\frac{9|w|^2}{16})^r}\int_0^\infty\int_0^\pi e^{-\frac{1}{8}\left(\left(2|w|\cos\theta+|\gamma|^2\right)^2+|\gamma|^2\right)}\sin\theta\mathrm{d}|\gamma|\mathrm{d}\theta\\
&\leq CC(d)2\,\|\beta\|_{L^1}\,\frac{(1+|w|^2)^{r+1/2}}{(1+\frac{9|w|^2}{16})^r}\int_0^\infty e^{-\frac{|\gamma|^2}{8}}\mathrm{d}|\gamma|(2\pi)^{1/2}|w|^{-1}\\
&= CC(d)2\,\|\beta\|_{L^1}\,\frac{(1+|w|^2)^{r+1/2}}{|w|(1+\frac{9|w|^2}{16})^r}
\end{aligned}$$

and for $|w|\geq 1$ this is of the order of a constant, and so can be bounded independently of w.

It is left to bound this for $|w|<1$. In this case one replaces $|w|$ by 1 and then observes that the integral is independent of w and finite, and so the whole term is independent of w, as required. □

Theorem 3.6 *The operator K is linear, bounded and compact on L^2, the latter meaning that the image of any closed set is relatively compact.*

Proof By Lemma 3.4 one has that the operator is bounded, and it is clearly linear. Thus it is left to prove that it is compact.

We first claim that

$$\begin{aligned}
\|(1-\chi_R)K\| &\leq (1+R^2)^{-1/4}\\
\|K(1-\chi_R)\| &\leq (1+R^2)^{-1/4}
\end{aligned}$$

where χ_R is the characteristic function of the set $\{w : |w|\leq R\}$.

Then the fact that $\chi_R K$ is a Hilbert Schmidt operator, and so bounded and compact (Lemma 8.20 and Theorem 8.83 in [RR04]), and this convergence gives the fact that K is compact, since the set of compact operators is closed and linear in $B(L^2)$ by Proposition 4.2 in [Con85].

We now prove the claim. We have that

$$\|(1-\chi_r)K\|^2 = \sup_{\|h\|_{L^2}=1}\left[\int(1-\chi_R)\left[\int k(w,\eta)h(\eta)\right]^2\mathrm{d}w\right]$$

$$
\begin{aligned}
&\leq \sup_{\|h\|_{L^2}=1} \left[\int (1-\chi_R) \left[\int k(w,\xi) \mathrm{d}\xi \right] \left[\int k(w,\eta) h(\eta)^2 \right] \mathrm{d}w \right] \\
&\leq \sup_{\|h\|_{L^2}=1} \left[\int (1-\chi_R)(1+|w|^2)^{-1/2} \left[\int k(w,\eta) h(\eta)^2 \right] \mathrm{d}w \right] \\
&\leq C(1+R^2)^{-1/2} \sup_{\|h\|_{L^2}=1} \left[\int \int k(w,\eta) h^2(\eta) \mathrm{d}\eta \mathrm{d}w \right] \\
&\leq C(1+R^2)^{-1/2} \sup_{\|h\|_{L^2}=1} \left[\int \int k(w,\eta) \mathrm{d}w h^2(\eta) \mathrm{d}\eta \right] \\
&\leq C(1+R^2)^{-1/2}
\end{aligned}
$$

where we first use Hölder's inequality, then Lemma 3.5 and then finally Lemma 3.4. Then for the other inequality one has

$$
\begin{aligned}
\|K(1-\chi_R)\|^2 &= \sup_{\|h\|_{L^2}=1} \int K\left((1-\chi_R)h\right)(w) \mathrm{d}w \\
&= \sup_{\|h\|_{L^2}=1} \left[\int \left[\int k(w,\eta)(1-\chi_R(\eta)) h(\eta) \mathrm{d}\eta \right]^2 \mathrm{d}w \right] \\
&\leq \sup_{\|h\|_{L^2}=1} \left[\int \left[\int k(w,\xi)(1-\chi_R(\eta)) \mathrm{d}\xi \right] \underbrace{\left[\int k(w,\eta) h^2(\eta) \mathrm{d}\eta \right]}_{\leq C\|h\|_{L^2}^2} \mathrm{d}w \right] \\
&\leq C \int \int k(w,\eta)(1-\chi_R(\eta)) \mathrm{d}\eta \mathrm{d}w \\
&\leq C \int (1+|\eta|^2)^{-1/2}(1-\chi_R(\eta)) \mathrm{d}\eta \\
&\leq C(1+R^2)^{-1/2}
\end{aligned}
$$

□

It should be noted that all this analysis can be repeated for the operator L^B, although it is not performed here. We now discuss the spectrum of L^B for dimension $d = 3$.

Lemma 3.7 *Suppose that the dimension $d = 3$. Then the spectrum of the operator L^B is discrete and can be described, for $n, l = 0, 1, 2, \ldots,$ by*

$$
\lambda_{nl} = 2\pi\rho_0 m^{-1} \int_0^{\pi/2} \left[P_l(\sin\theta) \sin^{2n+l}\theta \left[\beta(\theta) + \beta\left(\frac{\pi}{2} - \theta\right) \right] - (\delta_{n0}\delta_{l0} + 1)\beta(\theta) \right] \mathrm{d}\theta
$$

where P_l denotes the l-th Legendre polynomial.

Furthermore the eigenfunctions are given by

$$g_{nlm} = \left[\frac{2n!}{\Gamma(n+l+3/2)}\right]^{1/2} \left(\frac{\xi}{2RT_0}\right)^l L_n^{(l+1/2)}\left(\frac{\xi^2}{2RT_0}\right) Y_l^m(\theta,\phi)$$

where ξ, θ, ϕ are polar coordinates in velocity space, Γ denotes the gamma function, L are the associated Laguerre polynomials and Y_i^m are the spherical harmonics, which can be found in [AS65]. Furthermore these satisfy the orthogonality condition

$$(2RT_0)^{-3/2} \int_{\mathbb{R}^3} \overline{g}_{nlm}(w) g_{n'l'm'}(w) j e^{-|w|^2/2RT_0} \mathrm{d}w = \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

Observe that for $n = 0, l = 0$, $n = 1, l = 0$ and $n = 0, l = 1$ we have that $\lambda_{nl} = 0$ and furthermore the eigenfunctions are linear combinations of the collision invariants.

3.2 Preliminary Functional Analysis

First, a few functional analytic tools are introduced which will be used to consider the existence of a solution to the Boltzmann equation with the simplified time dependence. They are introduced without motivation.

Lemma 3.8 *[Con85, p35 Thm 2.19] Suppose that H is a Hilbert space and A is a bounded linear operator $H \to H$. Then*

$$\ker A = (\mathrm{Im}(A^\star))^\perp$$

where $A^\star$ is the adjoint of A.

3.2.1 Fredholm operators

The following introduction, with proofs, to Fredholm operators can be found in [Zei95, Ch 5]

Definition 3.9 *A* **linear Fredholm operator** $T\colon X \to Y$ *for X, Y normed spaces, is a bounded linear operator with*

$$\dim\ker(T) < \infty \qquad \mathrm{codim} R(T) < \infty$$

The **index** *of a linear Fredholm operator is*

$$\mathrm{ind}(T) = \dim\ker(T) - \mathrm{codim} R(T)$$

Theorem 3.10 (Riesz Schauder) *Suppose that X, Y are Banach spaces. Furthermore suppose that $B : X \to Y$ is a bounded linear and bijective operator and that $C : X \to Y$ is a linear and compact operator. Then*

$$B + C : X \to Y$$

is a linear Fredholm operator of index zero.

The usefulness of the index being zero is given by the following theorem

Theorem 3.11 (Fredholm Alternative) *Suppose that $A : X \to Y$ is a linear Fredholm operator and that X, Y are normed linear spaces. Then*

1. *A is surjective if and only if* $\mathrm{ind} A = \dim \ker(A)$
2. *A is injective if and only if* $\dim \ker(A) = 0$
3. *A is bijective if and only if* $\mathrm{ind} A = \dim \ker(A) = 0$
4. *Suppose that X, Y are Banach spaces. Then*

$$Au = b \qquad\qquad u \in X$$

is well posed if and only if $\mathrm{ind} A = \dim \ker(A) = 0$

3.2.2 Fréchet Derivatives and the Implicit Function Theorem

The theoretical part of this chapter is from [Zei95, Ch 4].

Definition 3.12 *The* **Fréchet Derivative** *of a function $f : X \to Y$ at the point $u \in X$ is a linear operator $Df(u) : X \to Y$ such that*

$$f(u + v) - f(u) = (Df(u))(v) + o(\|v\|)$$

holds for all v in some open neighbourhood of $v = 0$ in X.

Note that all the usual things hold, for example the chain rule, sum rule and product rule.

One can also define the **partial Fréchet Derivative** in the expected sense. The partial derivative is denoted by $D_v f$. One defines a function $g(v) := f(u, v)$ and then $D_v f(u, v)h = Dg(v)h$ and similarly for the other partial derivatives.

Lemma 3.13 *Suppose that $f : U \subset X \times Y \to Z$ be given on U an open neighbourhood of (u_0, v_0), where the spaces X, Y, Z are Banach. If the derivative $Df(u_0, v_0)$ exists then the partial derivatives $D_u f(u_0, v_0)$ and $D_v f(u_0, v_0)$ exist and*

$$Df(u_0, v_0)(h, k) = D_u f(u_0, v_0)h + D_v f(u_0, v_0)k \qquad \forall h \in X, k \in Y$$

Theorem 3.14 *Suppose that X, Y, Z are Banach spaces, and let $U \subset X \times Y$ with $(u_0, v_0) \in U$. Furthermore suppose that*

$$F : U \to Z$$

where F is a C^n map with $F(u_0, v_0) = 0$. Suppose that the operator

$$D_v F(u_0, v_0) : Y \to Z$$

is bijective. Then

1. *There exist numbers $r > 0$ and $\rho > 0$ such that, for each given $u \in B(u_0, \rho) \subset X$, the equation $F(u, v) = 0$ has a unique solution $v \in Y$ with $v \in B(v_0, r)$. Denote this v by $v(u)$.*
2. *The function $u \mapsto v(u)$ is C^n on U. In particular*

$$v'(u) = -D_v F(u, v(u))^{-1} D_u F(u, v(u)) \qquad \forall u \in U$$

3.3 Applied Functional Analysis in the case of O.M.D.

We first prove a result about the linearised collision operator.

Lemma 3.15 *The linearised collision operator L is a Fredholm operator from $L^2 \to L^2$, with $\operatorname{ind}(L) = 0$.*

Proof From the previous section, one can split $L = K - K_3 I$. By lemma 3.6 K is a linear and compact operator, and $K_3 I$ is a bounded linear operator, and so by Theorem 3.10 one has that L is a Fredholm operator of index zero. $\square$

3.3.1 Application to Simple Shearing

In order to use the linearised collision operator, one expands the function G about the Maxwellian μ in the form $G = \mu(1 + h)$ for some h. Then one wishes to solve, using the simplification of notation that $-C := \left(\widetilde{T}(0) \colon \widetilde{A}(t)\right) I + AB_t^{-1}$, since it doesn't depend on time, the equation

$$\nabla_v \cdot [(\mu + \mu h)(v) C v] = Q[\mu + \mu h, \mu + \mu h](v)$$

which is equation (2.31), with the above substitution for G. Then this simplifies to

$$\nabla_v \cdot [\mu(v) C v] + h(v) \nabla_v \cdot [\mu(v) C v] + \mu(v) \nabla_v \cdot [h(v) C v] = \mu L^B(h) + \mu \Gamma^B(\mu h, \mu h)$$

where Γ^B is the corresponding non linear part to the linear L^B.

Then the aim would be in some manner to solve this equation using the Implicit Function Theorem. There is significant difficulty in this however. This in part comes from the fact that a linearisation of this about $A = 0$, which is the value where one knows the distribution to be Maxwellian (here $h = 0$), results in having a Fréchet derivative of $-L^B$, which is fine if one defines the map from $L^2 \to L^2$. However, this space is just too large, since one requires at least regularity of H^1 to make any sense of the derivative of a function. Thus this discrepancy of spaces creates problems, which cannot be resolved easily.

The method of characteristics has been attempted in [MJ] with limited success. This is not repeated here.

Instead, we first analyse the equation

$$L^B(h) = \nabla_v \cdot [\mu(v) C v] = \frac{1}{(2\pi)^{d/2}} (\mathrm{tr} C - Cv \cdot v) e^{-|v|^2/2} \tag{3.5}$$

to gauge whether there is any hope of solutions to the full equation, since if this cannot be solved, there would be no chance to solve the whole equation.

In dimension $d = 3$, the operator L^B has eigenfunctions given by, up to a constant,

$$\xi^l L_n^{(l+1/2)}(\xi^2) Y_l^m(\theta, \phi)$$

for $l, n = 0, 1, \ldots$ and $-l \leq m \leq l$, this coming from Lemma 3.7.

The eigenfunctions in the kernel are those for $l, n = 0$, $n = 0, l = 1$ and $n = 1, l = 0$ and there are 5 of these, in $d = 3$. These are linear combinations of the collision invariants. Thus if the right hand side of (3.5) has a part in the kernel, since L^B is a self adjoint operator one has no solutions. The following theorem proves the lack of existence of a solution.

Theorem 3.16 *Suppose that L is a linearised collision operator with collision kernel $\mathcal{S}$ such that L is self adjoint on L^2. Furthermore suppose that the collision invariants* *belong to the kernel of L.*

Suppose that $A \in M^{d\times d}$ and $B \in M^{d\times d}$ with the same sign entries as A, and such that $B\colon A \neq 0$. Define $C = -(B\colon A)\, I - A$

Then the equation

$$L(h) = (\mathrm{tr}C - Cv\cdot v)\mu$$

has no solution $h \in L^2$.

Proof Since L is self adjoint on L^2, one can write $L^2 = \ker(L) \oplus \mathrm{Im}(L)$. Thus if $(\mathrm{tr}C - Cv\cdot v)\mu$ has a part in $\ker(L)$ then it is not in the image of L and so there are no solutions. Observe that the constant function and $|v|^2$ are both in the kernel.

First assuming that C has the same term on the diagonal, one can write $Cv\cdot v = \alpha|v|^2 + \sum_{i,j=1 i\neq j}^{d} C_{ij}v_iv_j$. Then one can see that

$$\mathrm{tr}C - Cv\cdot v = \alpha d - \alpha|v|^2 - \sum_{i,j=1 i\neq j}^{d} c_{ij}v_iv_j$$

and so since the first two are in the kernel, this itself must have a component in the kernel. Then since $\mathrm{tr}C - \alpha|v|^2$ is in the kernel, we take

$$\begin{aligned}\int_{\mathbb{R}^d}(\mathrm{tr}C - Cv\cdot v)(\mathrm{tr}C - \alpha|v|^2)\mu(v)\mathrm{d}v &= -\int_{\mathbb{R}^d}(\alpha d - \alpha|v|^2)\left(\sum_{\substack{i,j=1\\ i\neq j}}^{d} C_{ij}v_iv_j\right)\mu(v)\mathrm{d}v \\ &\quad + \frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d}(\alpha d - \alpha|v|^2)^2 e^{-|v|^2/2}\mathrm{d}v \\ &= \frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d}(\alpha d - \alpha|v|^2)^2 e^{-|v|^2/2}\mathrm{d}v \\ &\neq 0\end{aligned}$$

where we have used that $\int_{\mathbb{R}^d} v_iv_j|v|^2e^{-|v|^2/2}\mathrm{d}v = 0 = \int_{\mathbb{R}^d} v_iv_je^{-|v|^2/2}\mathrm{d}v$ for any $i \neq j$, and using the fact that we are integrating a positive function, the last term must be non zero.

The above can also be repeated if C has terms with the same sign, but not necessarily constant, on all the diagonal terms, since choosing the coefficient of $|v|^2$ to have that same sign is enough to give a non zero integral.

Due to the special form of C, one can never have diagonal terms that are exact opposite values of each other and so one can never have cancellations in different directions to give a value of zero. Thus, over all such cases of C, one can choose an element of the kernel such that the function $(\mathrm{tr}C - Cv\cdot v)e^{-|v|^2/2}$ has non zero projection onto the space spanned by that element, and so it always has a part in the kernel.

Thus there is a part of this function in the kernel, for any choice of C as specified, and so by Lemma 3.8 the function always has a part outside of the range of the operator L, and so the equation has no solution. $\square$

An alternative and shorter way to prove the non-existence of solutions to this problem is to first choose a basis for the system such that the matrix $B = I$. Then C has the form where it is a constant on the diagonal, with some off diagonal terms. Thus the first consideration in the above proof is enough to say there are no solutions.

Thus, since this simple equation fails to have a solution, we question whether the full equation

$$\nabla_v \cdot [\mu(v)Cv] + h(v)\nabla_v \cdot [\mu(v)Cv] + \mu(v)\nabla_v \cdot [h(v)Cv] = \mu L^B(h) + \mu\Gamma^B(\mu h, \mu h)$$

can have a solution or not. Considerations are made using the order of each term, in terms of the asymptotics with $C \to 0$. If h is a solution, then the order of h must be $h = o(1)$. This is due to the fact that as $C \to 0$, the solution $\mu + \mu h$ must converge to the Maxwellian μ, as one knows that this is the only solution in this case. This though results in the fact that as $C \to 0$, one also has that $h \to 0$, and so this order of h is the minimum one to ensure this. It must decay just a little more than a bounded function.

Then one has the following orders for the other terms

$$\begin{aligned}
\nabla_v \cdot [\mu(v)Cv] &\sim \mathcal{O}(C) \\
h(v)\nabla_v \cdot [\mu(v)Cv] &\sim o(C) \\
\mu(v)\nabla_v \cdot [h(v)Cv] &\sim o(C) \\
\mu L^B(h) &\sim o(1) \\
\mu\Gamma^B(\mu h, \mu h) &\sim \mathcal{O}(h^2)
\end{aligned}$$

which can be seen immediately, since all terms, except Γ^B, are linear in C and h. The non linear Γ^B is at least quadratic in h.

Since we are analysing the case where $C \to 0$, the dominant order on both sides of the equation is the slowest one. The slowest on the left hand side is $\mathcal{O}(C)$ and so the order on the right hand side must match this for solutions to exist. There are two possible ways for these orders to be equal.

The first is that $L^B(h) \sim \mathcal{O}(C)$ at which point the linear collision operator is the dominant term on the right hand side. However, if this is the case, then the dominating equation is

$$L^B(h) = \nabla_v \cdot [\mu(v)Cv]$$

and as we have shown above, there are no solutions to this equation in L^2.

The other case is that the non linear term is of order $\mathcal{O}(C)$, at which point $h \sim \sqrt{C}$. While one cannot directly show that there are no solutions in this case, this case is unlikely to be true. The analysis required to prove a solution existed here would also be considerably difficult. One does not expect this scaling due to the following. The linearisation taken was one about the Maxwellian μ, in such a way that the dominant term is the linear one, especially for small h. The linear term can be bounded by the norm of h, and the non-linear one by the norm of h squared. As such, when $h \to 0$ one would have the square of the norm converging to zero much faster than the norm itself, and so the non-linear term would not be the dominant term. We thus must discount this case from possibility.

We are thus lead to conclude that the expectation is for there not to be a solution to this equation.

We now give a suggestion as to the reason why such a scaling does not work in this case.

The scaling

$$\xi(t)G(\eta(t)w)$$

works well in the case of uniform dilation because it assumes that energy is dissipated uniformly in every direction, since the scaling is the same for every velocity direction. As should be expected, this is the case for uniform dilation, hence the ease in finding a solution of this form.

However, in the case of shearing, one has a specific direction in which the velocity increases, and thus one has a specific direction in velocity space over which the contribution to the energy moves in. This is far from dissipation uniformly in every direction, and so a scaling where one assumes such a fact wouldn't be expected to work.

The scaling used conserves mass, since it is of the form

$$g(t, w) = \frac{1}{\varepsilon^d} G\left(\frac{w}{\varepsilon}\right)$$

but for the energy one has

$$\mathcal{E} = \int w^2 \varepsilon^{-d} G(w/\varepsilon) \mathrm{d}w = \varepsilon^2 \int w^2 g \mathrm{d}w$$

and so in dilation, where the energy scales of the form of the velocity squared, this renormalising to the factor of ε^2 is suitable.

However, in shearing, this relation does not work, since the energy scales of the form of the velocity, since only one direction contributes heavily. Thus this form of scaling does not result in energy being conserved, and hence the lack of solutions for the case of shearing.

Chapter 4

Concluding Remarks

Firstly, general equations and their properties have been shown for a forced gas of atoms on the torus, and this has been linked to a periodically distributed gas in $\mathbb{R}^d$.

The simplest rescaling of velocity space has been considered to account for the lack of conservation of energy in the models, to derive an equation for the motion.

Under these simplifications, it has been shown that a Maxwellian is the stationary solution for the case of dilation. In the case of shearing, it has been justified that solutions of this particular form will be unlikely to be found.

Since in the case of shearing the energy is transported along a particular direction in velocity space, anisotropic scalings should be considered. It is thus future work to consider anisotropic scalings to try to prove the existence of solutions.

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