Mathematics and Statistics
Centre for Doctoral Training

Long Range Particle Dynamics and the Linear Boltzmann Equation

by

Matthew Egginton

Thesis

Submitted for the degree of

Doctor of Philosophy

Mathematics Institute
The University of Warwick

February, 2018

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Acknowledgments

I would firstly like to thank my supervisor Florian Theil for his invaluable help and suggestions over the course of the past three and a half years. I must also thank my second supervisor Ian Melbourne for additional advice which was kindly given.

Secondly I would like to thank my fellow MASDOC students for making my Ph.D. so enjoyable, as well as everyone in the mathematics department I have had the pleasure of knowing. I must also thank the Engineering and Physical Sciences Research Council for funding my research through MASDOC under grant number EP/H023364/1.

I finally would like to thank my parents Jim and Molly for their support throughout my life, and their encouragement to pursue whatever I wanted to do. Thanks also to my girlfriend Cindy for providing an iron fist of encouragement to finish this on time!

This thesis was typeset with LaTeX 2ε[1] by the author.

[1] LaTeX 2ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The style package *warwickthesis* was used.

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Every effort has been made to reference correctly any result that is not solely mine, however partially it has been used.

Chapter 3 bears some similarities to [39], and this reference was the inspiration for that chapter.

The subject material of the thesis has been extended and then been shortened and rewritten for the purpose of publication. A preprint is available online, see [25].

Abstract

This thesis aims to give full and complete details of the first proof that the particle density for a tagged particle interacting with a background of particles via a long range potential ϕ converges weak-$\star$ to a weak solution of the linear Boltzmann equation for ϕ. This convergence is shown to hold for potentials where there is a $\gamma > 0$, and such that for sufficiently large $|x|$ we have

$$\nabla\phi(x) \leq Ce^{-|x|^{\frac{3}{2}+\gamma}}.$$

The main difficulty in this context are the many grazing collisions in the particle dynamics which prevents a Markovian structure of the dynamics. We remove grazing collisions via the use of a regularisation parameter. This enables us to consider an associated short range evolution, which we describe on a space of marked trees to encode the collisional history of the tagged particle. This description then enables a specification of Markovian dynamics by removing a set of trees that exhibit recollisions. We then relate this evolution with the Markovian evolution of the linear Boltzmann equation on this space.

The difference between dynamics with and without grazing collisions are estimated by comparing the contribution from near collisions with a bound on the time of collision, and the contribution from grazing collisions by using an L^∞ estimate on the potential.

The remaining error for the contribution of the grazing collisions on solutions of the linear Boltzmann equation are estimated by estimating the difference between deviation angles with and without grazing collisions.

Chapter 1

Introduction

One of the problems postulated by Hilbert in the early 1900s was the mathematical treatment of the axioms of physics, and one interpretation of this is providing a justification of the macroscopic laws of motion from their underlying particle dynamics. One such famous open problem in this area is the justification of the Boltzmann equation for a dilute gas from its underlying physical principles of a Newtonian gas of interacting particles.

This is a well studied problem in the field of kinetic theory, see for instance [35, 32, 26, 47], although all these results study the justification problem for short range interactions, where the particles interact for some compact set in physical space. While these results are impressive, most physically relevant models use long range interactions, and the justification of the Boltzmann equation from an underlying long range particle system is something that has not been treated, with the exception of [4], in the mathematics literature. This we feel is a major disadvantage of the current knowledge, and a treatment of long range interactions is the primary focus of this thesis.

Analysing this non-linear system with long range interactions is however very complicated, but there are two natural ways to simplify. One could remove the non-linearity by starting near to equilibrium, as in [4], or one could instead simplify the interactions to remove the non-linear effects of the particle dynamics, and we consider the latter. We here consider a gaseous system akin to the Lorentz gas [37] or the Rayleigh Gas. This system then has a unique particle of one species interacting with a free flowing background of particles of another species, and so one thus removes the non-linearity in the system, then making the problem of proving the justification of the Boltzmann equation simpler.

This system with short range interactions has in fact been well studied, for instance [27, 48, 36, 13, 24] prove results for the Lorentz gas, where the background particles are stationary. Some recent work [39, 38] has proved the justification for the Rayleigh gas with hard sphere interactions, where the background is allowed to move under free flow.

We first introduce the Newtonian and Boltzmann models for this Rayleigh gaseous system, and briefly analyse the validity of such models. After introducing both, we formally state the main result this thesis will prove, and the discussion of the result will highlight the issues arising from the use of a long range potential.

1.1 The Newtonian Model

The Newtonian model of a gas we take is one where, for each time t, one records the positions and velocities of each particle in the gas. These positions and velocities evolve under the ordinary differential equations given by Newton's second law of motion. For the Rayleigh gas, these are given by

$$\begin{cases} \dot{x}(t) &= v(t), \\ \dot{v}(t) &= -\sum_{i=1}^{N} \nabla\phi\Big(x(t) - x_i(t)\Big), \\ \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= 0. \end{cases} \tag{1.1}$$

We have here two species. Firstly one has the un-indexed particle, which we call the tagged particle, and N other particles with index $i = 1, \ldots, N$ which are called the background particles. The function $\phi\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is the interaction potential. The nomenclature of a particle however is somewhat an abuse of intuition, since for a long range interaction one no longer has a well defined sense of a particle. Even so, we use this language throughout as it is a convenient manner to describe the system.

Remark 1.1. *Historically famous examples of interaction potentials are the following:*

- $\phi(x) = \begin{cases} \infty & |x| < 1 \\ 0 & \text{else} \end{cases}$*, though not a potential, does correspond to a hard sphere model.*
- $\phi(x) = \frac{1}{|x|^k}$ *for* $k \geq 1$ *which are power law potentials. The case of* $k = 4$ *is called the Maxwellian potential, and* $k = 1$ *is the Coulomb potential.*

- $\phi(x) = \frac{1}{|x|^{12}} - \frac{1}{|x|^6}$ *is called the Lennard-Jones potential, which models the atomic interaction between a pair of neutral particles.*

The phase space we take as $\mathcal{U} = \mathbb{T}^3 \times \mathbb{R}^3$ and we furthermore assume periodic boundary conditions on the torus. One completes this system by specifying the initial conditions. We here take the background particles to be distributed independently and identically according to a density which is uniform in the 3 dimensional torus $\mathbb{T}^3$ and according to a probability density $g \in L^1(\mathbb{R}^3, (1+|v|^2)\,\mathrm{d}v)$ on the velocity space $\mathbb{R}^3$. The weighted L^1 space is chosen to ensure that the expected value of the mass and energy of each particle is finite. The tagged particle is distributed independently from this background according to a probability density f_0 on the phase space $\mathcal{U} = \mathbb{T}^3 \times \mathbb{R}^3$.

Remark 1.2. *For the purposes of these introductory sections, we only assume general initial data, though we take a specific background distribution of a Maxwellian for the remainder of the thesis.*

To ensure these dynamics are well defined, it is natural to assume that the force $\nabla\phi$ is Lipschitz, since by making this assumption we can apply standard results in ODE theory to provide existence of solutions to these equations for all time. Most physically valid models also make the assumption that the potential ϕ is radially symmetric, and we also make this assumption. With this assumption of radial symmetry, it means that an interaction between two particles lies within a plane, and this greatly simplifies an analysis of scattering, as well as the form of the linear Boltzmann equation.

Straightforward calculations can show that both mass and energy are conserved by this system of equations. Momentum however is not, since the background has no change in velocity from the interaction. Therefore, in order to be physically valid, one must assume that the expected values of mass and energy are finite initially, and so one assumes that

$$f_0 \in L^1\left(\mathcal{U},\, (1+|v|^2)\,\mathrm{d}x\,\mathrm{d}v\right).$$

One should note however that the energy of the tagged particle is not conserved, because the background can impart energy onto the tagged particle. We must also impose restrictions on the interaction potential, and so we ensure that ϕ imparts finite energy onto each particle through interactions, meaning $\phi \in L^1_{\text{loc}}$.

Given a specific value for the state of the particles at time t, equations (1.1) give a unique evolution of the system. However, when considering the phase space density f^ε of the tagged particle, one does not have information about the positions of the N background particles, and so the evolution does not satisfy the Markov property. Indeed, if one assumes that $\nabla\phi(x) > 0$ for all $x \neq 0$, then a consequence of the long range interaction is the fact that an historic position of a background particle affects the historic position of the tagged particle and determines the present and future position of that background particle, and therefore the future evolution of the tagged particle. Therefore, since these recollisions will be present in the density f^ε for any $t \in [0, T]$, in the presence of a long range interaction, the density for the particle dynamics will never be Markovian.

This non-Markovian nature of the dynamics however is different if one assumes that $\nabla\phi$ is compactly supported. The compact support of the potential implies then that the future dynamics are dependent upon the historic dynamics only in situations where one has recollisions, namely where one background particle interacts with the tagged particle more than once. These considerations will be of importance in our proof.

It should be clear that, in situations where $\nabla\phi(x) > 0$ for all $x \neq 0$, the vast majority of interactions in this system occur when the positions of the two particles are far apart. One should expect that these so called "grazing collisions" produce a very small deviation upon the velocity of the tagged particle. They thus should be expected to alter the distribution of the particles in a small manner, and thus not affect the system to a great extent. These by themselves are not a critical issue for the particle system, but when one is striving for an evolution of the system in the linear Boltzmann equation where collisions are the fundamental action on the system, recording many small deviations which should then produce a negligible effect on the density seems overly complicated. As such, an in depth study of the many grazing collisions will be necessary to analyse the properties of the system.

Given an initial realisation of this system, the evolution is explicit, and one can completely determine the state of the system at any given time. However, for any physically relevant situation, the number of particles in $1m^3$ of an ideal gas is of the order of magnitude of 10^{25} and so the computation for any typical system prohibits the use of such a detailed model.

1.2 The Linear Boltzmann Equation

Since for typical systems, the number of particles is prohibitive for a Newtonian description of the gas, one aims to treat the system probabilistically. One introduces a probability density f on $\mathcal{U}$ so that

$$\int_{\Omega} f(t, x, v) \, \mathrm{d}x \, \mathrm{d}v$$

gives the probability of finding the tagged particle with positions and velocities in $\Omega \subset \mathcal{U}$ at time t.

This model encompasses collisions at its heart, and one thus has an underlying interaction structure, which here we assume to be given by a potential $\phi \colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$. One assumes the density f to be affected by the operations of both free transport and binary collisions, resulting in the evolution of the density being expressed in the linear Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = L(f) \\ f(0, x, v) = f_0(x, v) \end{cases} \tag{1.2}$$

with the collision operator

$$L(f) = \int_{\mathbb{R}^3} \int_{\mathcal{S}} \left(f' \, g'_\star - f \, g_\star \right) |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star, \tag{1.3}$$

where we use the shorthand notation of $g_\star = g(v_\star)$ for the evaluation at the velocity of the colliding particle, and $f' = f(v')$, $g'_\star = g(v'_\star)$ to represent the evaluation at the pre-collisional velocities in a two body interaction. $\mathcal{S}$ is a subset of the plane through x perpendicular to $v_\star - v$, and is the parameter space of possible interactions. The pre-collisional velocities v' and $v'_\star$ and the set $\mathcal{S}$ are determined by the potential ϕ from the underlying two body particle dynamics, and are thus functions of the potential, the parameters specifying $\mathcal{S}$ and the velocities v and $v_\star$. This relationship is deliberately left vague here, but will be exposed in much greater detail in Chapter 2.

Remark 1.3. *From a historical perspective, this equation has roots in the Boltzmann equation, which was postulated as a macroscopic description of a dilute gas in [14, 15]. This was also considered in [43], and so the equation is sometimes also called the Maxwell-Boltzmann equation.*

The equation describes the evolution as the density moving under free flow, until the particle encounters a collision. The collision operator then describes, at least on a formal level, a loss of density of the pre-collisional velocity at v, and a gain in density at v from a collision with particles with pre-collisional velocities v' and $v'_\star$ colliding to create particles moving with velocities v and $v_\star$.

We again take initial condition $f_0 \in L^1(\mathcal{U}, (1 + |v|^2)\,\mathrm{d}x\,\mathrm{d}v)$ and background distribution $g \in L^1(\mathbb{R}^3, (1 + |v|^2)\,\mathrm{d}v)$. This thus means that we can show that a solution has finite mass and energy for the time of existence of solutions. Furthermore, we have that mass is conserved in this system. As in the particle dynamics case, we do not have conservation of energy for the tagged particle density.

The first natural question is what conditions on the initial densities f_0 and g and the interaction potential ϕ are required to make sense of equations (1.2). For a long range potential we have $\mathcal{S} \simeq \mathbb{R}^2$, and integrating over this unbounded set means that the equation no longer makes sense in the strong sense given in equation (1.3) since, for $f \in L^1$ the collision operator will in general be infinite. For weak solutions however, given by the form in Definition 1.6, which we now state loosely, for h a Lipschitz test function, as

$$-\int (\partial_t h + v \cdot \nabla_x h)\, f \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = \int (h' - h) f\, g\, |v_\star - v| \,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t,$$

we observe that we only need $f \in L^1(\mathcal{U}, (1 + |v|^2)\,\mathrm{d}x\,\mathrm{d}v)$ for all time. This form, together with the precise definition of the pre-collisional velocity in (2.2) requires ϕ to be continuous and have sufficient decay for the integral to converge. For example, with $\phi(x) \leq |x|^{-2+}$ we will see later from Lemma 2.5 that $|v' - v| \sim r^{-2+}$ and so this term decays to 0 faster than the radial Jacobian in the integral of $\mathcal{S}$, and so the integral in the equation converges.

In stating the form of the collision operator in equation (1.3) one has implicitly made the assumption that all collisions are binary. From a physical point of view, this can be seen as a result of an assumption of a low density gas. This assumption thus affects the types of potential one considers to ensure this equation is physically valid. To be consistent with this assumption, it is necessary that the potential decays such that $\phi(x) \to 0$ as $|x| \to \infty$ at a rate that ensures the probability of more than two body collisions is 0. This will be made clearer in Section 1.4.

This formulation of the equation furthermore respects the fact that the majority of interactions in the particle dynamics are grazing. This is seen as follows. The space $\mathcal{S}$ parametrises the possible types of collision, and the sets far from the origin in this plane describe the grazing collisions. Since these have much greater measure than sets near the origin in the plane $\mathcal{S}$, this thus means that the majority of collisions that occur are grazing collisions. As was commented upon above, equation (1.2) does not make sense in a strong sense, and this is precisely because of the non-integrable singularity that occurs because of these grazing collisions.

The incorporation of grazing collisions into the Boltzmann dynamics has the conjectured effect, as remarked in [51], that the non-integrable singularity over the parameter space $\mathcal{S}$ produces a smoothing effect on the weak solutions of (1.2). This is furthermore reinforced by the study of the Landau equation, which is obtained similarly to the Boltzmann equation, but where one concentrates on grazing collisions.

The linear Boltzmann equation can be interpreted as defining Markovian dynamics. In kinetic theory literature, the notion of this evolution being Markovian is more commonly called propagation of chaos, and is a major difficulty in proving justification of macroscopic equations, as the underlying particle dynamics do not have this property. This Markovian nature can be seen when one rewrites the equation into the form of an equation describing the generator of a Lévy process, where the linear collision operator becomes an integral with respect to a Lévy measure. This has been performed in [7] for the linear Boltzmann equation and in [44, 34] for the non-linear Boltzmann equation.

The books [49], [20], [21], and the overview [52] give more information on the origins of the Boltzmann equation and its properties. The books [23] and [1] give information on the linear Boltzmann equation.

1.3 Relationship Between the Models

The two models should be considered as valid models of the underlying system when one views this system on differing spatial scales. The Newtonian model has the implicit assumption that the particles have some form of size associated to them, which is in particular seen when one considers hard sphere interactions or short range interactions, and the relevant physical scale where one observes this is on a microscopic level. The Boltzmann model however assumes that the collisions occur at a specific point, and so the particles have no size, which is consistent with a macroscopic model.

In order to relate the two models, we must compare the parameters in the two models. While the particle dynamics included a parameter of the number of background particles N in the system, together with an implicit parameter of the typical length scale of the system, which we shall henceforth call ε, neither of these two parameters were present in the Boltzmann equation. To remove the dependence on these parameters, one must simultaneously take the limit $N \to \infty$ as well as $\varepsilon \to 0$ to recover the macroscopic description of the system.

In the specification of the linear Boltzmann equation, one had made the assumption that the gas was at low density. The relationship between N and ε

thus must correspond to a low density limit of the Newtonian particle system. This thus means that particles undergo a finite number of collisions per unit time, and a scaling argument results in requiring $N\varepsilon^2 = 1$ for such a low density limit. This is the Boltzmann-Grad limit, as introduced in [28].

1.4 Precise Statement of Main Result

We now proceed to state precisely the theorem of which the remainder of this thesis will prove. This is preceded by formal definitions of the densities and interaction potential that are needed to complete the particle and linear Boltzmann descriptions of the gaseous system. Throughout this thesis, we have restricted our attention to three dimensions for notational simplicity. In principle, the method works for arbitrary dimension greater or equal to two, and the linear Boltzmann equation and the particle dynamics can be defined for any dimension $d \geq 2$.

We start by defining the properties required for the interaction potential ϕ. The inspiration for the conditions required of our potentials originates in [26], and the conditions we take are somewhat similar.

Definition 1.4. *A potential* $\phi \colon \mathbb{R}^3 \to \mathbb{R}$ *is an* **admissible long range potential** *if*

(1) ϕ *is radial, namely there is a function* $\psi \colon (0, \infty) \to \mathbb{R}$ *such that* $\phi(x) = \psi(|x|)$,

(2) $\psi \in C^2((0, \infty))$,

(3) ψ *is strictly decreasing,*

(4) $\lim_{\rho \to \infty} \psi(\rho) = 0$ *and* $\lim_{\rho \to 0} \psi(\rho) = \infty$,

(5) There is a $\rho_1 > 0$ *such that for* $\rho \in (0, \rho_1)$, *we have* $\psi(\rho) + \frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) \leq 0$.

We give some description of these conditions, as they are unmotivated at this point. Assumption (1) is natural from physical considerations, as for example Coulomb interactions are radial, and is used to enable the specification of the linear collision operator in (1.3), as was commented before. Without this assumption, the operator must depend upon the orientation of the two molecules, and so $\mathcal{S}$ would vary dependent upon this.

The second assumption gives sufficient regularity to make sense of equations (1.1), and necessary regularity so that later estimates are well defined. As stated before, one only needs Lipschitz continuity of $\frac{\mathrm{d}}{\mathrm{d}\rho}\psi$ to ensure Newton's laws are well defined. The extra second derivative implies this, and is in particular required for the estimate on the scattering time of collisions in Chapter 2.

Conditions (3) and (4) are assumed to ensure that the interaction occurs in a certain manner, and the aim is to have a two body interaction that evolves as follows. The particles interact by approaching each other, and obtain a unique point at a minimum distance away from each other, and then move away. One thus wishes to remove situations where the two particles coalesce, which is achieved by making the assumption

$$\lim_{\rho \to 0} \psi(\rho) = \infty.$$

Indeed, coalescence could only happen if the particles collide head on, with ingoing relative velocity $|w|^2 = \psi(0)$. With this asymptote, we avoid these singularities in the interactions.

Condition (3) ensures that $\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) \neq 0$ for all ρ and so one avoids singularities where the particles remain a fixed distance apart. Furthermore, (3) and (4) together ensure that $\psi(\rho) > 0$ which gives a repulsive interaction, which forces the particles apart at all distances.

Finally, assumption (5) is purely technical, but, in conjunction with (4) it means that ϕ has an asymptote at the origin similar to the asymptote in a power law interaction, and in particular this assumption is satisfied by a power law potential. Upon the face of it, one may expect that $\psi(\rho) \geq e^{\rho}$, being a typical solution of the constraint in (5) is implied by this condition. However, by assuming (4), we discount such a solution, and we instead have a power law relationship. Indeed, supposing that $\psi(\rho) = |x|^{-s}$ for $s > 0$, we then have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) = -s\rho^{-s-1} = -s\rho^{-1}\rho^{-s}$$

which thus satisfies (5). The reason for such a condition on the potential for small radius is to ensure the validity of the estimate in Chapter 2 on the scattering time of collisions.

Using such a potential, we evolve the position of an un-indexed tagged particle with indexed background on the phase space $\mathcal{U} = \mathbb{T}^3 \times \mathbb{R}^3$ at spatial scale $\varepsilon > 0$

under the equations

$$\begin{cases} \dot{x}(t) &= v(t), \\ \dot{v}(t) &= -\frac{1}{\varepsilon}\sum_{i=1}^N \nabla\phi\left(\frac{x(t)-x_i(t)}{\varepsilon}\right) \\ \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= 0. \end{cases} \tag{1.4}$$

The initial distributions are the following. The tagged particle is distributed according to a probability density f_0 on the state space $\mathcal{U}$ and the background particles are independent of the tagged particle, and distributed independently and identically according to a spatially uniform distribution and which is Maxwellian in velocity

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\,\pi\,\beta}^3} e^{-\frac{|v|^2}{2\,\beta}}$$

where $\beta > 0$ is the temperature. We denote by f^ε the density of the tagged particle.

The initial density of the tagged particle satisfies the following physically relevant assumptions. We first assume finite mass and energy, and that the density is initially comparable to the equilibrium density of the system.

Definition 1.5. *We consider initial densities $f_0 \colon \mathcal{U} \to \mathbb{R}$ that satisfy*

(1) The function $f_0 \in L^1(\mathcal{U}, (1+|v|^2)\,\mathrm{d}x\,\mathrm{d}v)$ and $f_0 \in L^\infty(\mathcal{U})$ and $f_0 \geq 0$ almost everywhere.

(2) There is a constant C such that $f_0 \leq C\mathcal{M}$ almost everywhere.

The second condition clearly implies the regularity in the first, though we state the former to emphasise that this is an important and required condition. While both conditions are somewhat natural to assume from a physical perspective, we highlight the mathematical reasons for assuming them. The first condition is used to ensure that solutions of the linear Boltzmann equation lie in the domain of the collision operator, and so that the particle density is in L^∞ for all time.

The second condition is used in Chapter 5 to provide estimates on solutions of the linear Boltzmann equation with a regularised potential that are independent of the regularisation.

In the low density limit, the tagged particle density is a weak solution of the linear Boltzmann equation on $[0, T]$ for some arbitrary time $T > 0$. By this we mean the following.

Definition 1.6. *A function* $f\colon [0,T]\times\mathcal{U}\to\mathbb{R}$ *is a weak solution of the linear Boltzmann equation with initial density* f_0 *if*

(1) For all $t\in[0,T]$

$$\int_0^t\int_{\mathcal{U}}(1+|v|^2)\,f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t<\infty.$$

(2) For all $h\in C_c^\infty([0,T)\times\mathcal{U})$ *we have*

$$-\int_0^T\int_{\mathcal{U}}(\partial_t h+v\cdot\nabla_x h)\;f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t-\int_{\mathcal{U}}f_0\,h(0)\,\mathrm{d}x\,\mathrm{d}v=\int_0^T\langle L(f),\,h\rangle\,\mathrm{d}t \tag{1.5}$$

where

$$\langle L(f),\,h\rangle:=\int_{\mathcal{U}}\int_{\mathbb{R}^3}\int_{\mathcal{S}}(h'-h)\,f\,\mathcal{M}_\star\,|v_\star-v|\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}v\,\mathrm{d}x. \tag{1.6}$$

For cleanness of notation, we have avoided the explicit specification of the velocity v' and of the plane $\mathcal{S}$ here. This can be found in Chapter 2. We remark that if $f\in C^1$ is a weak solution, then it is also in fact a strong solution. Indeed, by integrating by parts in (1.5) one can show that

$$-\int_0^T\int_{\mathcal{U}}(\partial_t h+v\cdot\nabla_x h)\;f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t-\int_{\mathcal{U}}f_0\,h(0)\,\mathrm{d}x\,\mathrm{d}v$$
$$=\int_0^T\int_{\mathcal{U}}h\,(\partial_t+v\cdot\nabla_x)\,f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

Furthermore, by changing coordinates in the gain term in (1.6) we obtain, since the Jacobian from pre to post collisional velocities is 1, the relation

$$\int_{\mathcal{U}}\int_{\mathbb{R}^3}\int_{\mathcal{S}}h'\,f\,\mathcal{M}_\star\,|v_\star-v|\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}v\,\mathrm{d}x$$
$$=\int_{\mathcal{U}}\int_{\mathbb{R}^3}\int_{\mathcal{S}}h\,f'\,\mathcal{M}_\star'\,|v_\star-v|\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}v\,\mathrm{d}x.$$

The arbitrariness of h then enables one to conclude the strong form of the linear Boltzmann equation from the fundamental theorem of the calculus of variations.

These previous definitions enable us to now state our main theorem.

Theorem 1. *Let* $T>0$, *and let* f^ε *be the phase space density for a tagged particle on* $[0,T]\times\mathcal{U}$, *evolving according to* (1.4) *with initial density given by* f_0 *satisfying Definition 1.5, with an admissible potential* ϕ *as in Definition 1.4 such that there is* *a $\rho_2 > 0$ and $\gamma > 0$ with*

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) \leq e^{-C\rho^{\frac{3}{2}+\gamma}} \tag{1.7}$$

for all $\rho > \rho_2$. Then as $\varepsilon \to 0$ with $N\varepsilon^2 = 1$, we have f^ε converges weak-$\star$ in L^∞ up to a subsequence to f which is a weak solution of the linear Boltzmann equation on $[0,T] \times \mathcal{U}$ associated to ϕ, as given by Definition 1.6.

1.5 Relationship with existing results

As stated earlier, the aim of the thesis is to give a full proof of this theorem, and it is the first of its kind for the linear Boltzmann equation associated to a long range potential. As such, the existing results are similar in aspects of the work, but by no means identical. The similarities are highlighted, and will enable us to describe the context of the theorem, and to motivate parts of the method of our proof.

1.5.1 Physical Considerations

We start by analysing the validity of the result from a physical point of view. We remark that the result is consistent with physical intuition.

In including long range interactions in the linear Boltzmann equation, as in [20, Ch. 2], one should in theory include a Vlasov term $X(x) \cdot \nabla_v f$ into the Boltzmann equation, where

$$X(x) = \int_{\mathbb{R}^3} \int_{|x-x_\star|>0} f(t, x_\star, v_\star)\, \nabla\phi(x - x_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star,$$

which corresponds to the grazing collisions creating a self consistent field in which they evolve. However, one can easily see that this term converges only for potentials ϕ which decay faster than ρ^{-2}. This is in agreement with [8], where they pass to the non-dimensional form of the Vlasov and Boltzmann operators, and conclude that in the class of potentials $\{C/\rho^s : s > 1\}$, both terms are of the same order of magnitude only for an inverse square power law potential. Furthermore, they conclude that, for potentials with $s > 2$ the Boltzmann term is dominant. Therefore, with the decay we assume in Theorem 1, our potentials are in the situation where the Boltzmann term is dominant. Therefore the linear Boltzmann equation without a Vlasov term is the relevant low density limit.

By including grazing collisions into the linear Boltzmann equation, one would expect the properties of the equation to agree with a generalised Fokker Planck equation

$$\partial_t f + v \cdot \nabla_x f = \nabla^2(f) \cdot D(f) - \nabla_v(Xf)$$

for diffusion tensor D, since this equation is derived from analysing the statistical effects of grazing collisions. Proceeding as in [19, Ch. 2], we first observe that for grazing collisions $|v' - v|$ is small, and so we can use a Taylor series expansion for h in (1.6) about v. This then enables one to derive equations for R and D in the Fokker-Planck equation from the weak form of the linear collision operator (1.6).

The derivation of this from the weak form of the collision operator thus suggests from a physics point of view that weak solutions are the relevant type of solution to encode the physical properties of the system. We detail the mathematical reasons for considering weak solutions explicitly in Section 2.4.

1.5.2 Short Range Interactions

While the primary focus of the thesis is an exposition of long range effects, we also provide an extension of the result in [39] from hard sphere dynamics to short range potentials. The added difficultly from hard sphere dynamics to short range interactions is primarily the fact that the interactions occur over an interval of time, as opposed to instantaneously. One is thus recourse to impose extra conditions on the particle dynamics to ensure that the evolution still has the form of a collection of two body interactions.

This adds onto the difficulty that the evolution of the particle dynamics is not Markovian, which was commented upon before, and is a result of recollisions. In this paper, as well as in [39], we use a method developed in [40, 41, 42] where one circumvents the issue of the non-Markovian structure of the particle dynamics by using a space of marked trees, which enlarges the state space for the particle dynamics, thus enabling a Markovian description of the dynamics for most evolutions. This then allows for a comparison of the particle and linear Boltzmann dynamics on this space.

For other particle systems, an alternative method is to use the BBGKY hierarchy (named after Bogoliubov [12], Born [16], Green and Kirkwood [33] and Yvon [53]), which is a hierarchy of equations for the marginals of the phase space density of the N-particle density, to describe the particle evolution, and then restrict the integration in the definition of the marginals to avoid recollisions. This avoidance of recollisions creates an evolution for the one particle density that is Markovian.

The BBGKY hierarchy is a well used methodology, and has been employed in the work of Lanford [35] and King [32], as well as in [26, 9, 47, 10, 11] in analysing the non-linear Boltzmann equation.

With regards to the Lorentz gas, where the background is stationary, many studies have analysed the low density limit for this particle system in the case where the background interacts with a tagged particle with short range or hard sphere interactions, see for example [27, 48, 36, 13]. Some analysis has furthermore been considered for the Rayleigh gas [50].

1.5.3 Long Range Potentials

We now highlight the current mathematical treatment of the justification of the linear Boltzmann equation with long range potentials.

The first such result is by Desvillettes and Pulvirenti [24], which gives a proof that the tagged particle density for particles evolving via a short range potential of the form

$$\phi^R(x) = \begin{cases} |x|^{-s} & |x| < R \\ R^{-s} & |x| \geq R \end{cases}$$

for some $s > 2$, converges weak-$\star$ to a solution of the linear Boltzmann equation for $\phi(x) = |x|^{-s}$. This result is an incomplete proof of the justification of the linear Boltzmann equation as it does not show convergence of the particle density for this short range potential ϕ^R to a corresponding density for the long range potential. In comparison with the decay we assume (equation (1.7)), this result is impressive in the weakness in the decay assumption required. This however is not so surprising, since we assume such strong decay solely to ensure the short range particle dynamics approximate the long range dynamics. This is observed in Chapter 4.

We conjecture that decay for $\psi(\rho) = \rho^{-2}$ is the minimal decay one can assume the potential to satisfy. This is due to the fact that, as described in the previous section, the linear Boltzmann equation should not be the macroscopic evolution for potentials decaying slower than ρ^{-2}, as the grazing collisions here provide extra effects that are not quantified by the collision operator.

The paper by Ayi [4] shows convergence of the one particle density function for a system of N interacting particles where the initial density is a perturbation of equilibrium only for the first particle, for potentials with decay of

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho} \psi(\rho) \right| \leq e^{-e^{e^{\lambda\left(1+\rho^{2(d-1)}\right)}}}$$

to a solution of the non-linear Boltzmann equation where the perturbation converges to the linear Boltzmann equation.

Similarly to [24], this paper also regularises the long range potential by using a smooth cut off of the interaction potential to compare the BBGKY hierarchy for the long range particle dynamics and a solution of the linear Boltzmann equation for this cut off potential. A compactness argument then shows the solution to this equation converges to the linear Boltzmann equation for ϕ.

In comparison, our decay condition on the potential is very weak, and this is to be expected, since here one has a fully interacting system, and so the collisional structure is much more complicated. Thus, in using the BBKGY hierarchy, one is required to use all marginals in the system, whereas in our case we can use marked trees to specify only the position of the tagged particle, and this encoding of much less information enables convergence for more slowly decaying potentials.

Other systems in kinetic theory have dealt with long range interactions. The Vlasov-Poisson system is a classic example. Here, one has an evolution equation of

$$\partial_t f + v \cdot \nabla_x f + (E + F)\nabla_v f = 0$$

for

$$E(t,x) = \int \frac{x-y}{|x-y|^3} \int f(t,y,v) \,\mathrm{d}v \,\mathrm{d}y,$$

which corresponds to an evolution of a charge density through its self consistent electric field. Pfaffelmoser [46] proved the existence of classical solutions to this equation by using a path based description of the solution. This method can be seen to be similar to the path based method in [39].

With the addition of a point charge density to this system, [22] proves existence of Lagrangian solutions. The proof proceeds by truncating the external self consistent field from the Coulomb potential, and then uses a compactness result to obtain convergence. This thus demonstrates that even in this non-linear setting, the removal of long range effects can be used to gain a non-linear Markov evolution for the dynamics which can be utilised.

1.6 Structure of the Proof

As was mentioned, since the linear Boltzmann equation describes an evolution via its collisional structure, and this evolution is also Markovian, we aim to exploit both these properties. The long range particle dynamics are however not Markovian, and so we aim to identify a Markovian evolution that is close to the long range dynamics. It is the long range nature of these dynamics that is prohibitive to enabling a Markov evolution, and so, as in [4] and [22], we truncate the evolution by introducing a regularisation parameter $R > 0$ and a smooth approximation Λ^R to the indicator function $\mathbb{1}_{B_R(0)}$ such that $\mathrm{supp}(\Lambda^R) \subseteq B_R(0)$.

We then define a short range potential $\phi^R = \Lambda^R \phi$, and probability densities $f^{\varepsilon,R}$ and f^R associated to the short range particle dynamics and to the linear Boltzmann equation associated to the potential ϕ^R.

We remark at this stage that this cut off is of a very different flavour to the Grad cut off, as in [28], for the Boltzmann equation. The Grad cut off is an analytic tool where one truncates the integration over the plane $\mathcal{S}$ to ensure that the integral

$$\int_{\mathcal{S}} \mathrm{d}S < \infty.$$

For our cut off, we instead truncate the interaction, which as a consequence ensures that this integral is also finite. We then have the advantage that our truncated system does indeed describe physically valid particle evolutions, but we have the downside that the formulation of the operators is less concise and less explicit.

We then compare the densities using

$$|f^\varepsilon - f| \leq |f^\varepsilon - f^{\varepsilon,R}| + |f^{\varepsilon,R} - f^R| + |f^R - f|$$

and we desire estimates as $\varepsilon \to 0$ with the regularisation parameter $R \to \infty$. The estimates we use require $R = \varepsilon^{-1/(3+\gamma)}$ where we use the exponent in comparing f^ε and $f^{\varepsilon,R}$.

Once we have performed the cut-off, the density for evolution with short range potential is Markovian up to recollisions, and we are in a situation where it is possible to describe the particle evolutions in terms of their collisional structure. Instead of describing $f^{\varepsilon,R}$ as a density on the phase space $\mathcal{U}$, we enlarge this space and describe the particle evolution on a space of marked trees of height one, where each branch of the tree describes a different collision. This space then enables the specification of those trajectories for which there are no recollisions, and therefore a specification of those dynamics for which the particle evolution is Markovian.

On the space where the particle evolution is Markovian, we are then able in Chapter 3 to compare the density $f^{\varepsilon,R}$ to the solution f^R of the linear Boltzmann equation for ϕ^R by explicitly estimating the propagation of error between the densities in between collisions, and the error in the jump at each collision.

We are thus left to quantify the contribution of the grazing collisions to the particle dynamics and to solutions of the linear Boltzmann equation.

In comparing the long and short range particle dynamics in Chapter 4, we identify those realisations of the background particles for which the short range and long range evolutions have the same collisional structures. Interpreting the long range dynamics for the background described in the tree as a random variable, we are then able to directly compare the long and short range dynamics for a specific tree.

This description then requires the analysis of two estimates. Firstly, one is required to describe the measure of the scatterers for which the long range evolution does not encounter the same collisions as the short range evolution, subject to both having the same initial conditions of the tagged particle. Secondly, one is required to estimate the deviation of the short and long range evolutions for the same collisions. One should see the choice of decay of the potential as a choice so that these estimates decay to zero in the limit $\varepsilon \to 0$.

One finally has to show convergence of a solution f^R of the linear Boltzmann equation for ϕ^R to a solution f of the linear Boltzmann equation for ϕ, which we perform in Chapter 5. This proceeds along similar lines to [24] and [4]. We use a compactness argument, where we use (2) in Definition 1.5 to uniformly bound the solutions f^R in R, and then use estimates on the collision operator to conclude that the limit satisfies the linear Boltzmann equation for ϕ.

1.7 Technical Details of the Proof

Given the structure provided in the previous section, we now describe on a technical level how these ideas manifest themselves. The aim is to give the reader a description of how the technicalities fit together through the rest of the thesis. It also gives a description of how the chapters fit together.

We start by showing in Chapter 3 how one can address the non-Markovian nature of the particle dynamics by the consideration of the short range evolution. Using the potential ϕ^R, we introduce the particle density $f^{\varepsilon,R}$ as the density of the tagged particle from evolution via the equations

$$\begin{aligned} \dot{x}(t) &= v(t), & \dot{v}(t) &= -\frac{1}{\varepsilon}\sum_{i=1}^{N} \nabla\phi^R\left(\frac{x(t)-x_i(t)}{\varepsilon}\right) \\ \dot{x}_i(t) &= v_i(t), & \dot{v}_i(t) &= 0, \end{aligned} \tag{1.8}$$

and introduce f^R a weak solution of the linear Boltzmann equation associated to ϕ^R, meaning that

$$-\int_0^T\int_{\mathcal{U}} (\partial_t h + v\cdot\nabla_x h)\ f^R \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t - \int_{\mathcal{U}} f_0\, h(0)\,\mathrm{d}x\,\mathrm{d}v = \int_0^T \langle L^R(f^R),\, h\rangle\,\mathrm{d}t$$

where

$$\langle L^R(f^R),\, h\rangle := \int_{\mathcal{U}}\int_{\mathbb{R}^3}\int_{B_R(0)} (h'^{,R} - h)\, f^R(v)\, \mathcal{M}(v_\star)\, |v_\star - v| \,\mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}v\, \mathrm{d}x$$

for all test functions h, and where $v'^{,R}$ is the pre-collisional velocity for evolution under ϕ^R.

With evolution of equations (1.8) under ϕ^R, as shown in Chapter 2, for a subset of initial positions and velocities, the dynamics are in the form of a sequence of two body interactions. This then allows for the specification of the ingoing velocities and geometric parameters of each collision. These parameters, as well as the initial position and velocity of the tagged particle, are sufficient to specify the position and velocity of the tagged particle, and all background that collide with it, for all time.

As in [39], we describe these parameters in a marked tree structure. Each non root node corresponds to a collision, and the markers on these nodes are the velocity and geometric parameters of the collision. The tree is completed by specifying the initial position and velocity of the tagged particle. We then have associated to each tree Φ an evolution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ which corresponds to the particle dynamics under potential ϕ^R. We can then place a probability density on the space $\mathcal{MT}$ of marked trees corresponding to these dynamics. The evolution of the particle density on this space is then governed by an equation of the form

$$\partial_t f^{\varepsilon,R}(\Phi) = \mathcal{C}^{\varepsilon,R}(\Phi) f^{\varepsilon,R}(\overline{\Phi}) - \mathcal{D}^{\varepsilon,R}(\Phi) f^{\varepsilon,R}(\Phi)$$

where $\overline{\Phi}$ is the tree representing the dynamics without the final collision. The important property of this equation is that for a subset of dynamics where one removes recollisions, and also ensures that each interaction is binary, the coefficients $\mathcal{C}^{\varepsilon,R}$ and $\mathcal{D}^{\varepsilon,R}$ depend only weakly on the tree itself. Furthermore, these coefficients only depend upon the state of the system at time t.

We then are required to express the linear Boltzmann equation as an evolution on $\mathcal{MT}$. Interpreting the linear Boltzmann equation as a generator of a Lévy process, we can then create a collisional structure from this equation, and so use the collision operator to quantify the change of density according to the jumps in density from encountering a collision. Thus the evolution of the linear Boltzmann equation on $\mathcal{MT}$ takes a similar form to the evolution for particle dynamics, with a gain and loss part to the density, although with differing coefficients. We have for all trees the evolution equation

$$\partial_t f^R(\Phi) = \mathcal{C}^R(\Phi) f^R(\overline{\Phi}) - \mathcal{D}^R(\Phi) f^R(\Phi).$$

The forms of these equations then enable one to compare the densities in an estimate of the form

$$\int \left(f^R(\Phi) - f^{\varepsilon,R}(\Phi)\right) \mathrm{d}\Phi \le \int \rho_t^{\varepsilon,R}(\Phi)\,\mathrm{d}\Phi$$

where $\rho_t^{\varepsilon,R}$ is a function dependent on the differences $\mathcal{C}^{\varepsilon,R} - \mathcal{C}^R$ and $\mathcal{D}^{\varepsilon,R} - \mathcal{D}^R$. Furthermore,

$$\rho_t^{\varepsilon,R}(\Phi) \to 0$$

as $\varepsilon \to 0$. This is then enough to conclude that the particle density converges to the solution of the linear Boltzmann equation for ϕ^R. Proving the estimates of this form then concludes Chapter 3.

In comparing the particle densities f^ε and $f^{\varepsilon,R}$, we first introduce solutions on the space $[0,T] \times \mathcal{U}$ to the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^\varepsilon(t) = \bar{v}^\varepsilon(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\bar{v}^\varepsilon(t) = -\frac{1}{\varepsilon}\sum_{i=1}^{k} \nabla\phi\left(\frac{\bar{x}^\varepsilon(t) - x_i(t)}{\varepsilon}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = v_i(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}v_i(t) = 0,$$

and then one estimates the maximum deviation from these dynamics that the solution to equation (1.4) can have. We find that this is of the form

$$|x^\varepsilon - \bar{x}^\varepsilon| + |v^\varepsilon - \bar{v}^\varepsilon| \le C(T)e^{NR^{1+\gamma/2}} \left\|(1 - \Lambda^R)\nabla\phi\right\|_{L^\infty}$$

and we then interpret the term on the right hand side as giving a parameter that quantifies the size of the set of background particles one must remove to ensure that the random dynamics $(x^\varepsilon, v^\varepsilon)$ encounter the same collisions as the short range dynamics $(x^{\varepsilon,R}, v^{\varepsilon,R})$ for the same background interactions.

We then quantify the maximum error between the solution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ of the short range dynamics (1.8) and the long range particle dynamics $(x^\varepsilon, v^\varepsilon)$, where the latter is conditioned so that it encounters the same collisions as the former evolution. For regions where both are within $R\varepsilon$ of a background particle, we use the estimate on the scattering time in Lemma 2.8 together with a Gronwall estimate. On the region where both are outside $R\varepsilon$ of all background particles that interact with the short range dynamics, the maximum error between the two is a multiple of $\left\|(1 - \Lambda^R)\nabla\phi\right\|_{L^\infty}$. These two observations then result in

$$|x^\varepsilon - x^{\varepsilon,R}| + |v^\varepsilon - v^{\varepsilon,R}| \le e^{NR^{1+\gamma/2}} \left\|(1 - \Lambda^R)\nabla\phi\right\|_{L^\infty}.$$

This inequality then enables one to estimate the error between $f^{\varepsilon,R}$ and f^{ε} by quantifying the spread of the supports of each density for each evolution.

The remarkable property of these estimates and the parameters defining the good particle dynamics is that they remain the same in this situation, and have thus been chosen sufficiently robustly to allow for the random deviations in the particle positions and velocities.

The crucial estimate for a quantitative analysis of the grazing collisions on a solution of the linear Boltzmann equation is an estimate on the difference of the outgoing velocities of the tagged particle under evolutions with ϕ and ϕ^R. One shows, for potentials with decay as in (1.7), that

$$|v' - v'^{,R}| \leq \overline{\kappa}(R)\, |v_\star - v|$$

where $v'^{,R}$ is the outgoing velocity from the potential ϕ^R. Furthermore, the term $\overline{\kappa}$ is integrable over the plane $\mathcal{S}$ with integral decaying to 0 as $R \to \infty$. These properties are discussed in Chapter 2.

This property then allows us to estimate in Chapter 5 the difference of the collision operators as

$$\langle L(f) - L^R(f), h\rangle \leq C\, \left\|(1+|v|^2)\, f\right\|_{L^1}\, \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1} \int_{\mathcal{S}} \overline{\kappa}(R)\, \mathrm{d}S.$$

Using the maximum principle for solutions f^R, together with (2) in Definition 1.5, enables one to extract a convergent subsequence of $\{f^R\}$, and the estimate above on the difference between the collision operators then allows us to conclude that the limit of the convergent subsequence is the solution f of the linear Boltzmann equation associated to ϕ.

We conclude the thesis with a brief chapter combining the results of Chapters 3, 4 and 5 into a proof of Theorem 1, and then a discussion of potential extensions of the work.

Chapter 2

Analysis Of Scattering Maps

In the preceding chapter, we introduced the pre-collisional velocities v' and $v'_\star$ and the plane $\mathcal{S}$ that describes the parameters of a collision, and commented that their form is derived from the interaction potential ϕ chosen for the particle dynamics. We now specify this relationship for general potentials.

This specification of the dependence of the outcome of scattering on the interaction potential will then be used to compare the scattering maps for an admissible long range potential ϕ as in Definition 1.4 and for an associated short range potential ϕ^R as introduced in Section 1.6. The primary purpose of this is to demonstrate that the choice of cut off we make produces an evolution in a binary interaction that is close to the evolution of the long range potential. The estimates we provide in comparing these evolutions then give us a control on the impact of the cut off, which is used in Chapters 4 and 5.

This regularisation of the interaction can be seen to be a removal of the grazing collisions of the system. We define the notion of grazing collision in the following natural manner.

Definition 2.1. *A* **grazing collision** *is an interaction for which the minimum distance between the tagged particle x^ε and an interacting background x_s satisfies*

$$\frac{1}{\varepsilon} \min |x^\varepsilon - x_s| \geq R$$

and is a **near collision** *otherwise.*

One should observe that for short range potential ϕ^R, this definition agrees with the usual notion of grazing collision.

While this definition requires the specification of the parameter R, it should be seen that this is precisely to enable a discussion of the interactions under ϕ but for which there is no interaction with ϕ^R. The remainder of the chapter then uses these two potentials to analyse the impact of grazing collisions on the particle dynamics.

The chapter concludes by proving existence of solutions to the linear Boltzmann equation for ϕ^R, which is necessary for the definition of the linear Boltzmann equation on the space of marked trees, and for proving existence of solutions to the linear Boltzmann equation for ϕ.

2.1 Two Body Collisions

The first aim is to describe the scattering map $(r, \zeta, v, v_\star) \mapsto (v', v'_\star)$ for an interaction potential ϕ. We here make no assumption on the support of ϕ and so this specification will hold for both ϕ and for ϕ^R. This consists in detailing the effect of the force on the two particles. The system one analyses is where one specifies that the velocities v and $v_\star$ are the asymptotic velocities of the two particles when $t \to -\infty$, and $v', v'_\star$ are the velocities as $t \to +\infty$. This thus enables one to consider a binary collision where the force is supported in $\mathbb{R}^3$. For potentials with compact support, outside of the support, the velocities are equal to the asymptotic velocities.

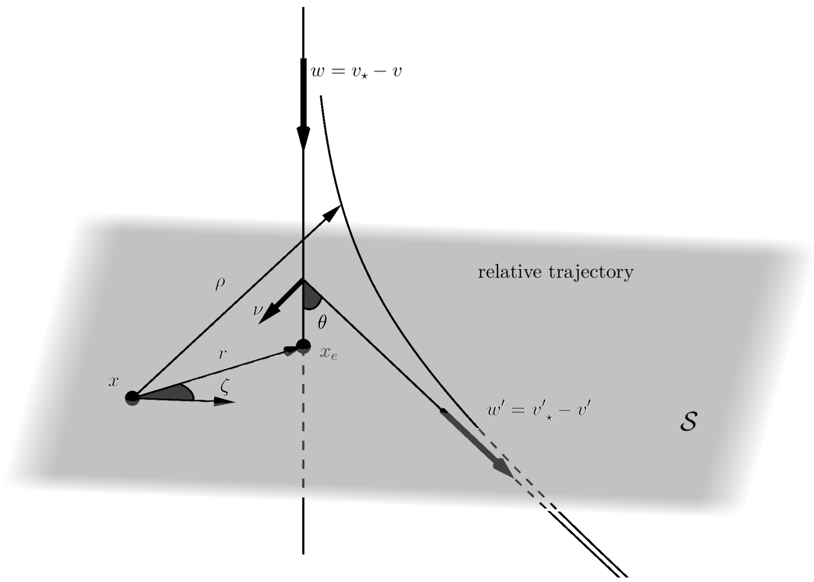


Figure 2.1: Description of parameters in a binary interaction

This situation can be simplified as follows, and is shown in Figure 2.1. Fix one particle at point $x \in \mathbb{R}^3$. Suppose the second particle has a trajectory relative to the point x. This trajectory then has asymptotic velocities $w = v_\star - v$ as $t \to -\infty$ and $w' = v'_\star - v'$ as $t \to \infty$. To describe the interaction one then defines the plane

$\mathcal{S}$ through x perpendicular to w, which is

$$\mathcal{S} := \{z \in \mathbb{R}^3 \colon z \cdot (v_\star - v) = x \cdot (v_\star - v)\}.$$

In this plane, one places the point x_e at the point through which the second particle would intersect $\mathcal{S}$ without the presence of the interaction potential. This point is specified by polar coordinates (r, ζ) in $\mathcal{S}$ centred at x. The distance r then corresponds to the minimum distance between the two particles without interaction, and ζ specifies the direction between them.

Remark 2.2. *We here consider $v, v_\star$ as the pre-collisional velocities, whereas in the specification of the linear Boltzmann equation they were described as post collisional. While this is an abuse of notation, the time reversibility of the binary system makes these equivalent.*

The radial symmetry of the interaction potential further means the description of the collision does not depend upon ζ, and so we no longer consider it, though full details can be found in [49, Ch.6].

To specify the relationship between the pre and post collisional velocities, we use the conservations of momentum and energy, which give that

$$\begin{aligned} v + v_\star &= v' + v'_\star \\ |v|^2 + |v_\star|^2 &= |v'|^2 + |v'_\star|^2 \end{aligned}$$

and are the same for all potentials. These are a system of 4 constraints for the 6 variables, and so one would expect the solution to have 2 free variables which depend upon the choice of interaction. This is indeed the case, and one can write the outgoing velocities $v', v'_\star$ in terms of the ingoing velocities as

$$\begin{aligned} v' &= v + ((v_\star - v) \cdot \nu(r, \zeta, v_\star - v))\, \nu(r, \zeta, v_\star - v), \\ v'_\star &= v_\star - ((v_\star - v) \cdot \nu(r, \zeta, v_\star - v))\, \nu(r, \zeta, v_\star - v). \end{aligned} \tag{2.1}$$

where the vector $\nu(r, \zeta) \in \mathbb{S}^2$ depends upon the potential. We define the map $\sigma \colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S} \to \mathbb{R}^3 \times \mathbb{R}^3$ by $\sigma(v, v_\star, r, \zeta) = (v', v'_\star)$ as the scattering map.

The projection of ν onto the plane $\mathcal{S}$ is given by

$$\nu \cdot (v_\star - v) = |v_\star - v| \sin\left(\frac{1}{2}\theta(r, v_\star - v)\right)$$

where θ is called the **deviation angle**, and is the angle between w' and w. It is given by the formula

$$\theta(r,w) = \pi - 2\int_{\rho_\star}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}}, \tag{2.2}$$

with $\rho_\star$ the largest root of the denominator. For a physical interpretation of θ, see Figure 2.1.

Remark 2.3. *An alternative and often used relationship are the equations*

$$\begin{aligned} v' &= \frac{v+v_\star}{2} + \tilde{\nu}(r,\zeta)\frac{|v-v_\star|}{2}, \\ v'_\star &= \frac{v+v_\star}{2} - \tilde{\nu}(r,\zeta)\frac{|v-v_\star|}{2}, \end{aligned}$$

and these differ from the choice we make by

$$\tilde{\nu}\cdot(v_\star - v) = \cos\theta\,|v_\star - v|$$

for the same deviation angle θ.

While we take this equation as the definition of θ, one can derive this form of the deviation angle from the equations of rotational momentum and energy in polar coordinates, as in [49, Ch.6]. We observe that this definition of the scattering angle requires minimal assumptions on the interaction potential. In particular, it is well defined for admissible long range potentials.

Remark 2.4. *Many authors take $\frac{1}{2}(\pi-\theta)$ as the deviation angle. We chose this angle θ because it results in $\theta(r,w)\to 0$ as $r\to\infty$, which is a property suggestive of the name.*

We first check some basic properties of θ. Firstly, the formula should be consistent with no interaction potential, namely we should have $\theta = 0$ for $\psi \equiv 0$. This is clearly the case since one has

$$\theta = \pi - 2\int_{\rho_\star}^{\infty}\frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}} = \pi - 2\int_{r}^{\infty}\frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}} = \pi - 2\frac{\pi}{2}.$$

Furthermore, if $r = 0$ then θ should be equal to π, and this is obvious from the formula. We finally desire $\theta\to 0$ as $r\to\infty$. This is dealt with in the next section, but is indeed the case. This suggests the formula satisfies the expected properties.

We now restrict our attention to two specific types of potential. We analyse the scattering for a potential ϕ which is an admissible long range potential as in

Definition 1.4, and for a related short range potential ϕ^R which is defined by

$$\phi^R = \Lambda^R \phi \tag{2.3}$$

for fixed $R > 0$ and with $\Lambda^R \in C^\infty(\mathbb{R}^3)$ a radial strictly decreasing function with

$$\Lambda^R(x) = \begin{cases} 1 & |x| \le R-1 \\ 0 & |x| \ge R. \end{cases}$$

For the potential ϕ^R, we add a superscript R to the deviation angle, as well as to the pre-collisional velocities obtained from this in equation (2.1), and so we write $v^{\prime,R}$ and $v_\star^{\prime,R}$, and call the scattering map in this case σ^R. We assume a radial cut off so that the potential ϕ^R is also radial, and we assume $\Lambda^R = 1$ in B_{R-1} to ensure that ϕ^R and ϕ are the same for most of the short range interaction. Finally we assume the cut off is decreasing to ensure that ϕ^R is monotonic on B_R.

2.2 Estimates on the Deviation Angle

We now aim to provide estimates on the deviation angles θ and θ^R which describe how the cut off on the interaction potential impacts the binary collision.

The first estimate one desires can be thought of in two ways. Firstly, as considered above, from physical intuition, one would expect that $\theta \to 0$ as $r \to \infty$, and so the estimate is the final test of physical validity of the formula (2.2). The estimate however we first give is more than just this. For grazing collisions, where the impact parameter $r > R$, this can be considered as providing an estimate on the difference between θ^R and θ, since in the range $r > R$ one has $\theta^R = 0$.

Once we have compared in this manner the scattering angles for grazing collisions, we are left to compare the angles for impact parameter $r < R$. This is the purpose of the second of the two estimates in this section.

We start with the estimate providing consistency with physical intuition.

Lemma 2.5. *Suppose that ϕ is an admissible long range potential such that there is a $\rho_2 > 0$ and $s > 2$ as well as $C > 0$ such that, for $\rho > \rho_2$ the relation,*

$$\psi(\rho) \le C\rho^{-s}$$

holds. Then, for all $r \in [0, \infty)$ and $w \in \mathbb{R}^3 \setminus \{0\}$ we have

$$\theta(r, w) \le \frac{C}{1 + |w|^2 r^s},$$

and for $r \in [0, R]$ we have, for a different constant

$$\theta^R(r, w) \leq \frac{C}{1 + |w|^2 r^s}.$$

Proof: We proceed as in the appendix to [24]. We have

$$\theta(r, |w|) = \pi - 2 \int_{\rho_\star}^{\infty} \frac{r \, \mathrm{d}\rho}{\rho^2 \sqrt{1 - \frac{2}{|w|^2}\psi(\rho) - \frac{r^2}{\rho^2}}}$$

and using the change of coordinates $u = r/\rho$ we obtain

$$\theta(r, |w|) = \pi - 2 \int_{0}^{r/\rho_\star} \frac{\mathrm{d}u}{\sqrt{1 - \frac{2}{|w|^2}\psi(\frac{r}{u}) - u^2}}$$

and then by letting

$$u^2 + \frac{2}{|w|^2}\psi\left(\frac{r}{u}\right) = \sin^2 \mu$$

we have

$$2\left(u + \frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}u}\psi\left(\frac{r}{u}\right)\right) \mathrm{d}u = 2 \sin\mu \cos\mu \, \mathrm{d}\mu.$$

This then enables us to obtain, with the change of coordinates given by the above, the equality

$$\begin{aligned}
\theta(r, |w|) &= \pi - 2 \int_0^{\pi/2} \frac{1}{u + \frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}u}\psi(\frac{r}{u})} \frac{\sin\mu\cos\mu \, \mathrm{d}\mu}{\sqrt{1 - \sin^2\mu}} \\
&= 2 \int_0^{\pi/2} \left(1 - \frac{\sin\mu}{u + \frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}u}\psi(\frac{r}{u})}\right) \mathrm{d}\mu \\
&= 2 \int_0^{\pi/2} \frac{u + \frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}u}\psi(\frac{r}{u}) - \sin\mu}{u + \frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}u}\psi(\frac{r}{u})} \mathrm{d}\mu.
\end{aligned}$$

Since $u \leq \sin\mu$ we have

$$\theta(r, |w|) \leq 2 \int_0^{\pi/2} \frac{1}{1 + \frac{u}{\frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}u}\psi(\frac{r}{u})}} \mathrm{d}\mu = 2 \int_0^{\pi/2} \frac{1}{1 - \frac{u^3}{\frac{1}{|w|^2} r \, \psi'(\frac{r}{u})}} \mathrm{d}\mu$$

by evaluating the derivative of the potential. The strictly decreasing nature of ψ, (condition (3) of Definition 1.4) ensures that the derivative of ψ is negative, and then combining this with the decay assumption on ψ in the statement of the lemma then implies that $-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) \le s\rho^{-s-1}$ and so

$$\frac{1}{1 - \frac{u^3}{r\,\frac{1}{|w|^2}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(r/u)}} \le \frac{1}{1 + \frac{u^3}{r\,\frac{1}{|w|^2}\,s\,(r/u)^{-s-1}}} = \frac{1}{1 + \frac{|w|^2\,r^s}{s\,u^{s-2}}}.$$

Inputting this into the inequality for θ one observes that

$$2\int_0^{\pi/2} \frac{1}{1 - \frac{u^3}{r\,\frac{1}{|w|^2}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(r/u)}}\,\mathrm{d}\mu \le 2\int_0^{\pi/2} \frac{1}{1 + \frac{|w|^2\,r^s}{s\,u^{s-2}}}\,\mathrm{d}\mu \le \frac{C}{1+|w|^2\,r^s}$$

as is required. Similar arguments show the form for θ^R. □

We now compare the deviation angles θ^R and θ for $r < R$. The detailed analysis of these deviation angles here demonstrates that the regularisation of the potential in (2.3) enables the comparison of the binary dynamics, and allows us to obtain relevant estimates for use later in Chapter 5.

Lemma 2.6. *Suppose that ϕ is an admissible long range potential with the condition that there is a $\rho_2 > 0$ and $s > 2$ such that for $\rho > \rho_2$ we have*

$$\psi(\rho) \le C\rho^{-s},$$

for some constant $C > 0$, and suppose that the relative velocity $|w| \ge \eta$ for some $\eta > 0$. Then for R such that $R - 1/\eta^2 > 1 + \rho_2$, we have

$$|\theta(r,w) - \theta^R(r,w)| \le \begin{cases} \frac{C}{1+\eta^2\,r^s} & \text{for } r > R - 1 - 1/\eta \\ \\ \frac{C\,\kappa(r,R)}{\eta^2} & \text{for } r < R - 1 - 1/\eta, \end{cases} \tag{2.4}$$

where the constants are independent of r and $|w|$ and

$$\int_0^{R-1-1/\eta} r\,\frac{\kappa(r,R)}{\eta^2}\,\mathrm{d}r \to 0.$$

Remark 2.7. *The useful property of the estimate in* (2.4) *is that the integral of the right hand side tending to zero implies that potentials with decay faster than ρ^{-2} have $\theta^R - \theta \to 0$ as $R \to \infty$, as well as it enables the respective collision operators to approximate each other, as will be seen in Chapter 5.*

The proof is simple, and involves rewriting the terms onto a common denominator, before using a property of the interaction potential to gain an upper bound of the required form. It is however rather unwieldy to write down.

Proof: Firstly, for $r > \rho_2$, proceeding as in the proof of the previous lemma, we obtain the estimate

$$\theta(r,w) \leq \frac{C}{1+\eta^2\, r^s}$$

and since $\theta^R \geq 0$, these two facts result in $\theta - \theta^R \leq \theta$, and so

$$\theta(r,w) - \theta^R(r,w) \leq \frac{C}{1+\eta^2\, r^s}$$

for $r > \rho_2$. The choice of R then ensures that we have $R-1-1/\eta > \rho_2$ and so this gives the estimate for $r > R-1-1/\eta$.

For $r < R-1-1/\eta$, we first show that $\rho_\star^R < R-1$. We have, by the decay on ψ^R, that

$$1 - \frac{\psi^R(\rho_\star^R)}{|w|^2} \geq 1 - \frac{1}{(\rho_\star^R)^2|w|^2}$$

and therefore, by using this in the equation for $\rho_\star^R$ we obtain

$$1 - \frac{1}{(\rho_\star^R)^2|w|^2} \leq \frac{r^2}{(\rho_\star^R)^2}$$

and by rearranging, we then obtain

$$(\rho_\star^R)^2 \leq r^2 + \frac{1}{|w|^2} = \left(r + \frac{1}{|w|}\right)^2 - \frac{2\,r}{|w|} \leq \left(r + \frac{1}{|w|}\right)^2$$

and then using the fact that $r < R-1-1/\eta$ we obtain

$$(\rho_\star^R)^2 \leq \left(R - 1 - \frac{1}{\eta} + \frac{1}{|w|}\right)^2 \leq (R-1)^2$$

since $\frac{1}{|w|} \leq \frac{1}{\eta}$. Furthermore, we observe that this means that $\rho_\star^R = \rho_\star$. Therefore, from equation (2.2), in analysing the difference

$$\theta^R(r,w) - \theta(r,w) = \int_{\rho_\star^R}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1 - \frac{2\psi(\rho)}{|w|^2} - \frac{r^2}{\rho^2}}} - \int_{\rho_\star}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1 - \frac{2\psi^R(\rho)}{|w|^2} - \frac{r^2}{\rho^2}}}$$

we observe that the integrands in both integrals for $\rho < R-1$ are the same.

We then split the difference $\theta - \theta^R$ into a difference corresponding to the long range nature of ϕ, and an error which comes from the choice of the smooth cut off

Λ^R. We have

$$\theta^R(r,w) - \theta(r,w) = \int_{\rho_\star}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}} - \int_{\rho_\star}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{2\psi^R(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}}$$

$$= \int_{R}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}} - \int_{R}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}}$$

$$+ \int_{R-1}^{R} \left(\frac{r}{\rho^2\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}} - \frac{r}{\rho^2\sqrt{1-\frac{2\psi^R(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}} \right) \mathrm{d}\rho$$

We consider these terms separately. For the long range term, by combining the terms into a single fraction, we obtain

$$\int_{R}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}} - \int_{R}^{\infty} \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}}$$

$$= \int_{R}^{\infty} \frac{r\frac{2\psi(\rho)}{|w|^2}\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}\left(\sqrt{1-\frac{r^2}{\rho^2}}+\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}\right)}.$$

We then observe that for $\rho > R$ we have $\psi(\rho) \leq \psi(R)$ and so

$$\frac{1}{\sqrt{1-\frac{\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}} \leq \frac{1}{\sqrt{1-\frac{\psi(R)}{|w|^2}-\frac{r^2}{R^2}}}$$

and furthermore that, for $r < R$ we have, for some constant,

$$\sqrt{1-\frac{2\psi(R)}{|w|^2}-\frac{r^2}{R^2}} \geq C\sqrt{1-\frac{r^2}{R^2}}$$

and so we obtain

$$\int_R^\infty \frac{r\,\frac{2\psi(\rho)}{|w|^2}\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}\left(\sqrt{1-\frac{r^2}{\rho^2}}+\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}\right)}$$

$$\leq \int_R^\infty \frac{r\,\frac{2\psi(\rho)}{|w|^2}\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}\sqrt{1-\frac{2\psi(R)}{|w|^2}-\frac{r^2}{R^2}}\left(\sqrt{1-\frac{r^2}{R^2}}+\sqrt{1-\frac{2\psi(R)}{|w|^2}-\frac{r^2}{R^2}}\right)}$$

$$\leq \frac{C}{1-\frac{r^2}{R^2}}\frac{\sup\limits_{\rho\in(R,\infty)} 2\psi(\rho)}{|w|^2}\int_R^\infty \frac{r\,\mathrm{d}\rho}{\rho^2\sqrt{1-\frac{r^2}{\rho^2}}}$$

$$\leq \frac{C}{1-\frac{r^2}{R^2}}\frac{\left\|(1-\Lambda^R)\,\phi\right\|_{L^\infty}}{|w|^2}\arcsin\left(\frac{r}{R}\right)$$

One then observes that

$$\int_0^{R-1-1/\eta}\frac{C\,r\,R^{-s}}{1-\frac{r^2}{R^2}}\arcsin\left(\frac{r}{R}\right)\,\mathrm{d}r<\infty$$

as required.

For the cut off error term, by rearranging and bounding terms similarly to before, we obtain

$$\int_{R-1}^R\left(\frac{r}{\rho^2\sqrt{1-\frac{2\psi(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}}-\frac{r}{\rho^2\sqrt{1-\frac{2\psi^R(\rho)}{|w|^2}-\frac{r^2}{\rho^2}}}\right)\,\mathrm{d}\rho$$

$$\leq \frac{C\,r\,\left\|(1-\Lambda^R)\,\phi\right\|_{L^\infty}}{|w|^2\,R\,(R-1)\,\left(1-\frac{r^2}{(R-1)^2}\right)^{3/2}}.$$

The term on the right hand side also has the required form. □

We observe that we take

$$\kappa(r,R)=R^{-s}\left(\frac{1}{1-\frac{r^2}{R^2}}+\frac{1}{R\,(R-1)\,\left(1-\frac{r^2}{(R-1)^2}\right)}\right)$$

in the previous lemma. We use this form later.

2.2.1 Scattering Time

The main aim for this section is to estimate the difference in position and velocity between long range and short range evolutions for a binary interaction. This proceeds by providing an estimate on the scattering time, the length of time for which a collision occurs, before using this on the solutions of the particle evolutions.

We first turn to analysing dynamics solely under the potential ϕ^R. Unlike evolution under ϕ, the scattering for two particles under ϕ^R takes a finite time, with time given by

$$\tau_\star(r, w, R) = 2\int_{\rho_\star}^{R} \frac{\mathrm{d}\rho}{|w|\sqrt{1 - 2\frac{\psi^R(\rho)}{|w|^2} - \frac{r^2}{\rho^2}}},$$

which we call the scattering time. This can be derived similarly to the equation for θ from the conservations of angular momentum and energy. The first observation to make with this formula is that it is undefined for a long range potential ϕ as it is always ∞, and so is consistent with our comment that one requires short range interactions to have localised collisions.

The use of this formula is that, for potentials with $\nabla\phi \neq 0$, one has that $\tau_\star$ is bounded on compact sets of $[0, R] \times \mathbb{R}^3$, as shown in [26, Prop 8.2.1]. Without this condition, one has situations where the particles move together at a fixed distance apart. Furthermore, one can specify an upper bound for $\tau_\star$ on these sets in the form given in the following lemma.

Lemma 2.8. *Let $\eta > 0$. The time of collision for evolution under ϕ^R can be bounded, for $r \in [0, R)$ and $w \in \mathbb{R}^3 \setminus B_\eta(0)$, by*

$$\tau_\star(r, w, R) \leq C\,\frac{R}{\eta}.$$

Remark 2.9. *The form of the estimate in this lemma should not be surprising. Indeed with the absence of the potential, the velocity is then constant, and so this inequality is sharp with $C = 1$.*

The remarkable property here is that this form is unchanged, up to a constant, when one includes the interaction potential.

The proof of the lemma is somewhat long. This is mainly because it uses three methods, two of which have been used before in [47] and [4], and the third of which is new.

Proof: In order to gain a suitable bound on the time of collision, by setting

$$i_0 = \frac{1}{2\sqrt{2}}\psi^{-1}(|w|^2/4),$$

we split collisions into situations where the impact parameter is in the three regions $\left[\frac{R}{2}, R\right]$, $\left[i_0, \frac{R}{2}\right]$ and in $[0, i_0]$.

By a simple extension of [47, Lemma 1] to potentials supported in $B_R(0)$, we can bound the time of collision in the desired manner for $r \in \left[\frac{R}{2}, R\right]$. This is performed as follows.

We rewrite

$$\tau_\star = \sqrt{2} \int_{\rho_\star}^{R} \frac{1}{\left(\frac{|w|^2}{2} - \frac{|w|^2 r^2}{2\rho^2} - \psi^R(\rho)\right)^{1/2}} \, \mathrm{d}\rho,$$

and we observe that this is evolution under an effective potential of

$$\psi_e^R(\rho) = \frac{|w|^2 r^2}{2\,\rho^2} + \psi^R(\rho) - \frac{|w|^2 r^2}{2}.$$

Therefore

$$\tau_\star = \sqrt{2} \int_{\rho_\star}^{R} \frac{1}{\left(\psi_e^R(\rho_\star) - \psi_e^R(\rho)\right)^{1/2}} \, \mathrm{d}\rho \leq \frac{1}{\sqrt{\min_{(0,R)}\left(-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi_e^R \rho\right)}} \int_{\rho_\star}^{R} \frac{\mathrm{d}\rho}{\sqrt{\rho - \rho_\star}}.$$

The integral in the above is given exactly by $2\sqrt{R - \rho_\star}$, and

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\psi_e^R(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\rho) - \frac{r^2|w|^2}{\rho^3}$$

and so

$$\min_{(0,R)}\left(-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi_e^R(\rho)\right) = \min_{(0,R)}\left(-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\rho) + \frac{r^2|w|^2}{\rho^3}\right) \geq \frac{r^2|w|^2}{R^3}$$

since ψ^R is decreasing for ρ increasing thus $\frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\rho)$ is itself negative. Combining these facts gives

$$\tau_\star \leq \frac{1}{\sqrt{\min_{(0,R)}\left(-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi_e^R(\rho)\right)}} \int_{\rho_\star}^{R} \frac{\mathrm{d}\rho}{\sqrt{\rho - \rho_\star}} \leq \sqrt{\frac{R^3}{|w|^2\, r^2}}\, 2\sqrt{R - \rho_\star} \leq \frac{2R^2}{r|w|}$$

and since $r > R/2$ we obtain

$$\tau_\star \leq \frac{4R^2}{R|w|} \leq \frac{CR}{\eta}$$

as required.

Furthermore, the conditions required for admissible long range potentials ensures that one can proceed as in the proof of [4, Prop. 2] in the interval $[0, i_0)$ as follows. We split

$$\tau_\star = \tau_\star^1 + \tau_\star^2$$

for

$$\tau_\star^1 = 2\int_{\rho_\star}^{\gamma} \frac{\mathrm{d}\rho}{\sqrt{|w|^2 - \frac{|w|^2 r^2}{\rho^2} - 2\psi^R(\rho)}}$$

for $\gamma = \psi^{-1}(|w|^2/8)$.

Firstly, since $\psi^R(\rho_\star) = \frac{|w|^2}{2}\left(1 - \frac{r^2}{\rho^2}\right) \leq \frac{|w|^2}{2}$ we have $\rho_\star \geq \psi^{-1}\left(|w|^2/2\right)$. Thus using the bounds on r and $\rho_\star$ we obtain

$$\frac{|w|^2\, r^2}{2\,\rho_\star^2} \leq \frac{|w|^2\, i_0^2}{2\,\rho_\star^2} \leq \frac{|w|^2\, i_0^2}{16\, i_0^2} = \frac{|w|^2}{16}$$

and so

$$\psi^R(\rho_\star) = \frac{|w|^2}{2} - \frac{|w|^2\, r^2}{2\,\rho_\star^2} \geq \frac{4\,|w|^2}{16} - \frac{|w|^2}{16} = \frac{3\,|w|^2}{16} \geq \frac{|w|^2}{8}$$

meaning

$$\rho_\star \leq \psi^{-1}\left(\frac{|w|^2}{8}\right) = \gamma$$

and so the integral $\tau_\star^1$ is well defined.

Setting $y = |w|^2 - \frac{|w|^2\, r^2}{\rho^2} - 2\,\psi^R(\rho)$ we observe that

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}\rho} &= \frac{2\,|w|^2\, r^2}{\rho^3} - 2\,\frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\rho) \\
&\geq -2\,\frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\rho) \\
&\geq 2 \inf_{i_0<\rho<\gamma} \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\rho)\right|
\end{aligned}$$

and since ψ^R is non-increasing,

$$\inf_{i_0<\rho<\gamma} \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\gamma)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\phi^{-1}(|w|^2/8))\right|.$$

Then assumption (5) in Definition 1.4 together with the assumption on the sign of the derivative of Λ^R ensures that

$$\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\psi^R(\psi^{-1}(|w|^2/8))\right| \geq |w|^2/8,$$

and therefore

$$\frac{\mathrm{d}y}{\mathrm{d}\rho} \geq \frac{|w|^2}{4}.$$

Inputting this change of coordinates into the equation for $\tau_\star^1$ results in

$$\begin{aligned}\tau_\star^1 &\leq \frac{2}{|w|^2/4}\int_0^{|w|^2-|w|^2\,r^2/\gamma^2} \frac{\mathrm{d}y}{\sqrt{y}} \\ &= \frac{\sqrt{|w|^2-|w|^2\,r^2/\gamma^2}}{|w|^2/4} \\ &\leq \frac{4}{|w|}.\end{aligned}$$

For $\tau_\star^2$, the integrand is bounded by

$$\frac{1}{\sqrt{|w|^2-\frac{|w|^2\,r^2}{\rho^2}-2\psi^R(\rho)}} \leq \frac{1}{\sqrt{|w|^2-\frac{|w|^2\,r^2}{\gamma^2}}}$$

and so

$$\tau_\star^2 \leq \frac{2\,R}{\sqrt{|w|^2-\frac{|w|^2\,r^2}{\gamma^2}}}$$

and since

$$\frac{|w|^2\,r^2}{\gamma^2} \leq \frac{|w|^2\,i_0^2}{\gamma^2} = \frac{|w|^2}{8}\left(\frac{\psi^{-1}\left(\frac{|w|^2}{4}\right)}{\psi^{-1}\left(\frac{|w|^2}{8}\right)}\right)^2 \leq \frac{|w|^2}{8}$$

we have

$$\tau_\star^2 \leq \frac{4\sqrt{2}\,R}{\sqrt{7}|w|}$$

as required.

We are thus left to analyse for $r \in \left[\frac{1}{2\sqrt{2}}\psi^{-1}\left(\frac{|w|^2}{4}\right), \frac{R}{2}\right]$. We have, for any parameter r in this region, that

$$\tau_\star \leq \frac{\text{max distance}}{\text{min velocity}}$$

where the terms on the right hand side are the maximum and minimum for all r in the desired region. Clearly the maximum distance is at most $2R$, and so we are left to calculate the minimum velocity for a particle in this region.

The point at which the particle has lowest speed is the point at which it has maximal potential energy, which is the closest point, namely $\rho_\star$. Then, letting the minimum velocity be $w_\star$, we have from conservation of energy

$$\frac{1}{2}(|w|^2 - |w_\star|^2) = \psi^R(\rho_\star) - \psi^R(R) = \psi^R(\rho_\star)$$

since $\psi^R(R) = 0$ by assumption. Rearranging, we obtain

$$|w_\star| = \sqrt{|w|^2 - 2\psi^R(\rho_\star)},$$

and since

$$\psi^R(\rho_\star) = |w|^2 \left(1 - \frac{r^2}{\rho_\star^2}\right),$$

this results in

$$|w_\star| = \sqrt{|w|^2 - 2\psi^R(\rho_\star)} = |w|\sqrt{1 - 1 + \frac{r^2}{\rho_\star^2}} = |w|\frac{r}{\rho_\star}.$$

We conclude by finding a lower bound on $r/\rho_\star$. We split this interval into two. On the interval $[\psi^{-1}\left(\frac{|w|^2}{4}\right), \frac{R}{2}]$, we have $r > \psi^{-1}\left(\frac{|w|^2}{4}\right)$, and so

$$\psi^R(\rho_\star) < \psi^R\left(\psi^{-1}\left(\frac{|w|^2}{4}\right)\right) = \frac{|w|^2}{4},$$

and plugging this into the equation for $\rho_\star$ we obtain

$$\frac{r}{\rho_\star} = 1 - \frac{2\psi^R(\rho_\star)}{|w|^2} \geq 1 - \frac{2|w|^2}{4|w|^2} = \frac{1}{2}.$$

The monotonicity of ψ ensures that, on the interval $\left[i_0, \psi^{-1}\left(\frac{|w|^2}{4}\right)\right]$, the minimum radius for impact parameter r is smaller than the minimum radius for $\psi^{-1}\left(\frac{|w|^2}{4}\right)$, and so

$$\rho_\star \leq 2\psi^{-1}\left(\frac{|w|^2}{4}\right).$$

Since $r > \frac{1}{2\sqrt{2}}\psi^{-1}\left(\frac{|w|^2}{4}\right)$ we have $\rho_\star \leq 4\sqrt{2}r$ as required. □

The use of this estimate is that it enables a simple estimate on the outgoing positions and velocities of particles interacting via ϕ^R with different ingoing positions and velocities.

Lemma 2.10. *Suppose one has two different initial conditions y_1^0, z_1^0 and y_2^0, z_2^0 for* *the equations*

$$\begin{cases} \dot{y} = z \\ \dot{z} = -\frac{1}{\varepsilon}\nabla\phi^R\left(\frac{y}{\varepsilon}\right). \end{cases}$$

Furthermore suppose that $|z_1^0|, |z_2^0| \geq \eta$. *Then the respective positions after scattering satisfy*

$$|y_1 - y_2| + |z_1 - z_2| \leq C\frac{e^{CR/\eta}}{\varepsilon}\left(|y_1^0 - y_2^0| + |z_1^0 - z_2^0|\right)$$

where C *depends upon* $\nabla\phi$.

The proof of this lemma is found in [4], and is simply a combination of Gronwall's lemma, the previous bound on the time of collision, and a rescaling of space via $y \mapsto x/\varepsilon$. We include for completeness.

Proof: We first analyse the dynamics with $\varepsilon = 1$. The equations of motion are then

$$\begin{cases} \dot{y} = z \\ \dot{z} = -\frac{1}{\varepsilon}\nabla\phi^R\left(\frac{y}{\varepsilon}\right). \end{cases}$$

and we note that the map from

$$\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ -\nabla\phi(y) \end{pmatrix}$$

is Lipschitz with Lipschitz constant given by $\max\{1, C_{\nabla\phi}\}$ where $C_{\nabla\phi}$ is the Lipschitz constant for $\nabla\phi$. Then by Gronwall's lemma we obtain

$$|y_1(t) - y_2(t)| + |z_1(t) - z_2(t)| \leq e^{Ct}\left(|y_1(0) - y_2(0)| + |z_1(0) - z_2(0)|\right).$$

Applying the transformation $y \mapsto y/\varepsilon$ and $z \mapsto z$ results in

$$\frac{1}{\varepsilon}|y_1(t) - y_2(t)| + |z_1(t) - z_2(t)| \leq e^{Ct}\left(\frac{1}{\varepsilon}|y_1^0 - y_2^0| + |z_1^0 - z_2^0|\right)$$

and using the fact that $1 < 1/\varepsilon$ together with the estimate on the scattering time $\tau_\star$ results in the desired inequality. □

2.3 Many Body Dynamics

The study so far has been solely regarding binary collisions and the errors between the outgoing velocities for differing interactions. For a general particle system however, the dynamics consist of many interacting particles, and we now briefly focus our attention on this. We aim to estimate the difference between the positions and velocities for the tagged particle where interactions are given by a long range and short range potential. This is given here to add context, and so that one can see much more readily the convenience of the use of marked trees in the next chapter for describing the particle dynamics. In particular, the estimates calculated here are similar in nature to those in Chapter 4.

For a particle system with long range interactions, when considering many background particles, one should note that the system cannot be considered as a sequence of binary collisions, since the particles are always interacting. However, with a short range potential, for a subset of initial conditions, one does have this structure, namely that for each time, there is at most one background particle colliding with the tagged particle.

Indeed, the estimate in Lemma 2.8 on the scattering time implies that there is a compact interval of time for which the tagged particle and a background particle are interacting. Therefore the set of dynamics with only binary interactions has strictly positive probability. One can furthermore quantify this probability,

For these systems, we can use the estimates on binary collisions to infer an estimate on the difference of positions and velocities under long and short range evolutions. This consideration enables a tractable comparison between these two dynamics. However, to provide a detailed estimate, one furthermore is required to know the number of collisions in each evolution. This is unknown due to the existence of recollisions in the system.

It should be clear however that there are no straightforward conditions one can impose on the background to ensure that the evolution is recollision free, and certainly no a priori conditions. Desvillettes and Pulvirenti [24] specify conditions for fixed background to remove recollisions, and these are that

$$\min_{i=1,\dots,N} \min_{j=i+2,\dots,N} \inf_{t_j+\tau_\star^j \le s \le t_{j+1}} |x(s) - x_i| \ge R\varepsilon.$$

In the case of moving background, the conditions are much less straightforward. This comment should be seen as a first brief motivation for the use of marked trees, as the constraint (see Definition 3.3 point (6)) used on these trees to remove recollisions is simple in contrast.

For these dynamics where we have solely binary collisions, We use this structure to compare the evolution under ϕ^R to the evolution under ϕ. The main reason for this is the following. In comparing solutions of the particle dynamics for various potentials, it is necessary to control both the position of the tagged particle and the velocity of the tagged particle, for the same background particles. The estimates on the deviation angle in Lemma 2.6 can in theory produce this, but in practise this would produce an incredibly messy analysis. The analysis we provide here is much simpler, and is based on Gronwall's lemma for solutions of differential equations. This does however mean that the estimate is much coarser, since it ignores almost all structure of the physics of each interaction.

Lemma 2.11. *Suppose that one has a collection $x_1, \ldots, x_N \in \mathbb{R}^3$ of scatterers such that for all $i \neq j$, we have $B_{R\varepsilon}(x_i) \cap B_{R\varepsilon}(x_j) = \emptyset$. Furthermore, suppose that the tagged particle encounters exactly M collisions with these background particles. Then let $(x^\varepsilon, v^\varepsilon)$ solve on $[0, T]$ the following system of ODEs*

$$\begin{cases} \dot{x}^\varepsilon = v^\varepsilon \\ \dot{v}^\varepsilon = -\frac{1}{\varepsilon} \sum_{i=1}^{k} \nabla\phi\left(\frac{x^\varepsilon - x_i}{\varepsilon}\right) \end{cases}$$

and let $(x^{\varepsilon,R}, v^{\varepsilon,R})$ solve

$$\begin{cases} \dot{x}^{\varepsilon,R} = v^{\varepsilon,R} \\ \dot{v}^{\varepsilon,R} = -\frac{1}{\varepsilon} \sum_{i=1}^{N} \nabla\phi^R\left(\frac{x^{\varepsilon,R} - x_i}{\varepsilon}\right) \end{cases}$$

with the same initial conditions.

Then under the assumption that for all $t \in [0, T]$ we have $v^\varepsilon(t), v^{\varepsilon,R}(t) \in \mathbb{R}^3 \setminus B_\eta(0)$, the solutions of these equations satisfies

$$|x^{\varepsilon,R}(t) - x^\varepsilon(t)| + |v^{\varepsilon,R}(t) - v^\varepsilon(t)| \leq C\,N \frac{e^{C\,R\,\eta^{-1}\,M}}{\varepsilon^M} \left\|(1 - \Lambda^R)\,\nabla\phi\right\|_\infty .$$

The proof is similar to [4, Lem 2], and we use the methodology from there in our proof.

Proof: We proceed by induction on the number of collisions already encountered. We first consider the base case.

If the short range tagged particle has encountered no collisions, then since all the background particles are at least $R\varepsilon$ from it, by directly estimating the error on the right hand side of the ordinary differential equations we obtain

$$|v^{\varepsilon,R} - v^\varepsilon| \leq N\,T \left\|(1 - \Lambda^R)\nabla\phi\right\|_\infty$$

and by integrating the above we furthermore obtain

$$|x^{\varepsilon,R} - x^{\varepsilon}| \le N\,T^2 \,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}.$$

If the short range tagged particle then encounters a collision, then at the time of collision, these errors can then be used to estimate the difference of initial conditions in Lemma 2.10 and so one obtains an error of

$$|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \le C\frac{e^{CR/\eta}}{\varepsilon} N\,T^2 \,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}$$

up to the end of the first collision. This concludes the base case of the argument.

Suppose now for the inductive hypothesis that the tagged particles have encountered $k-1$ collisions and that the error is bounded by

$$|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \le C\frac{e^{C\,R\,(k-1)/\eta}}{\varepsilon^{k-1}} N\,T^2 \,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}.$$

Then, since the short range evolution proceeds through free flow, we have, after the $k-1$th collision, that

$$|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \le C\frac{e^{C\,R\,(k-1)/\eta}}{\varepsilon^{k-1}} N\,T^2 \,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty} + N\,T\,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}.$$

Then another application of Lemma 2.10 gives the error during the kth collision as

$$\begin{aligned}|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \le C\frac{e^{C\,R\,k/\eta}}{\varepsilon^{k}} N\,T^2 \,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty} \\ + C\frac{e^{CR/\eta}}{\varepsilon} N\,T\,\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty},\end{aligned}$$

which concludes the proof of the lemma. □

2.4 Solutions of the Linear Boltzmann Equation

This specification of the two body interaction then allows us to comment on the types of solution one can obtain. The question one must ask is, for a solution to the linear Boltzmann equation, in what sense do the cancellation effects of the gain and loss parts of the collision operator manifest themselves.

For an admissible long range potential ϕ, we claim that this cancellation can only be considered in a Wasserstein sense, as opposed to the total variation sense given by the strong collision operator

$$L(f) = \int_{\mathbb{R}^3} \int_{\mathcal{S}} \left(f' \, \mathcal{M}'_\star - f \, \mathcal{M}_\star \right) \, |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star.$$

Indeed, the difference $f' - f$ in this form is required to decay sufficiently fast to compensate the unbounded integration over $\mathcal{S}$. Suppose naively that

$$(f' \, \mathcal{M}'_\star - f \, \mathcal{M}_\star) \sim \mathcal{M}_\star \, (f' - f) \sim C \mathcal{M}_\star \, r^{-s} \, |v_\star - v|$$

for some s. Then one formally obtains

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathcal{S}} \left(f'\mathcal{M}'_\star - f\mathcal{M}_\star \right) \, |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star &= C \int_{\mathbb{R}^3} \int_0^\infty \mathcal{M}_\star \, r^{1-s} |v_\star - v|^2 \, \mathrm{d}r \, \mathrm{d}v_\star \\ &\leq \int_{\mathbb{R}^3} \mathcal{M}_\star (|v|^2 + |v_\star|^2) \, \mathrm{d}v_\star \int_0^\infty r^{1-s} \, \mathrm{d}r \end{aligned}$$

and so we must have $s > 2$ for this integral to converge. Thus with $f \in L^1$ this will not in general decay fast enough to imply that the integral converges. This thus implies that strong solutions in this setting would not be expected to exist, and therefore we only consider a weak solution to the equation.

Remark 2.12. *Weak solutions have been shown, as in [20, Ch.2], to demonstrate the same effects as a Fokker-Planck term, and so in this sense are the physically relevant type of solution.*

This situation is in contrast to the evolution for the potential ϕ^R. Here one can split the collision operator into

$$L^R = L^R_+ - L^R_-$$

for

$$\begin{cases} L^R_+(f) &= \int_{\mathbb{R}^3} \int_{B_R} f(v'^{,R}) \, \mathcal{M}(v'^{,R}_\star) \, |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star \\ L^R_-(f) &= f(v) \int_{\mathbb{R}^3} \int_{B_R} \mathcal{M}(v_\star) \, |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star, \end{cases} \tag{2.5}$$

since due to the cut off, the integration over the parameter space $\mathcal{S}$ is finite, and so for f with finite mass and energy, we have $L^R_-(f) < \infty$ meaning that this formulation is well defined.

This splitting enables one to use the machinery of semi-group theory, since one can show that the operator $-v \cdot \nabla_x - L^R_-$ is a closed operator from its domain to L^1. Then by setting $\mathcal{T}$ to be semi-group generated by $-v \cdot \nabla_x - L^R_-$ we define a mild solution of the linear Boltzmann equation as a function $f^R \in L^1(\mathcal{U}, (1 + |v|^2) \, \mathrm{d}x \, \mathrm{d}v)$

such that

$$f^R(t,x,v) = \mathcal{T}(t) f_0(x,v) + \int_0^t \mathcal{T}(t-s) L_+^R(f^R)(s,x,v)\,\mathrm{d}s. \tag{2.6}$$

Since the purpose of this thesis is to relate densities for long range dynamics, we ultimately aim to compare weak solutions for the interaction potentials ϕ^R and ϕ. As such, we must first check that a mild solution is a weak solution.

Lemma 2.13. *Suppose that f^R is the unique mild solution of the linear Boltzmann equation for ϕ^R, as in* (2.6)*. Then it is also the unique weak solution as in* (1.5)*.*

Proof: We aim to apply [5] which states that we have the equivalence above if the operator $v \cdot \nabla_x + L^R$ is the generator of a strongly continuous semigroup of bounded linear operators.

By [6, Ch. 10] we have that $-v \cdot \nabla_x + L^R$ is the generator of a substochastic semi-group if L_+^R can be written in the form of an integral operator. This argument is carried out in the proof of the next lemma. □

We can also prove this directly. For instance, using the form of a mild solution in [6, Prop 3.31], we write

$$f^R(t) = f_0 + \left(-v \cdot \nabla_x - L_-^R\right) \int_0^t f^R(s)\,\mathrm{d}s + \int_0^t L_+^R(f^R)(s)\,\mathrm{d}s$$

which is well defined since the regularity $f^R \in L^1(\mathcal{U}, (1+|v|^2)\,\mathrm{d}x\,\mathrm{d}v)$ ensures that the expressions involving f^R are in the domains of the relevant operators.

We then take $h \in C_c^\infty([0,T) \times \mathcal{U})$ and integrate f^R in the above equation against $\partial_t h$ over $[0,T) \times \mathcal{U}$. We consider the terms separately. We first observe that the initial condition can be rewritten using the fundamental theorem of calculus to obtain

$$\begin{aligned}\int_0^T \int_{\mathcal{U}} \partial_t h\, f_0 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t &= \int_{\mathcal{U}} (h(T) - h(0))\, f_0 \,\mathrm{d}x\,\mathrm{d}v \\ &= -\int_{\mathcal{U}} h(0)\, f_0 \,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

since we choose test functions h with compact support and so $h(T) = 0$. Furthermore, by Fubini's theorem, we interchange the integrations over s and t in the following expression to obtain,

$$\begin{aligned}\int_0^T \int_{\mathcal{U}} \partial_t h \int_0^t L_+^R(f^R)(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t &= \int_{\mathcal{U}} \int_0^T L_+^R(f^R)(s) \int_s^T \partial_t h \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \\ &= -\int_{\mathcal{U}} \int_0^T L_+^R(f^R)(s)\, h(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

Finally, by performing the same operations as before, we can obtain

$$\begin{aligned}
\int_0^T \int_{\mathcal{U}} \partial_t h \,(-v\cdot\nabla_x - L_-^R) \int_0^t f^R(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&= \int_0^T \int_{\mathcal{U}} \int_0^t \partial_t h\,(-v\cdot\nabla_x - L_-^R)\, f^R(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&= -\int_0^T \int_{\mathcal{U}} h(s)\,(-v\cdot\nabla_x - L_-^R)\, f^R(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_0^T \int_{\mathcal{U}} v\cdot\nabla_x f^R(s)\, h(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v + \int_0^T \int_{\mathcal{U}} h(s)\, L_-^R(f^R)(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \\
&= -\int_0^T \int_{\mathcal{U}} f^R(s)\, v\cdot\nabla_x h(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad + \int_0^T \int_{\mathcal{U}} h(s)\, L_-^R(f^R)(s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

Combining these together ensures that

$$-\int_0^T \int_{\mathcal{U}} \partial_t + v\cdot\nabla_x) f^R\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t - \int_{\mathcal{U}} f_0\, h(0)\,\mathrm{d}x\,\mathrm{d}v = \int_0^T \langle L^R(f^R),\, h\rangle\,\mathrm{d}t$$

which is the weak form of the linear Boltzmann equation (1.5).

We complete this chapter with a proof of existence and uniqueness of mild solutions. By the previous lemma, this implies that weak solutions for short range potential exist and are unique.

Proposition 2.14. *For any $T > 0$, there exists a unique mild solution* (2.6) *to the linear Boltzmann equation on* $[0, T]$ *with interaction potential ϕ^R such that*

$$\int_{\mathcal{U}} (1 + |v|^2)\, f^R(t, x, v)\,\mathrm{d}x\,\mathrm{d}v < \infty$$

for all $t \in [0, T]$.

Proof: We aim to apply [6, Thm. 10.28] which is stated in Appendix A which will imply that $-v\cdot\nabla_x + L^R$ generates an honest semi-group, meaning that the semi-group conserves mass and energy. This then implies existence and uniqueness due to [2, Thm 3.1.12].

We now demonstrate how we satisfy the conditions required to apply [6, Thm. 10.28]. Conditions (A_1), (A_2) and (A_4) are trivially solved as they refer to an external force, which is 0 in our situation.

To show condition (A_3), observe that $L_-^R(v)$ is positive, and is locally integrable in v. Furthermore, since for any V with $|v| \le V$ we have

$$|L_-^R(v)| \le C\, V\, R^2 \left\|(1+V^2)\,\mathcal{M}\right\|_{L^1}$$

and so the loss operator satisfies condition (A_5). To satisfy the final conditions, we are thus required to show that

$$L_+^R(f) = \int_{\mathbb{R}^3} k(v,v')\, f(v')\, \mathrm{d}v'$$

where the kernel k satisfies the following. There exists $C > 0$ such that for all $V > 0$ we have

$$\int_{|v|>V} k(v,v')\, \mathrm{d}v \le C$$

for all $|v| < V$.

Using the Carleman representation, as in [17, 18, 51], we can rewrite $L_+^R(f)$ as

$$L_+^R(f) = \int_{\mathbb{R}^3} \frac{1}{|v-v'|^2} \int_{E_{vv'}} f(v')\, \mathcal{M}(v'_\star)\, b^R\left(\frac{|v'-v|}{|v'-v'_\star|},\, |v'-v'_\star|\right) \mathrm{d}v'_\star\, \mathrm{d}v'$$

where

$$E_{vv'} = \{w \in \mathbb{R}^3 : w \cdot (v'-v) = v \cdot (v'-v)\}$$

and b^R is the cross section for the potential ϕ^R in terms of the deviation angle θ^R and the relative velocity w. Defining

$$k(v,v') = \frac{1}{|v-v'|^2} \int_{E_{vv'}} \mathcal{M}(v'_\star)\, b^R\left(\frac{|v'-v|}{|v'-v'_\star|},\, |v'-v'_\star|\right) \mathrm{d}v'_\star$$

we have the desired form of the gain part of the collision operator.

Proceeding as in [3, Thm 2.1] or [45], we can rewrite

$$\begin{aligned} k(v,v') = {} & \frac{\mathcal{M}\left(|v-v'| + \frac{|v|^2-|v'|^2}{|v-v'|}\right)}{|v-v'|^2} \\ & \times \int\limits_{V_2\cdot(v'-v)=0} \mathcal{M}(u+V_2)\, b^R\left(\frac{|v'-v|}{|v'-v-V_2|},\, |v'-v-V_2|\right) \mathrm{d}V_2 \end{aligned}$$

where u is the part of $\frac{1}{2}(v+v')$ perpendicular to $v-v'$. Transforming coordinates of b^R into $\theta, |w|$ from $r, |w|$ we observe that

$$b^R(\theta, |w|) \leq C \sin\theta \, |w|$$

and therefore

$$\begin{aligned}
&\int\limits_{V_2\cdot(v'-v)=0} \mathcal{M}(u+V_2)\, b^R\left(\frac{|v'-v|}{|v'-v-V_2|},\, |v'-v-V_2|\right) \mathrm{d}V_2 \\
&\qquad\qquad \leq \int\limits_{V_2\cdot(v'-v)=0} \mathcal{M}(u+V_2)\, \sin\left(\frac{|v'-v|}{|v'-v-V_2|}\right) |v'-v-V_2| \, \mathrm{d}V_2 \\
&\qquad\qquad \leq \int\limits_{V_2\cdot(v'-v)=0} \mathcal{M}(u+V_2)\, |v'-v| \, \mathrm{d}V_2 \\
&\qquad\qquad \leq C|v'-v|
\end{aligned}$$

which enables one to use arguments in [6, Ch. 10] to conclude existence of solutions.

Arguments in [39] can be used to show the estimate

$$\int_{\mathcal{U}} (1+|v|^2)\, f^R(t,x,v) \, \mathrm{d}x \, \mathrm{d}v < \infty$$

which concludes the proof. □

Chapter 3

Marked Trees and Short Range Dynamics

The results on short range scattering for potential ϕ^R enable the specification of an evolution with local in time collisions. This evolution is however not Markovian due to the existence of recollisions in the dynamics. As was suggested in the preceding chapter, conditions on the dynamics to ensure they are recollision free, as in [24], are not straightforward, and any specification in fact requires the historic evolution. Therefore, on the state space $\mathcal{U}$ one can never describe Markovian dynamics; the space is too small.

This is the principal motivation for describing the dynamics on the space of marked trees. These trees enlarge the state space by describing the evolution of the tagged particle for all times up until the present. In this sense history is in fact part of the present, and therefore enables us to describe those systems that are recollision free by conditions on the current state of the system.

Even without this specification of recollisions, we can still describe the particle density via an evolution equation on the space of marked trees, the downside of this being that the form of the equation would be complex. However, for those evolutions where one removes recollisions, the evolution equation takes a particularly simple form (in equation (3.2)), and depends only weakly upon the tree itself.

Once we have described the evolution of the linear Boltzmann equation with an equation taking a similar form to equation (3.2), we are able to compare the evolutions in a straightforward manner. This result is carefully stated in Theorem 2, and the proof encompasses a careful description of how the errors from colliding at different times and in different manners provide only a small deviation on the particle and linear Boltzmann densities.

3.1 Marked Trees

We now introduce the space of marked trees. One starts with the particle evolution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ which is a solution of the equations

$$\begin{cases} \dot{x}^{\varepsilon,R} &= v^{\varepsilon,R} \\ \dot{v}^{\varepsilon,R} &= -\frac{1}{\varepsilon}\sum_{i=1}^{N} \nabla\phi^R\left(\frac{x^{\varepsilon,R}(t)-x_i(t)}{\varepsilon}\right) \\ \dot{x}_i &= v_i \\ \dot{v}_i &= 0. \end{cases}$$

From this evolution, one can specify the geometric parameters (r, ζ) of each collision, as well as the velocity v_i of the background particle. Together with these parameters we include the time of collision, which we take to be the start of the collision. By this we mean, if the tagged particle is colliding with background j, the time of collision is the time t for which

$$\begin{cases} |x(s) - x_j(s)| > R\varepsilon & s < t \\ |x(s) - x_j(s)| < R\varepsilon & s > t \end{cases}$$

at least for some small interval around t.

These parameters are then encoded in a marked tree, where each node corresponds to a collision, and the markers for the node are the parameters of the collision as described in Section 2.1. The root node of the tree then describes the initial position and velocity of the tagged particle. The space is then defined formally as follows.

Definition 3.1. *The set of collision trees, which we denote by $\mathcal{MT}$, is defined by*

$$\mathcal{MT} := \Big\{(x_0, v_0), (t_1, r_1, \zeta_1, v_1), \dots, (t_n, r_n, \zeta_n, v_n) \,|\, (x_0, v_0) \in \mathcal{U},\ t_i \in [0, T],\ r_i \in [0, R],\ \zeta_i \in \mathbb{R}/(2\pi\mathbb{Z}),\ v_i \in \mathbb{R}^3,\ n \in \mathbb{N} \cup \{0\}\Big\}$$

and we furthermore define, for a tree $\Phi \in \mathcal{MT}$, the function $n(\Phi) = n$ to be the number of collisions, as well as

$$\mathcal{MT}_k = \Big\{\Phi \in \mathcal{MT} : n(\Phi) = k\Big\}.$$

Also, defining

$$\tau(\Phi) := \begin{cases} 0 & n(\Phi) = 0 \\ \max_{1 \le j \le n} t_j & \text{else,} \end{cases}$$

we typically denote the final marker by

$$(\tau, \bar{r}, \bar{\zeta}, \bar{v}) := (t_n, r_n, \zeta_n, v_n).$$

Before discussing the relationship of $\mathcal{MT}$ with the particle dynamics, we digress briefly to describe the topological properties of the space. While it is not important for the analysis of the thesis, one can think of the topology of $\mathcal{MT}$ as a topology on càdlàg functions where these càdlàg functions correspond to the possible particle dynamics. Instead of taking one of the Skorokhod topologies, we take one that is more physically relevant for our analysis, where one is close if one has the same number of collisions, and if the parameters describing those collisions are close.

Lemma 3.2. *Defining*

$$\mathrm{d}(\Phi, \Psi) = \begin{cases} 1 & n(\Phi) \neq n(\Psi) \\ \min\{1, \max_{0 \le j \le n} |\Phi_j - \Psi_j|_\infty\} & \textit{else} \end{cases}$$

we have $(\mathcal{MT}, \mathrm{d}_{\mathcal{MT}})$ *is a complete separable metric space.*

Proof: There is a bijection between $\mathcal{MT}$ and the space

$$\bigcup_{k \ge 0} \left(\mathcal{U} \times \left([0,T] \times [0,R] \times \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}^3 \right)^k \right)$$

by identifying each tree with its node labels and suppressing the tree structure.

Let x_k be a Cauchy sequence in this space. Then there exists $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$, we have $n(x_k) = n(x_{k+1}) = n(x_{k_0})$, by the nature of $\mathrm{d}_{\mathcal{MT}}$. Then by the completeness of the space $\mathcal{U} \times \left([0,T] \times [0,R] \times \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}^3 \right)^{n(x_{k_0})}$, the shifted sequence x_{k_0+i} is convergent, as required.

The separability follows from the separability of each space, and since a product of separable spaces is itself separable we can conclude. □

Furthermore, the natural measure to endow on such a space is the Lebesgue measure on $\mathcal{MT}$, and we denote this by λ.

3.1.1 Representing Particle Dynamics on Marked Trees

While the space of marked trees was created from particle dynamics, we aim to provide an inverse to this process and specify the dynamics from a tree Φ. Given a tree $\Phi \in \mathcal{MT}$, we can define functions

$$(x^{\varepsilon,R},\, v^{\varepsilon,R})\colon [0,T) \times \mathcal{MT} \to \mathcal{U}$$

that correspond to evolution of the tagged particle which encounters collisions at the times and parameters specified by Φ. Outside of these collisions one assumes that the tagged particle evolves via free flow.

We first remark that not every tree $\Phi \in \mathcal{MT}$ can represent physically valid particle dynamics. For instance, if we assume the background is stationary, the dynamics in Figure 3.1 can be derived from a tree, yet are unphysical. This is because the inferred dynamics "miss" a collision with particle a_6, and fail to encode the second collision with particle a_1.

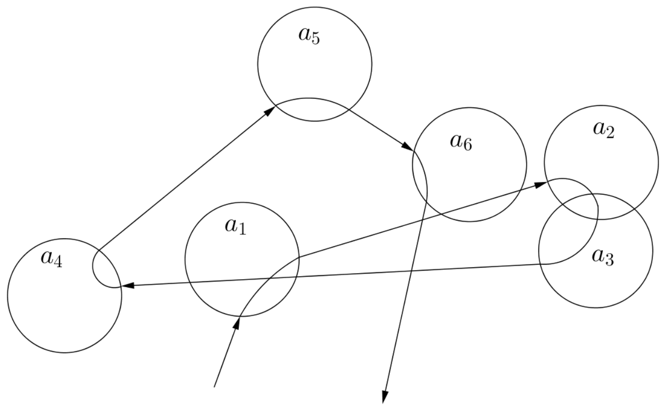


Figure 3.1: Example of un-feasible dynamics obtained from a tree

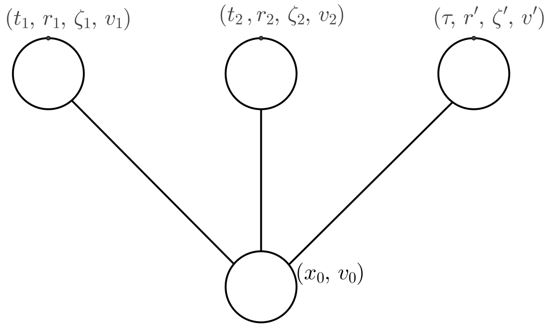
The second possible error from the evolution of a tree is that the inferred dynamics allow for recollisions. Absent from the marked trees is a record of which background particle is colliding with the tagged particle, and so, in order to determine the dynamics of the background from the evolution of the tagged particle, one must assume that each collision is with a distinct background particle. In theory one can determine if a particle recollides, but one would not know if they are two overlapping particles or not.

In dealing with the first error, there are two possible solutions. Firstly one can ensure that one restricts the space $\mathcal{MT}$ onto a space of admissible dynamics. The second solution is somewhat more straightforward, where we observe that by construction, the probability density on $\mathcal{MT}$ corresponding to the particle dynamics is not supported on those trees for which the dynamics are unphysical, and we therefore use this method.

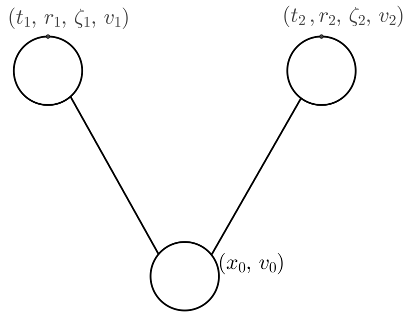
By itself, the second issue is not important, indeed, we can define for all $i = 1, \ldots, n(\Phi)$ the functions

$$(x_i,\, v_i)\colon [0, T) \times \mathcal{MT} \to \mathcal{U}$$

as the position and velocity of the background particle with collision specified by the i-th node. Recall that since these trees allow for recollisions, some of the functions x_i, v_i may be repeated. This however becomes an issue when one is describing the evolution of the probability density corresponding to particle dynamics on $\mathcal{MT}$, as allowing for recollisions of the background particles introduces a strong dependence on all background particles for each collision.



(a) Example of a marked tree Φ, in 3 dimensions, representing dynamics with three collisions



(b) The pruned tree $\overline{\Phi}$

Figure 3.2: Sketches to illustrate the process of pruning.

The evolutions of the probability densities on $\mathcal{MT}$ requires, for tree Φ, the specification of the tree that represents the dynamics associated to Φ with the node representing the final collision removed. This process is called pruning, and the pruned tree is denoted by $\overline{\Phi}$. This is demonstrated in Figure 3.2.

3.2 Particle Density on Marked Trees

The aim of the section is to derive the evolution equation for the particle density on the space of marked trees. We calculate this however only for those trees which are recollision free. Thus, we first describe the trees for which the underlying dynamics are recollision free. Once this is achieved, we describe the evolution equation in Lemma 3.4.

Having described the identification of marked trees from particle dynamics, we can relate the phase space density $f^{\varepsilon,R}$ to a density on $\mathcal{MT}$ as follows. We define the probability density $P_t^{\varepsilon,R}$, for a subset $\Omega \subset \mathcal{U}$, to be given by

$$\int_\Omega f^{\varepsilon,R}(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\,\mathrm{d}\Phi \tag{3.1}$$

where

$$S_t^{\varepsilon,R}(\Omega) := \{\Phi :\ (x^{\varepsilon,R}(t),\, v^{\varepsilon,R}(t)) \in \Omega\}.$$

The functions $x^{\varepsilon,R}, v^{\varepsilon,R}$ depend upon the tree Φ, although we suppress this dependence in the notation for cleanness of statements. The meaning of the set $S_t^{\varepsilon,R}(\Omega)$ is to restrict to those trees for which the particle evolution ends in Ω at time t. In this sense the relation (3.1) is a change of coordinates from an Eulerian viewpoint in $f^{\varepsilon,R}$ to a Lagrangian system in $P^{\varepsilon,R}$, where we have a different coordinate change for each tree.

As was described above, one can uniquely specify the background particle for trees that are recollision free. We now introduce a space of trees for which one has this recollision free property. This however is not the only assumption we make on the dynamics. To ensure the dynamics are physical, we are recourse to restrict trees further. The following definition contains the complete requirements for such trees. Some of the conditions in the form given are not necessary at this stage, but are used later when relating the particle and linear Boltzmann densities.

The space of trees we restrict to is the following.

Definition 3.3 (Good Trees). *Let*

$$M, V_2 \colon (0,1) \to \mathbb{R}_+$$

be given decreasing functions, and

$$V_1, \delta\colon (0,1) \to \mathbb{R}_+$$

be given increasing functions all with decay specified in Theorem 2. The set $\mathcal{G}(\varepsilon)$ of good trees is then the set of trees $\Phi \in \mathcal{MT}$ that satisfy the following:

(1) The maximum velocity is bounded above, meaning

$$\max\left\{\sup_{t\in[0,\tau]} |v^{\varepsilon,R}(t)|, \max|v_j|\right\} \leq V_2(\varepsilon).$$

(2) The velocities have a minimum separation, meaning that

$$\min_{i=1,\dots,n(\Phi)} |v^{\varepsilon,R}(t_i) - v_i| \geq V_1(\varepsilon).$$

(3) The number of collisions is bounded above by $M(\varepsilon)$, i.e.

$$n(\Phi) \leq M(\varepsilon).$$

(4) The collisions are separated by δ, meaning for all $i = 2, \dots, n(\Phi)$ we have

$$|t_i - t_{i-1}| > \delta(\varepsilon)$$

(5) There is no initial overlap at diameter ε if for all $j = 1, \dots, N$,

$$|x_0 - x_j(0)| > R\varepsilon.$$

(6) The trees are recollision free at diameter ε meaning, for all $0 \leq \varepsilon' \leq R\varepsilon$, for all $1 \leq j \leq n(\Phi)$ and for all $t \in [0,T] \setminus [t_j, t_j + \tau_\star^j]$, one has

$$|x^{\varepsilon,R}(t) - (x_j + tv_j)| > \varepsilon'.$$

Principally the aim of the space of good trees is to ensure that the tree specifies a unique evolution of particles. This is achieved by ensuring that the collisions are only binary interactions via (2) and (4), albeit indirectly, and condition (6) ensures that the background do not recollide with the tagged particle, so each background particle is distinct from the others.

Condition (5) should be thought of as a condition ensuring that the tree describes valid dynamics. This is performed by removing those situations where one is initially colliding with a background particle, and for which one does not have a node describing this. The other constraints are technical and solely used for analytical reasons in the convergence proof.

We leave unspecified for the moment the explicit form of the parameters introduced in the previous definition. Some conditions are required on these in the derivation of the evolution equation for $P^{\varepsilon,R}$, and we specify these when we make them. Otherwise, we describe the explicit form used in the convergence section of this chapter.

We are now able to state the evolution equation for the particle dynamics. This is similar in nature to the evolution equation in [39], but different because here the collisions now occur over an interval of time, and furthermore the particles are now considered to have radius $R\varepsilon$. As a result the evolution equation is subtly different.

Lemma 3.4. *The tagged particle density function on $\mathcal{G}(\varepsilon)$ evolves via the equation, for $\Phi \in G(\varepsilon)$, by*

$$\begin{cases} \partial_t P_t^{\varepsilon,R}(\Phi) = (1-\gamma(t,\varepsilon)) \left(\mathcal{Q}^{\varepsilon,+}[P_t^{\varepsilon,R}](\Phi) - P_t^{\varepsilon,R}(\Phi) \mathcal{Q}_t^{\varepsilon,-}(\Phi) \right) \\ P_0^{\varepsilon,R}(\Phi) = \xi(\varepsilon,R)\, \mathbb{1}_{\mathcal{MT}_0}(\Phi)\, f_0(x_0(\Phi), v_0(\Phi)), \end{cases} \tag{3.2}$$

where

$$\mathcal{Q}^{\varepsilon,+}[P_t^{\varepsilon,R}](\Phi) = \begin{cases} \delta(t-\tau(\Phi))\, \mathbb{1}_{t-\tau(\bar{\Phi})>\delta} \frac{\bar{r}\,|\bar{v}-v^{\varepsilon,R}(t)|\,\mathcal{M}(\bar{v})\,P_t^{\varepsilon,R}(\bar{\Phi})}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\,\mathbb{1}_\tau^{\varepsilon,R}[\Phi](x_\star,v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star} & n(\Phi)>0 \\ 0 & n(\Phi)=0 \end{cases}$$

$$\mathcal{Q}_t^{\varepsilon,-}(\Phi) = \mathbb{1}_{t-\tau(\Phi)>\delta} \frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}} \mathcal{M}(v_\star)\,|v_\star - v^{\varepsilon,R}(t)|\,\mathrm{d}S\,\mathrm{d}v_\star - c(\varepsilon)}{1-\eta_t^{\varepsilon,R}(\Phi)}$$

with

$$\eta_t^{\varepsilon,R}(\Phi) = \int_{\mathcal{U}} \mathcal{M}(v_\star) \left(1 - \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star,v_\star)\right) \mathrm{d}x_\star\,\mathrm{d}v_\star$$

for

$$\mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star,v_\star) = \begin{cases} 1 & \textit{for all } s\in(0,t) \textit{ we have } |x^{\varepsilon,R}(s)-(x_\star+sv_\star)|>R\varepsilon \\ 0 & \textit{else.} \end{cases}$$

We also have $c(\varepsilon) \geq 0$, *and*

$$\xi(\varepsilon, R) = \left(1 - \frac{4}{3}\pi\varepsilon^3 R^3\right)^N,$$

$$\gamma(t, \varepsilon) = \begin{cases} n(\overline{\Phi})\varepsilon^2 & t = \tau \\ n(\Phi)\varepsilon^2 & t > \tau \end{cases}.$$

We remark here that for these trees in $\mathcal{G}(\varepsilon)$, the dependency upon the tree itself is weak. It depends explicitly on the parameters of the final collision, which is a dependency one would expect, and in comparison with Lemma 3.12 is entirely natural. The dependency upon the rest of the tree is implicit, and depends only upon $(x^{\varepsilon,R}, v^{\varepsilon,R})$ in requiring that the particle in the latest collision does not collide with the tagged particle throughout its history.

We prove this lemma in three parts. Firstly we show that the initial condition agrees with the dynamics, then the gain and loss terms are justified separately.

We first however prove that $P_t^{\varepsilon,R}$ is absolutely continuous with respect to the Lebesgue measure.

Lemma 3.5. $P_t^{\varepsilon,R}$ *is absolutely continuous with respect to the Lebesgue measure* λ *for almost every* $\Phi \in \mathcal{G}(\varepsilon)$ *and for* $t \in [0, T]$.

Proof: The proof is identical to [39], with one exception. We elucidate.

Since $\mathcal{G}(\varepsilon)$ is open, for $\Psi \in \mathcal{G}(\varepsilon)$ there is an $h > 0$ such that $B_h(\Psi) \subset \mathcal{G}(\varepsilon)$. Then define the function $\varphi\colon B_h(\Psi) \to \mathcal{MT} \times \mathcal{U}$ by

$$\varphi(\Phi) = (\overline{\Phi}, x(\tau + \nu(r, \zeta)\varepsilon - \tau\,\bar{v}, \bar{v})$$

where the final component maps to the initial position of the final background particle. An easy calculation shows that

$$\det \nabla\varphi = r\,\varepsilon\,|v^{\varepsilon,R}(\tau^-) - \bar{v}|.$$

We aim to show absolute continuity by showing that the Radon-Nikodym derivative exists, and we thus show that the limit

$$\lim_{h\to 0} \frac{P_t^{\varepsilon,R}(C_h)}{\lambda(C_h)}$$

exists for suitable sets C_h.

To that end, we first note that, if $n(\Psi) = 0$ we have

$$P_t^{\varepsilon,R}(\Psi) \leq f_0(x_0, v_0)$$

which proves the existence of a derivative.

Now suppose that $n(\Psi) > 0$ and define the function $\tilde{\varphi}\colon B_h(\Psi) \to \mathcal{U}^{n+1}$ as the function that maps tree Φ to the initial positions of all the particles described. Note that we have

$$\det \nabla\tilde{\varphi}(\Phi) = \prod_{j=1}^{n} r\,\varepsilon\,|v^{\varepsilon,R}(\tau^-) - v_j|$$

by repeated application of the above formula. We also define $C_{h,j}(\Phi)$ to be the cube centred at $\tilde{\varphi}_j(\Phi)$ with side length h, and define

$$C_h(\Phi) = \prod_{j=0}^{n} C_{h,j}(\Phi).$$

We then observe that

$$\lambda(\tilde{\varphi}^{-1}(C_h))) = \frac{h^{6(n+1)}}{\prod_{j=1}^{n} r\,\varepsilon\,|v^{\varepsilon,R}(\tau^-) - v_j|}(1 + o(1))$$

by directly estimating the area, and that

$$P_t^{\varepsilon,R}(\tilde{\varphi}^{-1}(C_h))) \leq \int_{C_{h,0}} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \prod_{i=1}^{n(\Phi)} \int_{C_{h,j}} \mathcal{M}(v_\star)\,\mathrm{d}x\,\mathrm{d}v_\star$$

since the probability of finding the particles at time t is no more than the probability of finding the particles at time 0.

This thus results in

$$\frac{P_t^{\varepsilon,R}(B_h(\Psi))}{\lambda(B_h(\Psi))} \leq \frac{1}{h^6}\int_{C_{h,0}} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \prod_{i=1}^{n(\Phi)} \frac{r|v^{\varepsilon,R}(t_j) - v_j|}{h^6(1 + o(1)} \int_{C_{h,j}} \mathcal{M}(v)\,\mathrm{d}x\,\mathrm{d}v_\star$$

from which one observes that by the Lebesgue differentiation theorem the right hand side is bounded as $h \to 0$, as required. □

For cleanness of notation, we drop the indices on the density $P^{\varepsilon,R}$ throughout the following proofs. We start by analysing the initial condition. It is a straightforward calculation of the restriction of the condition in Definition 3.3.

Lemma 3.6. *The initial condition for the particle evolution on $\mathcal{G}(\varepsilon)$ satisfies*

$$P_0^{\varepsilon,R}(\Phi) = \xi(\varepsilon, R)\, f_0(x_0, v_0)\, \mathbb{1}_{\mathcal{MT}_0}$$

Proof: We observe that P_0 is concentrated on those trees for which $n(\Phi) = 0$. Therefore for a good tree, we have by using (5) in Definition 3.3,

$$\begin{aligned} P_0(\Phi) &= \mathbb{P}[\text{no initial overlap}]\, f_0(x_0, v_0)\, \mathbb{1}_{\mathcal{MT}_0} \\ &= \mathbb{P}[\cap_{j=1}^{N}\{|x_0 - x_j| > R\varepsilon\}]\, f_0(x_0, v_0)\, \mathbb{1}_{\mathcal{MT}_0} \\ &= \prod_{j=1}^{N} \mathbb{P}[\{|x_0 - x_j| > R\varepsilon\}]\, f_0(x_0, v_0)\, \mathbb{1}_{\mathcal{MT}_0} \end{aligned}$$

the last line following from independence of the background. We then have

$$\begin{aligned} \mathbb{P}[\{|x_0 - x_1| > R\varepsilon\}] &= 1 - \mathbb{P}[\{|x_0 - x_1| < R\varepsilon\}] \\ &= 1 - \frac{4}{3}\pi\, \varepsilon^3\, R^3 \end{aligned}$$

as required. □

We next analyse the jump part of the density, namely the form of the equation for $t = \tau$. This formalises the fact that the density jumps onto the tree Φ at a rate proportional to the density of the pruned tree. In order to formalise this, we are recourse to calculate the probability of finding a background particle in a position that collides with the tagged particle and such that it ensures the tree is in $\mathcal{G}(\varepsilon)$.

Lemma 3.7. *The instantaneous evolution at $\Phi \in \mathcal{G}(\varepsilon)$ at the time $\tau(\Phi)$ is given by*

$$P_\tau^{\varepsilon,R}(\Phi) = (1 - \gamma(\tau, \varepsilon))\, \delta(t - \tau(\Phi))\, \mathbb{1}_{t - \tau(\overline{\Phi}) > \delta} \frac{\bar{r}\, |\bar{v} - v^{\varepsilon,R}(t)|\, \mathcal{M}(\bar{v})\, P_t^{\varepsilon,R}(\overline{\Phi})}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_\tau^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}$$

Proof: We remove the superscript ε, R to keep the notation clean. We first remark that

$$P_\tau(\Phi) = P_\tau(\Phi |\, \overline{\Phi}) P_\tau(\overline{\Phi})$$

and we analyse the conditional probability. Defining

$$U_h = \{\Psi \in \mathcal{MT} : \tau(\Psi) = \tau(\Phi),\ \overline{\Psi} = \overline{\Phi} \text{ and } \Psi \in B_{h/2}(\Phi)\}$$

and by noting that $U_0 = \{\Phi\}$ we observe that by using the absolute continuity of

P_t with respect to the Lebesgue measure, we have

$$P_\tau(\Phi|\,\overline{\Phi}) = \lim_{h\to 0} h^{-6} P_\tau(U_h|\,\overline{\Phi}).$$

For $\Psi \in U_h$ define $V_h(\Psi) \in \mathcal{U}$ to be the initial position of the background particle giving a final collision in Ψ and define

$$V_h = \bigcup_{\Psi \in U_h} V_h(\Psi)$$

and note that $V_0 = \{(x^{\varepsilon,R}(\tau) + R\,\varepsilon\,\nu - \tau\,\bar{v}, \bar{v})\}$ where recall the notations used here from Definition 3.1. We obtain with a change of coordinates

$$\begin{aligned} P_\tau(U_h|\,\overline{\Phi}) &\leq \sum_{i=1}^{N-(n(\Phi)-1)} P_\tau\left((x_i,v_i) \in V_h|\,\overline{\Phi}\right) \varepsilon^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}| \\ &= (N - n(\overline{\Phi}))\,\varepsilon^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}|\,P_\tau((x_1,v_1) \in V_h|\,\overline{\Phi}) \\ &= (1 - \gamma(\tau,\varepsilon))\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}|\,P_\tau((x_1,v_1) \in V_h|\,\overline{\Phi}) \end{aligned}$$

and then using the absolute continuity of P_t almost everywhere, as we have shown in Lemma 3.5, one obtains

$$\begin{aligned} P_\tau(\Phi|\,\overline{\Phi}) &= \lim_{h\to 0} h^{-6}\,P_\tau(U_h|\,\overline{\Phi}) \\ &\leq \lim_{h\to 0}\; h^{-6}\,(1 - \gamma(\tau,\varepsilon))\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}|\,P_\tau((x_1,v_1) \in V_h|\,\overline{\Phi}) \\ &= (1 - \gamma(\tau,\varepsilon))\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}|\,P_\tau((x_1,v_1) \in V_0|\,\overline{\Phi}) \end{aligned}$$

We now strive for a lower bound. The inclusion exclusion principle, with the same change of coordinates, gives

$$\begin{aligned} P_\tau(U_h|\,\overline{\Phi}) \geq & \sum_{i=1}^{N-(n(\Phi)-1)} P_\tau\left((x_i,v_i) \in V_h|\,\overline{\Phi}\right)\,\varepsilon^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}| \\ & - \sum_{1 \leq i < j \leq N-(n(\Phi)-1)} P_\tau((x_i,v_i),(x_j,v_j) \in V_h|\,\overline{\Phi})\,\varepsilon^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}| \end{aligned}$$

and the first term can be simplified as before, and the second can be bounded by

$$\begin{aligned} \sum_{1 \leq j < j \leq N-(n(\Phi)-1)} & P_\tau((x_i,v_i),(x_j,v_j) \in V_h|\,\overline{\Phi})\,\varepsilon^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}| \\ & \leq N(N-1) P_\tau((x_1,v_1),(x_2,v_2) \in V_h|\,\overline{\Phi})\,\varepsilon^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}| \\ & = (N-1) P_\tau((x_1,v_1) \in V_h|\,\overline{\Phi})^2\,\bar{r}\,|v^{\varepsilon,R}(\tau) - \bar{v}| \end{aligned}$$

The quadratic probability term, together with the absolute continuity of P_t almost everywhere results in this times h^{-6} tending to zero as $h \to 0$. Then one has

$$\begin{aligned} P_\tau(\Phi|\,\overline{\Phi}) &= \lim_{h\to 0} h^{-6}\, P_\tau(U_h|\overline{\Phi}) \\ &\geq \lim_{h\to 0} h^{-6}\,(1-\gamma(\tau,\varepsilon))\,\bar{r}\,|v^{\varepsilon,R}(\tau)-\bar{v}|\,P_\tau((x_1,v_1)\in V_h|\,\overline{\Phi}_\tau) \\ &= (1-\gamma(\tau,\varepsilon))\,\bar{r}\,|v^{\varepsilon,R}(\tau)-\bar{v}|\,P_\tau((x_1,v_1)\in V_0|\,\overline{\Phi}) \end{aligned}$$

and thus these two are in fact equal.

We are thus left to calculate $P_\tau((x_1,v_1)\in V_0|\overline{\Phi}_\tau)$. The nature of V_0 gives

$$P_\tau((x_1,v_1)\in V_0|\,\overline{\Phi}) = P_\tau\Big((x_1,v_1) = (x(\tau)+R\,\varepsilon\,\nu-\tau\,\bar{v},\bar{v})|\,\overline{\Phi}\Big)$$

and then observe that since we are conditioning on $\overline{\Phi}$, we must rule out a region of initial position and velocity of the background particle since we know it cannot have initial data that will lead to having a collision with the tagged particle in the interval $[0,\tau)$. This region is exactly given by when $\mathbb{1}_\tau^{\varepsilon,R}[\Phi]=1$, and so we have

$$P_\tau((x_1,v_1)\in V_0|\,\overline{\Phi}) = \frac{\mathcal{M}(\bar{v})}{\int_{\mathcal{U}}\mathcal{M}(v_\star)\mathbb{1}_\tau^{\varepsilon,R}[\Phi](x_\star,v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star}.$$

Since $\Phi\in\mathcal{G}(\varepsilon)$ the indicator function is 1 as required. □

Once the density jumps “onto” tree Φ, we must now quantify how it decays over time. This occurs if the tagged particle encounters a future collision, given that the future collision ensures that the future collision produces dynamics that are recollision free.

Lemma 3.8. *The loss of density, for $t>\tau(\Phi)+\delta$, is given by*

$$\partial_t P_t^{\varepsilon,R}(\Phi) = -(1-\gamma(t,\varepsilon))\,P_t^{\varepsilon,R}(\Phi)\,\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(\tau)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - c(R\,\varepsilon)}{1-\eta_t^{\varepsilon,R}(\Phi)}$$

We observe that the rate of decay of the density is given by the rate at which the tagged particle collides with a background particle. Therefore, it should be clear that we must first calculate the probability of encountering one or many collisions in some time interval $[t,t+h)$ and in $(t-h,t]$. These will then directly be used to find the time derivative.

Proposition 3.9. *Define*

$$W_h(t) = \{(x,v) \in \mathcal{U} : \exists (r',\zeta',t') \in [0,R) \times [0,2\pi) \times (t,t+h)$$
$$\text{such that } x^{\varepsilon,R}(t') + \varepsilon\,\nu(r',\zeta') = x + t'\,v \text{ and } (v^{\varepsilon,R}(t') - v)\cdot\nu > 0\}.$$

For ε sufficiently small, and for V_2 in the definition of $\mathcal{G}(\varepsilon)$ decaying with the relation

$$\varepsilon^2\,V_2(\varepsilon) \to 0,$$

and for $\Phi \in \mathcal{G}(\varepsilon)$ and $t > \tau+\delta$ we have, for ω the initial conditions of the background,

$$\lim_{h\downarrow 0}\frac{1}{h}P_t^{\varepsilon,R}\left(\#(\omega\cap W_h(t)) \geq 2|\,\Phi\right) = 0 = \lim_{h\downarrow 0}\frac{1}{h}P_{t-h}^{\varepsilon,R}\left(\#(\omega\cap W_h(t-h)) \geq 2|\,\Phi\right)$$

and furthermore we have

$$\lim_{h\downarrow 0}\frac{1}{h}P_t^{\varepsilon,R}\left(\#(\omega\cap W_h(t)) = 1|\Phi\right)$$
$$= (1-\gamma(t,\varepsilon))\,P_t^{\varepsilon,R}(\Phi)\,\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(\tau)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - C_2(\varepsilon)}{1-\eta_t^{\varepsilon,R}(\Phi)}$$

and

$$\lim_{h\downarrow 0}\frac{1}{h}P_{t-h}^{\varepsilon,R}\left(\#(\omega\cap W_h(t-h)) = 1|\Phi\right)$$
$$= (1-\gamma(t,\varepsilon))\,P_t^{\varepsilon,R}(\Phi)\,\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(\tau)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - C_2(\varepsilon)}{1-\eta_t^{\varepsilon,R}(\Phi)}$$

Remark 3.10. *It seems strange that the derivation of the evolution equation for spatial scale $\varepsilon > 0$ should depend upon the decay of parameters specifying $\mathcal{G}(\varepsilon)$ for all values of ε. It in fact does not depend upon the decay, only that the velocities are bounded above.*

This proposition is essentially formally describing an expected property of the dynamics, that in any small time interval, the probability of encountering more than one collision approaches zero, and the probability of encountering exactly one collision is quantifiable, and can be given explicitly.

Proof: For the first limit, we remark that the probability of at least 2 collisions occurring in the interval $[t,t+h]$ is less than the probability of exactly 2 collisions occurring, so

$$P_t\left(\#(\omega\cap W_h(t))\geq 2|\,\Phi\right)\leq \sum_{1\leq i<j\leq N-n(\Phi)} P_t\left((x_i,v_i)\in W_h(t),\,(x_j,v_j)\in W_h(t)|\,\Phi\right)$$

and then the independence of the background enables one to write this as

$$\sum_{1\leq i<j\leq N-n(\Phi)} P_t\left((x_i,v_i)\in W_h(t),\,(x_j,v_j)\in W_h(t)|\,\Phi\right)$$
$$= \sum_{1\leq i<j\leq N-n(\Phi)} P_t\left((x_1,v_1)\in W_h(t)|\,\Phi\right)P_t\left((x_2,v_2)\in W_h(t)|\,\Phi\right)$$

and the identical distribution of the background gives

$$\sum_{1\leq i<j\leq N-n(\Phi)} P_t\left((x_1,v_1)\in W_h(t)|\,\Phi\right)P_t\left((x_2,v_2)\in W_h(t)|\,\Phi\right)$$
$$\leq N\,(N-1)\,\left(P_t\left((x_1,v_1)\in W_h(t)|\,\Phi\right)\right)^2.$$

We now write

$$P_t\left((x_1,v_1)\in W_h(t)|\Phi\right)=I_h(t)=\frac{\int_{\mathcal{U}}\mathcal{M}(v_\star)\,\mathbb{1}_{W_h(t)}(x_\star,v_\star)\,\mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star,v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star}{\int_{\mathcal{U}}\mathcal{M}(v_\star)\,\mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star,v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star}$$

where the second equality comes from the following. The probability of experiencing a particle that is in $W_h(t)$ with $\Phi\in\mathcal{G}(\varepsilon)$ is the probability of a background particle being distributed such that $\mathbb{1}_{W_h(t)}=1$, with $\mathbb{1}_t^{\varepsilon,R}[\Phi]=1$ also.

We now observe that, by estimating the size of the cylinder that $x_\star$ lies in for fixed $v_\star$, we obtain

$$\int_{\mathcal{U}}\mathcal{M}(v_\star)\mathbb{1}_{W_h(t)}(x_\star,v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star$$
$$\leq\int_{\mathbb{R}^3}\mathcal{M}(v_\star)\,\pi\,\varepsilon^2\,R^2\int_t^{t+h}|v^{\varepsilon,R}(t)-v_\star|\,\mathrm{d}s\,\mathrm{d}v_\star$$
$$\leq\int\mathcal{M}(v_\star)\,\pi\,\varepsilon^2\,R^2\int_t^{t+h}\left(|v^{\varepsilon,R}(t)|+|v_\star|\right)\,\mathrm{d}s\,\mathrm{d}v_\star$$
$$\leq\int\mathcal{M}(v_\star)\,\pi\,\varepsilon^2\,R^2\,h\,(V_2(\varepsilon)+|v_\star|)\,\mathrm{d}v_\star$$
$$\leq\pi\,\varepsilon^2\,R^2\,h\,\left(V_2(\varepsilon)+\int\mathcal{M}(v_\star)(1+|v_\star|)\,\mathrm{d}v_\star\right)$$
$$\leq\pi\,\varepsilon^2\,R^2\,h\,\left(V_2(\varepsilon)+\left\|(1+|v|^2)\mathcal{M}\right\|_{L^1}\right).$$

Furthermore, using the fact that

$$\begin{aligned}\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi]\, \mathrm{d}x_\star\, \mathrm{d}v_\star &= \int_{\mathcal{U}} (1 - \mathbb{1}_{W_t(0)}) \mathcal{M}(v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star \\ &= 1 - \int_{\mathcal{U}} \mathbb{1}_{W_t(0)} \mathcal{M}\, \mathrm{d}x_\star\, \mathrm{d}v_\star,\end{aligned}$$

together with the decay of V_2 ensures that for ε small enough, we have

$$1 - \int \mathbb{1}_{W_t(0)}\, \mathcal{M}\, \mathrm{d}x_\star\, \mathrm{d}v_\star \geq \frac{1}{2}.$$

Therefore

$$\begin{aligned}I_h(t) &= \frac{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_{W_h(t)}(x_\star, v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star} \\ &\leq 2 \int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_{W_h(t)}(x_\star, v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star \\ &\leq 2 \int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_{W_h(t)}(x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star \\ &\leq 2\, \pi\, \varepsilon^2\, R^2\, h\, \left(V_2(\varepsilon) + \left\|(1 + |v|^2)\mathcal{M}\right\|_{L^1}\right).\end{aligned}$$

Finally, using all these, we have

$$\begin{aligned}&\lim_{h\to 0} \frac{1}{h} P_t\Big(\#(\omega \cap W_h(t)) \geq 2 |\, \Phi\Big) \\ &\qquad \leq \lim_{h\to 0} \frac{1}{h}\, 4\, N\, (N-1)\, \pi^2\, \varepsilon^4\, R^4\, h^2\, \left(V_2(\varepsilon) + \left\|(1 + |v|^2)\mathcal{M}\right\|_{L^1}\right)^2 \\ &\qquad \leq \lim_{h\to 0} \frac{1}{h}\, 4\, \pi^2\, R^4 h^2\, \left(V_2(\varepsilon) + \left\|(1 + |v|^2)\mathcal{M}\right\|_{L^1}\right)^2 \\ &\qquad \leq \lim_{h\to 0} 4\, \pi^2\, R^4\, h\, \left(V_2(\varepsilon) + \left\|(1 + |v|^2)\mathcal{M}\right\|_{L^1}\right)^2 \\ &\qquad = 0\end{aligned}$$

which concludes the proof of the first limit. For the second term, we observe that by the same arguments we obtain

$$P_{t-h}\Big(\#(\omega \cap W_h(t-h)) \geq 2 |\, \Phi\Big) \leq 4\pi^2\, h\, R^4\, \left(V_2(\varepsilon) + \left\|(1 + |v|^2)\mathcal{M}\right\|_{L^1}\right)$$

which concludes the limit for times before t.

For the third limit, we have

$$P_t\left(\#(\omega \cap W_h(t)) = 1 |\, \Phi\right) = \sum_{1 \le i \le N-n(\Phi)} P_t\left((x_i, v_i) \in W_h(t), \text{ and } (x_j, v_j) \notin W_h(t) |\, \Phi\right)$$

and then the independence and identical distribution of the background enables one to write this as

$$\sum_{1 \le i \le N-n(\Phi)} P_t\left((x_i, v_i) \in W_h(t), \text{ and } j \ne i, \ \ (x_j, v_j) \notin W_h(t) |\, \Phi\right) = (N - n(\Phi))\, P_t\left((x_1, v_1) \in W_h(t) |\, \Phi\right) \prod_{j=2}^{N-n(\Phi)} P_t\left((x_j, v_j) \notin W_h(t) |\, \Phi\right)$$

and recalling the definition of $I_h(t)$ above results in

$$(N - n(\Phi))\, P_t\left((x_1, v_1) \in W_h(t) |\, \Phi\right) \prod_{j=2}^{N-n(\Phi)} P_t\left((x_i, v_i) \notin W_h(t) |\, \Phi\right) = (N - n(\Phi))\, I_h(t)\, (1 - I_h(t))^{N-n(\Phi)-1}$$

and using Taylor's formula enables this to be rewritten as

$$(N - n(\Phi)) I_h(t)\, (1 - I_h(t))^{N-n(\Phi)-1} = (N - n(\Phi)) \sum_{j=0}^{N-n(\Phi)-1} (-1)^j \binom{N - n(\Phi) - 1}{j} I_h(t)^{j+1}$$

and we remark that we showed above that

$$\lim_{h \to 0} \frac{1}{h} I_h(t)^2 = 0$$

and an easy extension to that argument shows that, for any $k \ge 2$ we have

$$\lim_{h \to 0} \frac{1}{h} I_h(t)^k = 0.$$

We are thus left to consider the term $(N - n(\Phi)) I_h(t)$ in the above expression. Setting

$$B_{h,t}(\Phi) = \{(x, v) \in \mathcal{U} : \mathbb{1}_t^{\varepsilon,R}[\Phi] = 0 \text{ and } \mathbb{1}_{W_h(t)} = 1\}$$

we can write

$$\int_{\mathcal{U}} \mathcal{M}\, \mathbb{1}_{W_h(t)} = \int_{\mathcal{U}} \mathcal{M}\, \mathbb{1}_{W_h(t)}\, \mathbb{1}_t^{\varepsilon,R}[\Phi] + \int_{\mathcal{U}} \mathcal{M}\, \mathbb{1}_{W_h(t)}\, (1 - \mathbb{1}_t^{\varepsilon,R}[\Phi])$$
$$= \int_{\mathcal{U}} \mathcal{M}\, \mathbb{1}_{W_h(t)}\, \mathbb{1}_t^{\varepsilon,R}[\Phi] + \int \mathbb{1}_{B_{h,t}(\Phi)} \mathcal{M}\, \mathbb{1}_{W_h(t)}.$$

Changing coordinates in the first term on the right hand side from initial position to position of collision enables one to rewrite this as

$$\int_{\mathcal{U}} \mathcal{M}\, \mathbb{1}_{W_h(t)}\, \mathbb{1}_t^{\varepsilon,R}[\Phi] = h\, \varepsilon^2 \int_{B_R} \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\, |v_\star - v^{\varepsilon,R}(t)|\, \mathrm{d}v_\star\, \mathrm{d}S$$

and we are left to analyse $\int \mathbb{1}_{B_{h,t}(\Phi)} \mathcal{M}$, and Lemma 3.11 shows that

$$\int \mathbb{1}_{B_{h,t}(\Phi)} g = h\, R^2\, \varepsilon^2\, c(\varepsilon)$$

where $c(\varepsilon) = o(1)$. We now use all these facts to conclude. We obtain

$$\lim_{h\to 0} h^{-1} P_t^{\varepsilon,R}\, (\#(\omega \cap W_h(t)) = 1 |\, \Phi)$$
$$= \lim_{h\to 0} \frac{(N - n(\Phi))}{h} \sum_{j=0}^{N-n(\Phi)-1} (-1)^j \binom{N - n(\Phi) - 1}{j} I_h(t)^{j+1}$$
$$= \lim_{h\to 0} \frac{(N - n(\Phi))}{h}\, I_h(t)$$

and then using the form of $I_h(t)$ we can write this as

$$\lim_{h\to 0} \frac{N - n(\Phi)}{h}\, I_h(t)$$
$$= \lim_{h\to 0} \frac{(N - n(\Phi))}{h}\, \frac{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_{W_h(t)}(x_\star, v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}$$
$$= \lim_{h\to 0} \frac{(N - n(\Phi))}{h}\, \frac{h\, \varepsilon^2 \int_{\mathcal{S}} \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\, |v_\star - v^{\varepsilon,R}(t)|\, \mathrm{d}v_\star\, \mathrm{d}S - \int \mathbb{1}_{B_{h,t}(\Phi)} \mathcal{M}}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}$$
$$= \lim_{h\to 0} \frac{(N - n(\Phi)) h \varepsilon^2}{h}\, \frac{\int_c S \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\, |v_\star - v^{\varepsilon,R}(t)|\, \mathrm{d}v_\star\, \mathrm{d}S - c(\varepsilon)}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}$$
$$= \lim_{h\to 0} (1 - \gamma(t, \varepsilon)) \frac{\int_{\mathcal{S}} \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\, |v_\star - v^{\varepsilon,R}(t)|\, \mathrm{d}v_\star\, \mathrm{d}S - c(\varepsilon)}{\int_{\mathcal{U}} \mathcal{M}(v_\star)\, \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}$$
$$= (1 - \gamma(t, \varepsilon)) \frac{\int_{\mathcal{S}} \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\, |v_\star - v^{\varepsilon,R}(t)|\, \mathrm{d}v_\star\, \mathrm{d}S - c(\varepsilon, R)}{\int_{\mathcal{U}} \mathcal{M}(v_\star) \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star)\, \mathrm{d}x_\star\, \mathrm{d}v_\star}$$

which is the desired limit. For times $t-h$ a similar argument shows that

$$\lim_{h\downarrow 0}\frac{1}{h}P_{t-h}\left(\#(\omega\cap W_h(t-h))=1\right)=\lim_{h\downarrow 0}\frac{1}{h}(N-n(\Phi))I_h(t-h)$$

which gives the statement of the lemma. □

In the last proof, we introduced a set $B_{h,t}$. We now prove the estimate required on it.

Lemma 3.11. *Suppose that $\Phi\in\mathcal{G}(\varepsilon)$, and that $\varepsilon^2V_2(\varepsilon)^5=o(1)$, and recalling that*

$$B_{h,t}(\Phi)=\{(x,v)\in\mathcal{U}:\mathbb{1}_t^{\varepsilon,R}[\Phi]=0 \text{ and } \mathbb{1}_{W_h(t)}=1\}$$

we have

$$\int_{B_{h,t}(\Phi)}\mathcal{M}(v_\star)\,\mathrm{d}v_\star\,\mathrm{d}x_\star=h\,(R\,\varepsilon)^2\,c(\varepsilon)$$

where $c(\varepsilon)=o(1)$.

Proof: We aim to estimate the size of the set of $B_{h,t}(\Phi)$. Given a fixed velocity v for the background particle, the set of admissible positions x in $B_{h,t}$ lies within a cylinder of radius ε and of length $h|v^{\varepsilon,R}(t)-v|$. We denote this cylinder by $\mathrm{Cyl}(v,h,\varepsilon)$, and remark that it has volume given by $C\,h\,(R\,\varepsilon)^2|v^{\varepsilon,R}(t)-v|$.

We are thus left to estimate the size of admissible velocities. These velocities in $B_{h,t}$ are characterised by the times and impact parameters of the two collisions they must encounter. Denote these parameters by $t_1\in(0,t)$ and $t_2\in(t,t+h)$ and ν_1,ν_2. We first distinguish two situations.

We separate into two cases dependent upon whether $t_1\in(0,t-\alpha)$ or whether $t_1\in(t-\alpha,t)$, and denote these sets by $B^1_{h,t}$ and $B^2_{h,t}$, and we let $\alpha=\frac{1}{2V_2(\varepsilon)}$.

Suppose to start with that t_1 is in the former. We can then write that

$$v=\frac{x(t_2)-x(t_1)}{t_2-t_1}+R\varepsilon\frac{\nu_2-\nu_1}{t_2-t_1}$$

and this ensures that v lies within a cylinder about the curve $\frac{x(t_2)-x(t_1)}{t_2-t_1}$ with radius at most $2R\varepsilon/\alpha$, where the radius over approximates by using α as a lower bound on the difference t_2-t_1.

To enable the central axis of the vector to have a single variable parametrisation over the range $(0,t-\alpha)$, we replace the value t_2 with t. This however adds extra to the radius of the cylinder. The maximum distance travelled between t and t_2 is given by $|x(t_2)-x(t)|\le hV_2(\varepsilon)$. If we assume that $hV_2(\varepsilon)\ll R\varepsilon$ then the radius of the cylinder is at most $4\,R\,\varepsilon/\alpha$.

Defining the curve $\mathcal{C}$ by the parametrisation $r(s) = \frac{x(t)-x(s)}{t-s}$ we must estimate the length of this curve. We first calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} r(s) &= \frac{\frac{\mathrm{d}}{\mathrm{d}s}(x(t)-x(s))}{t-s} + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{t-s}\right)(x(t)-x(s)) \\ &= \frac{-v(s)}{t-s} + \frac{1}{(t-s)^2}(x(t)-x(s))\end{aligned}$$

and we then have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}s} r(s)\right| &= \left|\frac{-v(s)}{t-s} + \frac{1}{(t-s)^2}(x(t)-x(s))\right| \\ &\leq \frac{|v(s)|}{t-s} + \frac{1}{(t-s)^2}|x(t)-x(s)| \\ &\leq \frac{V_2(\varepsilon)}{t-s} + \frac{C}{(t-s)^2}\end{aligned}$$

where C is an upper bound on the maximum distance possible between $x(s)$ and $x(t)$. On the torus this is at most 2. We then have

$$\begin{aligned}l(\mathcal{C}) &= \int_0^{t-\alpha} \left|\frac{\mathrm{d}}{\mathrm{d}s} r(s)\right| \mathrm{d}s \\ &\leq \int_0^{t-\alpha} \frac{V_2(\varepsilon)}{t-s} + \frac{C}{(t-s)^2}\,\mathrm{d}s \\ &= \frac{C}{\alpha} - \frac{C}{t} + V_2(\varepsilon)\left(\log\alpha - \log t\right).\end{aligned}$$

We then write

$$\mathrm{Vol}(\mathcal{C}) = \left\{x \in \mathbb{R}^3\colon\ \exists y \in l(\mathcal{C}) \text{ s.t. } |x-y| \leq \frac{4R\varepsilon}{\alpha}\right\}.$$

This thus results in

$$\begin{aligned}
\int_{B^1_{h,t}(\Phi)} \mathcal{M}(v)\,\mathrm{d}x\,\mathrm{d}v &\le \int_{\mathrm{Vol}(\mathcal{C})}\int_{\mathrm{Cyl}(v,h,\varepsilon)} \mathcal{M}\,\mathrm{d}x\,\mathrm{d}v \\
&\le C\,h\,(R\,\varepsilon)^2 \int_{\mathrm{Vol}(\mathcal{C})} |v^{\varepsilon,R}(t)-v|\mathcal{M}\,\mathrm{d}v \\
&\le C\,h\,(R\,\varepsilon)^2 \int_{\mathrm{Vol}(\mathcal{C})} \mathrm{d}v (V_2(\varepsilon) + \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}) \\
&= C\,h\,(R\,\varepsilon)^2\,(V_2(\varepsilon) + \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}) \left(\frac{R\varepsilon}{\alpha}\right)^2 \\
&\qquad\qquad \times \left(\frac{C}{\alpha} - \frac{C}{t} + V_2(\varepsilon)\,(\log\alpha - \log t)\right) \\
&\le C\,h\,(R\,\varepsilon)^2\,(V_2(\varepsilon) + \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1})\,(R\,\varepsilon)^2\,V_2(\varepsilon)^2 \\
&\qquad \times (C\,V_2(\varepsilon) + V_2(\varepsilon)\,(\log V_2(\varepsilon) + \log t)) \\
&\le C\,h\,(R\,\varepsilon)^4\,V_2(\varepsilon)^4\,\log V_2(\varepsilon)
\end{aligned}$$

and the assumptions on V_2 ensure that this tends to 0.

Suppose now that $t_1 \in (t-\alpha, t)$. Then either $v = v^{\varepsilon,R}(t)$ or $|v| \gg |v^{\varepsilon,R}(t)|$ so that the background particle traverses the entirety of the torus in between the collisions. The latter forces $|v - v^{\varepsilon,R}(t)| \gg \frac{1}{\alpha+h}$, and if we take $h \le \alpha/4$ then we obtain

$$|v - v^{\varepsilon,R}(t)| \gg \frac{4}{5\alpha}$$

and so, by estimating the size of the relevant cylinder for the x coordinate,

$$\begin{aligned}
\int_{B^2_{h,t}(\Phi)} \mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}x &\le \int_{\mathbb{R}^3 \setminus B_{\frac{4}{5\alpha}}(v^{\varepsilon,R}(t))} \mathcal{M}(v) \int_{\mathrm{Cyl}(v,h,\varepsilon)} \mathrm{d}x\,\mathrm{d}v \\
&\le C(R\,\varepsilon)^2\,h \int_{\mathbb{R}^3 \setminus B_{\frac{4}{5\alpha}}(v^{\varepsilon,R}(t))} \mathcal{M}(v)\,|v^{\varepsilon,R}(t) - v|\,\mathrm{d}v.
\end{aligned}$$

Since $|v^{\varepsilon,R}(t)| \le V_2(\varepsilon) = 1/2\alpha$, we obtain

$$\begin{aligned}
C(R\,\varepsilon)^2 h &\int_{\mathbb{R}^3 \setminus B_{\frac{4}{5\alpha}}(v^{\varepsilon,R}(t))} \mathcal{M}(v)\,|v^{\varepsilon,R}(t) - v|\,\mathrm{d}v \\
&\qquad\qquad \le C(R\,\varepsilon)^2 h \int_{\mathbb{R}^3 \setminus B_{\frac{3}{10\alpha}}(0)} \mathcal{M}(v)\,|v^{\varepsilon,R}(t) - v|\,\mathrm{d}v
\end{aligned}$$

and then by estimating this integral, using the fact that the velocity is bounded below by$|v| \geq 3/10\alpha \geq 1/10\alpha$, we obtain

$$
\begin{aligned}
C(R\varepsilon)^2 h & \int_{\mathbb{R}^3 \setminus B_{\frac{3}{10\alpha}}(0)} \mathcal{M}(v)\, |v^{\varepsilon,R}(t) - v| \,\mathrm{d}v \\
& \leq C\,(R\varepsilon)^2 h \int_{\mathbb{R}^3 \setminus B_{\frac{3}{10\alpha}}(0)} \mathcal{M}(v)\,(V_2(\varepsilon) + |v|) \,\mathrm{d}v \\
& \leq C(R\varepsilon)^2 h \int_{\mathbb{R}^3 \setminus B_{\frac{3}{10\alpha}}(0)} \mathcal{M}(v)\,(100\alpha^2 |v|^2 V_2(\varepsilon) + 10\alpha |v|^2) \,\mathrm{d}v \\
& = 10C(R\varepsilon)^2 h \left(10\alpha^2 V_2(\varepsilon) + \alpha\right) \left\|(1 + |v|^2)\,\mathcal{M}\right\|_{L^1} \\
& \leq 10C(R\varepsilon)^2 h \frac{11}{V_2(\varepsilon)} \left\|(1 + |v|^2)\,\mathcal{M}\right\|_{L^1}
\end{aligned}
$$

and this thus gives the order of magnitude as required. □

Proof: (of Lemma 3.8) We first claim that as a function of t with $t \in (\tau, T]$ the density P_t is continuous. Indeed, by the above we have for $h > 0$ that

$$
|P_{t+h}(\Phi) - P_t(\Phi)| = P_t\left(\#(\omega \cap W_h(t)) > 0 | \Phi\right) P_t(\Phi) \leq h\,(N - n(\Phi))\, I_h(t)
$$

which tends to 0 as $h \to 0$. Furthermore we have

$$
|P_t(\Phi) - P_{t-h}(\Phi)| \leq P_{t-h}\left(\#(\omega \cap W_h(t-h)) > 0 | \Phi\right) P_\tau(\Phi)
$$

which by the arguments before also tends to 0 as $h \to 0$.

One then observes that

$$
\begin{aligned}
P_{t+h}(\Phi) &= \left(1 - P_t\left(\#(\omega \cap W_h(t)) > 0 | \Phi\right)\right) P_t(\Phi) \\
P_t(\Phi) &= \left(1 - P_{t-h}\left(\#(\omega \cap W_h(t-h)) > 0 | \Phi\right)\right) P_{t-h}(\Phi)
\end{aligned}
$$

and therefore using this relation one obtains

$$
\begin{aligned}
\partial_t P_t(\Phi) &= \lim_{h \to 0} h^{-1} (P_{t+h}(\Phi) - P_t(\Phi)) \\
&= \lim_{h \to 0} \left(-h^{-1} P_t\left(\#(\omega \cap W_h(t)) > 0 |\, \Phi\right) P_t(\Phi)\right).
\end{aligned}
$$

Using Proposition 3.9 enables one to rewrite this as

$$\begin{aligned}\lim_{h\to 0}\Big(- h^{-1} P_t\left(\#(\omega\cap W_h(t))>0|\,\Phi\right) P_t(\Phi)\Big)&\\ = -P_t(\Phi)\lim_{h\to 0} P_t\left(\#(\omega\cap W_h(t))=1|\,\Phi\right)&\\ = -P_t(\Phi)(1-\gamma(t,\varepsilon))\times&\\ \times\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(\tau)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - c(\varepsilon)}{\int_{\mathcal{U}}\mathcal{M}(v_\star)\,\mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star,v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star}&\end{aligned}$$

as required. □

3.3 Linear Boltzmann Equation on Marked Trees

Since the particle dynamics have been shown to solve an effective evolution on a proper subset of $\mathcal{MT}$, we now aim to show that the linear Boltzmann equation describes a similar evolution equation on the space of marked trees.

To describe the evolution on $\mathcal{MT}$, one is required to infer some notion of collision from a solution f^R of the linear Boltzmann equation (2.6). From a probabilistic viewpoint one can see this coming from using the linear Boltzmann equation as the equation for the generator of a Lévy process. This then enables one to split the solution f^R into densities for which the Lévy process has encountered k jumps. These densities for each number of jumps can then be interpreted as the corresponding density on $\mathcal{MT}_k$, and from this one can obtain the form of the evolution equation. This is the underlying motivation for the functional analytic approach we take throughout the rest of this chapter.

Prior to this, we need to define the effective dynamics from a tree which represents the Boltzmann dynamics. We define

$$(x^R, v^R)\colon [0,T]\times\mathcal{MT}\to\mathcal{U}$$

by

$$\begin{cases} v^R(t) = v_0 & t\in[0,t_1)\\ v^R(t) = \sigma_1^R(v^R(t_{i-1}, v_i, r_i,\zeta_i) & t\in[t_i,t_{i+1})\\ x^R(t) = x_0 + \int_0^t v^R(s)\,\mathrm{d}s. \end{cases}$$

where σ^R is the scattering map from equation (2.1) and $i = 1,\dots,n(\Phi)$. We remark that these have instantaneous collisions, and the velocity is piecewise constant and therefore a càdlàg representation. Furthermore, the collisions are local in space since the particles are assumed to have zero radius. We can now prove the following.

Lemma 3.12. *There exists a solution* $P\colon [0,T] \to L^1(\mathcal{MT})$ *of the equation*

$$\begin{cases} \partial_t P_t^R(\Phi) = \mathcal{Q}^+[P_t^R](\Phi) - P_t^R(\Phi)\mathcal{Q}_\tau^-(\Phi) \\ P_0(\Phi) = f_0(x_0, v_0)\, \mathbb{1}_{\mathcal{MT}_0}(\Phi) \end{cases} \tag{3.3}$$

for

$$\mathcal{Q}^+[P_t^R](\Phi) = \delta(t - \tau(\Phi))\, P_t^R(\overline{\Phi})\, \mathcal{M}(\bar{v})\, \bar{r}\, |v^R(\tau^-) - \bar{v}|$$
$$\mathcal{Q}_t^-(\Phi) = \int_{\mathbb{R}^3} \int_{\mathcal{S}} \mathcal{M}(v_\star)\, |v^R(t) - v_\star|\, \mathrm{d}S\, \mathrm{d}v_\star$$

and for f_0 *an initial density satisfying Definition 1.5. Then by defining, for* $\Omega \subset \mathcal{U}$, *the set*

$$S_t(\Omega) = \{\Phi \in \mathcal{MT} \,:\, (x^R(t),\, v^R(t)) \in \Omega\}$$

we have

$$\int_\Omega f^R(t,x,v)\, \mathrm{d}x\, \mathrm{d}v = \int_{S_t(\Omega)} P_t^R(\Phi)\, \mathrm{d}\Phi \tag{3.4}$$

for f^R *the unique mild solution of the linear short range Boltzmann equation as given by* (2.6).

Remark 3.13. *One should immediately observe the similarity of the forms of equations* (3.3) *and* (3.2)*, and the obtaining of these forms is the useful part of this methodology.*

Remark 3.14. *The discussion on recollisions and removal of certain trees is not required here, as the Boltzmann dynamics are considered to be undertaken by particles with radius* 0*, and so particles have probability* 0 *of re-colliding.*

Before commencing the proof, we first state two formulae from [2, Ch.3] for mild solutions of Cauchy problems. Suppose that T is a C_0-semigroup with associated generator A. The Cauchy problem

$$\begin{cases} u'(t) &= Au(t) \\ u(0) &= x \end{cases} \tag{3.5}$$

has a mild solution of

$$u(t) = T(t)x = x + A\int_0^t T(s)x\, \mathrm{d}s.$$

Furthermore, the inhomogeneous Cauchy problem

$$\begin{cases} u'(t) &= Au(t) + f(t) \\ u(0) &= x \end{cases} \tag{3.6}$$

has a mild solution of

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

We comment here that the proof of Lemma 3.12 is similar to proofs in [39]. The difference is the form of the collision operator, and the functions x^R and v^R have different discontinuities when compared with the evolutions in that paper. These differences however do not change the functional analytic properties of the gain and loss parts of the collision operator, and so the proof works along the same lines. It is included for completeness sake.

The lemma is proved in an inductive manner on the number of nodes of a tree. We thus start with $\mathcal{MT}_0$. We define $P^{(0)}$ to be the unique mild solution to the equation

$$\partial_t P^{(0)} + v \cdot \nabla_x P^{(0)} = -L_-^R P^{(0)}.$$

where we recall from (2.5) that

$$L_-^R f = f(v) \int_{\mathbb{R}^3} \int_{B_R} \mathcal{M}(v_\star)\,|v_\star - v|\,\mathrm{d}S\,\mathrm{d}v_\star$$

We remark that this exists, since by the proof of Proposition 2.14 we have $-v \cdot \nabla_x - L_-^R$ is a closed operator $L^1 \to L^1$, and so using [2, Thm. 3.1.12] we have a unique mild solution.

We then define P_t^R iteratively in the following manner. For $\Phi \in \mathcal{MT}_0$ we define

$$P_t^R(\Phi) = P_t^{(0)}(x^R(t), v^R(t)) = P_t^{(0)}(x_0 + tv_0, v_0)$$

and otherwise define

$$P_t^R(\Phi) = \mathbb{1}_{t \geq \tau(\Phi)}\, e^{-(t-\tau(\Phi))\,\mathcal{Q}_\tau^-(\Phi)}\, P_\tau^R(\overline{\Phi})\,\mathcal{M}(\bar{v})\,\bar{r}\,|v^R(\tau^-) - \bar{v}|.$$

The existence of $P^{(0)}$ implies that such a function exists, since the existence of P_t^R on $\mathcal{MT}_1$ requires only P_t^R to exist on $\mathcal{MT}_0$ and so on.

We now turn to showing that P_t^R as defined here satisfies the remaining properties of Lemma 3.12, namely that it satisfies equation (3.4) and we have $P_t^R \in$

$L^1(\mathcal{MT})$. To show both of these, we first define

$$P_t^{(j)}(x,v) = \int_{S_t(x,v)\cap \mathcal{MT}_j} P_t^R(\Phi)\,\mathrm{d}\Phi. \tag{3.7}$$

and we proceed to analyse the evolution of this function for $j \geq 1$. We have somewhat abused notation here. In fact, we should define a measure $P_t^{(j)}$ analogously, show it is absolutely continuous with respect to the Lebesgue measure, and then define the above. This is easily shown from the formula for P_t^R by changing coordinates from pre collisional variables to post collisional variables.

Lemma 3.15. *For $j \geq 1$, $P_t^{(j)}$ as defined in equation* (3.7) *is a mild solution of*

$$\begin{cases} \partial_t P_t^{(j)}(x,v) = -v\cdot\nabla_x P_t^{(j)}(x,v) - L_-^R(P_t^{(j)})(x,v) + L_+^R(P_t^{(j-1)})(x,v) \\ P_0^{(j)}(x,v) = 0 \end{cases} \tag{3.8}$$

Proof: Let $\mathcal{T}$ be the semigroup associated to the generator $-v\cdot\nabla x - L_-^R$. We demonstrate the method for $j=1$ for notational simplicity. The method for arbitrary $j \geq 1$ follows then similarly.

The mild solution for $j=1$ satisfies

$$\begin{aligned} P_t^{(1)} &= \int_0^t \mathcal{T}(t-s) L_+^R(P_s^{(0)})\,\mathrm{d}s \\ &= \int_0^t \mathcal{T}(t-s) L_+^R(P_s^{(0)})(x-(t-s)v, v)\,\mathrm{d}s \end{aligned}$$

and we are thus left to massage the formula for $P^{(1)}$ into this form. It is essentially a Lagrangian to Eulerian change of coordinates. We have from the definition of $P^{(1)}$ that

$$\begin{aligned} \int_{S_t(x,v)\cap\mathcal{MT}_1} P_t^R(\Phi)\,\mathrm{d}\Phi &= \int_0^t \int_{\mathcal{S}}\int_{\mathcal{U}}\int_{\mathbb{R}^3} e^{-(t-\tau(\Phi))\,\mathcal{Q}_\tau^-(\Phi)}\, P_s^{(0)}(x_0+sv_0, v_0) \\ &\qquad \times \mathcal{M}(v_\star)\,|v_0-v_\star|\mathbb{1}_{x=x_0+s\,v_0+(t-s)\,v}\,\mathbb{1}_{v=\sigma_1^R(v_0,v_\star)}\,\mathrm{d}v_\star\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}S\,\mathrm{d}s \\ &= \int_0^t e^{-(t-\tau(\Phi))\,\mathcal{Q}_\tau^-(\Phi)} \\ &\qquad \times \int_{\mathcal{S}}\int_{\mathcal{U}}\int_{\mathbb{R}^3} P_s^{(0)}(x_0+sv_0,v_0)\mathcal{M}(v_\star)\,|v_0-v_\star|\,\mathbb{1}_{v=\sigma_1^R(v_0,v_\star)} \\ &\qquad\qquad \times \mathbb{1}_{x=x_0+s\,v_0+(t-s)\,v}\,\mathrm{d}v_\star\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}S\,\mathrm{d}s \end{aligned}$$

where the indicators ensure that v is the relevant post collisional velocity from v_0 and $v_\star$ from the scattering map σ^R, and the final position agrees with the initial conditions and undergoing a collision at the right time. We thus need to interpret the last two lines of this as $L_+^R P_0^{(0)}(x-(t-s)v, v)$.

We have, by rearranging the condition on the position, that $x_0 + sv_0 = x-(t-s)v$ and so

$$\int_{\mathcal{S}}\int_{\mathcal{U}}\int_{\mathbb{R}^3} P_s^{(0)}(x_0+sv_0, v_0)\,\mathcal{M}(v_\star)\,|v_0-v_\star|\mathbb{1}_{x=x_0+s\,v_0+(t-s)\,v}\,\mathbb{1}_{v=\sigma_1^R(v_0,v_\star)} \times \,\mathrm{d}v_\star\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}S$$
$$= \int_{\mathcal{S}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} P_s^{(0)}(x-(t-s)v, v_0)\mathcal{M}(v_\star)\,|v_0-v_\star| \mathbb{1}_{v=\sigma_1^R(v_0,v_\star)}\,\mathrm{d}v_\star\,\mathrm{d}v_0\,\mathrm{d}S.$$

The gain operator L_+^R is written in terms of post-collisional velocities, whereas the velocities $v_0, v_\star$ are pre-collisional.

We observe that $v_0, v_\star$ are pre collisional velocities for post-collisional $v, w_\star$. We then change coordinates from pre-collisional velocities $v_\star, v_0$ to pre collisional $w', w'_\star$ by

$$w' = v - ((w_\star - v)\cdot\omega)\omega$$
$$w'_\star = w_\star + ((w_\star - v)\cdot\omega)\omega$$

and we obtain

$$\int_{\mathcal{S}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} P_s^{(0)}(x-(t-s)v, v-\omega\cdot(v_0-v_\star)\omega)\,\mathcal{M}(v_\star) \times |v-\omega\cdot(v_0-v_\star)\omega - v_\star|\,\mathrm{d}v_\star\,\mathrm{d}S$$
$$= \int_{\mathcal{S}}\int_{\mathbb{R}^3} P_s^{(0)}(x-(t-s)v, w')\,\mathcal{M}(w'_\star)\,|v-w_\star|\,\mathrm{d}w_\star\,\mathrm{d}S$$

which is exactly the form of $L_+^R P_s^{(0)}$ required. □

Lemma 3.16. *For all $j \geq 0$ we have*

$$P_t^{(j)}(x,v) \leq f^R(t,x,v).$$

Proof: We proceed inductively, and assume that this is the case for $k \leq j-1$. The difference $F^j = f^R - P^{(j)}$ is a mild solution to the equation

$$\begin{cases} \partial_t F^j & = -v\cdot\nabla_x F + L_+^R(f^R - P^{(j-1)}) \\ F_0 & = 0 \end{cases}$$

Using [2, Prop. 3.1.16] we have

$$F^j = \int_0^t \mathcal{T}(t-s) L_+^R (f^R - P^{(j-1)})(s)\,\mathrm{d}s$$

and the inductive assumption ensures that this is positive. For $j = 0$ we furthermore observe that F^0 solves a similar equation, and with $P^{(-1)} \equiv 0$. In this case, the maximum principle (Lemma 5.2) gives positivity of F^0. □

Lemma 3.17. *For all $t \in [0,T]$ and for all $(x,v) \in \mathcal{U}$ we have*

$$\sum_{j=0}^{\infty} P_t^{(j)}(x,v) = f^R(t,x,v)$$

where f is the unique mild solution to the linear Boltzmann equation.

Proof: The existence theory for f^R in Proposition 2.14 shows that $f^R \in D(-v \cdot \nabla_x - L_-^R)$, and so by Lemma 3.16 we have $P_t^{(j)} \in D(-v \cdot \nabla_x - L_-^R)$ for all $j \geq 0$. By [6, Sec 10.4.3] we have $D(-v \cdot \nabla_x - L_-^R) \subset D(L_+^R)$ and so since $P_s^{(j)}$ is a mild solution of equation (3.8) we have

$$\int_0^t P_s^{(j)}\,\mathrm{d}s \in D(-v \cdot \nabla_x - L_-^R) \cap D(L_+^R)$$

for all j, and so

$$\int_0^t P_s^{(j)}\,\mathrm{d}s \in D(-v \cdot \nabla_x - L_-^R + L_+^R).$$

By the proof of Proposition 2.14, this operator is closed, and so

$$\sum_{j=0}^{\infty} \int_0^t P_s^{(j)}\,\mathrm{d}s \in D(-v \cdot \nabla_x - L_-^R + L_+^R).$$

Therefore using the forms of the solutions in (3.5) and (3.6) for $P_t^{(j)}$ we obtain

$$\begin{aligned}\sum_{j=0}^{\infty} P_t^{(j)}(x,v) &= f_0(x,v) + \sum_{j=0}^{\infty}\left(-v\cdot\nabla_x - L_-^R\right)\int_0^t P_s^{(j)}(x,v)\,\mathrm{d}s \\ &\qquad + \sum_{j=1}^{\infty}\int_0^t L_+^R P_s^{(j-1)}(x,v)\,\mathrm{d}s \\ &= f_0(x,v) + \left(-v\cdot\nabla_x - L_-^R\right)\int_0^t \sum_{j=0}^{\infty} P_s^{(j)}(x,v)\,\mathrm{d}s \\ &\qquad + \int_0^t L_+^R \sum_{j=1}^{\infty} P_s^{(j-1)}(x,v)\,\mathrm{d}s \\ &= f_0(x,v) + \left(-v\cdot\nabla_x - L_-^R + L_+^R\right)\int_0^t \sum_{j=0}^{\infty} P_s^{(j)}(x,v)\,\mathrm{d}s\end{aligned}$$

where we have used linearity and Fubini in the final line.

Using the form given by (3.6), we can interpret this sum as a mild solution of the linear Boltzmann equation, and by uniqueness it must be given by $f^R(t,x,v)$. □

These tools then enable us to conclude the existence proof of this section.

Proof: (of Lemma 3.12) By the formula for $P_t^{(j)}$ and the previous lemma, we have

$$\begin{aligned}\int_{\mathcal{MT}} P_t^R(\Phi)\,\mathrm{d}\Phi &= \sum_{j=0}^{\infty}\int_{\mathcal{MT}_j} P_t^R(\Phi)\,\mathrm{d}\Phi \\ &= \sum_{j=0}^{\infty}\int_{\mathcal{U}} P_t^{(j)}(\Phi)\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathcal{U}} f^R(t,x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty\end{aligned}$$

and so $P_t^R \in L^1(\mathcal{MT})$. We now show that P_t^R solves the requisite equation. The initial conditions are immediate. By construction it is supported only on those trees for which $\tau(\Phi) = 0$. The initial conditions of $P_t^{(0)}$ then ensure that it is equal to f_0 there.

For $t > 0$ we have, for $\Phi \in \mathcal{MT}_0$, the relationships $v^R(t) = v_0$ and $x^R(t) = x_0 + tv_0$, and so the definition of P_t^R gives

$$P_t^R(\Phi) = P_t^{(0)}(x^R(t), v^R(t)) = e^{-t\mathcal{Q}_\tau^-(\Phi)} f_0(x_0, v_0)$$

and then since the only term involving t is in the exponential, and f_0 is in the domain of L_-^R, we have by differentiating this term that

$$\begin{aligned}\partial_t(P_t^R(\Phi)) &= \partial_t\left(e^{-t\mathcal{Q}_\tau^-(\Phi)}f_0(x_0,v_0)\right)\\ &= -\mathcal{Q}_\tau^-(\Phi)e^{-t\mathcal{Q}_\tau^-(\Phi)}f_0(x_0,v_0)\\ &= -\mathcal{Q}_\tau^-(\Phi)P_t^R(\Phi)\end{aligned}$$

which is the required form of the equation.

For $\Phi \in \mathcal{MT}_k$ we have, for $t = \tau$ the relationship

$$P_\tau^R(\Phi) = P_\tau^R(\overline{\Phi})\,\mathcal{M}(\bar{v})\,\bar{r}\,|v^R(\tau^-) - \bar{v}|$$

as required. As before, in the formula for P_t^R we notice that the only dependence in t is in the exponential, and so we have, for $t > \tau$ that

$$\begin{aligned}\partial_t P_t^R(\Phi) &= \partial_t\left(e^{-(t-\tau)\mathcal{Q}_\tau^-(\Phi)}P_\tau^R(\overline{\Phi})\,\mathcal{M}(\bar{v})\,\bar{r}\,|v^R(\tau^-) - \bar{v}|\right)\\ &= e^{-(t-\tau)\mathcal{Q}_\tau^-(\Phi)}\left(-\mathcal{Q}_\tau^-(\Phi)\right)\,P_\tau^R(\overline{\Phi})\,\mathcal{M}(\bar{v})\,\bar{r}\,|v^R(\tau^-) - \bar{v}|\\ &= -P_t^R(\Phi)\mathcal{Q}_\tau^-(\Phi)\end{aligned}$$

since $\mathcal{Q}^-$ is a multiplication operator. This is again the required form of the equation.

Finally, we have, using the relationships in previous lemmas and Fubini,

$$\begin{aligned}\int_\Omega f^R(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_\Omega \sum_{j=0}^\infty P_t^{(j)}(x,v)\,\mathrm{d}x\,\mathrm{d}v\\ &= \sum_{j=0}^\infty \int_\Omega P_t^{(j)}(x,v)\,\mathrm{d}x\,\mathrm{d}v\\ &= \sum_{j=0}^\infty \int_{S_t^j(\Omega)} P_t^R(\Phi)\,\mathrm{d}\Phi\\ &= \int_{S_t(\Omega)} P_t^R(\Phi)\,\mathrm{d}\Phi\end{aligned}$$

which gives the desired form of the relationship for the density on $\mathcal{MT}$ to the linear Boltzmann equation. □

3.4 Convergence

Having derived the forms of the evolution equations for $P^{\varepsilon,R}$ and P^R in equations (3.2) and (3.3), we now turn to comparing the two. The similarity of those forms should suggest a comparison is tractable and straightforward. The evolution equation suggests, for a specific tree Φ that there are two errors from the evolution. Firstly, the jump onto tree Φ has differing forms, and so will give differing contributions to the density, and secondly the speed of decay for both densities is different. The proof of convergence quantifies both of these effects explicitly.

We now state the theorem which is the main result of this chapter.

Theorem 2. *Suppose that $P^{\varepsilon,R}$ and P^R are the probability densities on $\mathcal{G}(\varepsilon)$ and $\mathcal{MT}$ respectively corresponding to the short range particle dynamics in equations* (3.2) *and the linear Boltzmann equation associated to ϕ^R. Let the functions V_1, V_2, δ and M in the definition of $\mathcal{G}(\varepsilon)$ (Definition 3.3) be defined as*

$$\begin{cases} V_1(\varepsilon) & = \frac{1}{|\log \varepsilon|} \\ V_2(\varepsilon) & = |\log \varepsilon| \\ \delta(\varepsilon) & = \sqrt{\varepsilon} \\ M(\varepsilon) & = |\log \varepsilon|. \end{cases}$$

Then we have for all $t \in [0, T]$ that

$$\left\| P_t^{\varepsilon,R} - P_t^R \right\|_{TV} \to 0$$

as $\varepsilon \to 0$ with $N\varepsilon^2 = 1$.

Some comments are necessary here.

Remark 3.18. *Comparing this theorem with the analogous one in [39], we see that the result is similar, with a modification of the parameters defining $\mathcal{G}(\varepsilon)$, and most notably the use of δ which was absent in [39]. This similarity is reflected in the similarity of the proofs.*

Remark 3.19. *The major difference with [39] is that the background is specified to be Maxwellian. This can be relaxed to general $g \in L^1(\mathbb{R}^3, (1 + |v|^2)\,\mathrm{d}v)$, see [25].*

We now elucidate the method of the proof. We first quantify the evolution of the error between $P^{\varepsilon,R}$ and P^R for times $t > \tau(\Phi)$. By comparing the evolution equations, we can write an ordinary differential equation for the difference in terms of $\mathcal{Q}_t^-(\Phi)$. We use the variation of constants formula on this to express the change in error in terms of the initial error at time τ.

This situation however is complicated by the fact that $\partial_t P_t^{\varepsilon,R}(\Phi) = 0$ for $\tau < t < \tau + \delta$ and so we must split our considerations into the time intervals

$[\tau, \tau+\delta)$ and $[\tau+\delta, T)$. This then results naturally in observing the propagation of the error from separating the collisions into the evolution equation for the error. These estimates are found in Lemma 3.21.

The second estimate we require is a comparison of the size of jump at $t = \tau$. This is performed by estimating the difference between the gain terms in the respective equations. This is calculated in Lemma 3.23, where we prove an explicit bound on this difference.

These two estimates then enable one to quantify the error between the densities $P^{\varepsilon,R}$ and P^R at time t based upon the evolution from time 0. This is performed by an iterated combination of these two estimates. This is the purpose of Lemma 3.24.

This analysis then produces a pointwise estimate of the form

$$P_t^{\varepsilon,R}(\Phi) - \xi\, P_t^R(\Phi) \geq -\rho_t(\Phi) P_t^R(\Phi)$$

and we then bound ρ_t by a function that is uniform in $\Phi \in \mathcal{G}(\varepsilon)$. Furthermore, the decay we choose of the functions M, V_1, V_2, δ in the definition of good trees then ensures that this uniform bound tends to 0 as $\varepsilon \to 0$. We then conclude by using this pointwise estimate on the difference $P_t^{\varepsilon,R}(S) - P_t^R(S)$ for some subset $S \subset \mathcal{MT}$, from which we can deduce convergence in total variation.

Before proving these considerations, we first show that the conditioning of the dynamics in $\mathcal{G}(\varepsilon)$ in Definition 3.3 is small enough so that its P_t^R measure is 0 in the limit $\varepsilon \to 0$. This should be thought of as the choice of $\mathcal{G}(\varepsilon)$ is made sufficiently carefully so that the constraints restrict onto a set of measure 0. The validity of this restriction should be clear. Indeed, each restriction should be seen to be asymptotically small in the limit $\varepsilon \to 0$. However, the proof is a little more involved.

Lemma 3.20. *For any $R > 0$ fixed, we have*

$$\lim_{\varepsilon \to 0} P_t^R(\mathcal{MT} \setminus \mathcal{G}(\varepsilon)) = 0$$

Proof: We first show that $\mathcal{G}(0)$ has measure 1. First, note that a tree in $\mathcal{MT}_0$ cannot include a recollision, and so those trees in $\mathcal{MT}_0$ that have recollisions is an empty set.

Now suppose that $\Phi \in \mathcal{MT}_1$, and suppose that it has a recollision. This means that there is a $s \in (\tau, T]$ and $m \in \mathbb{Z}^3$ such that $x(s) + m = x_1(s)$. We show that this restricts the impact parameters onto a zero measure set.

From the dynamics in Φ, we know that

$$\begin{aligned} x(s) &= x_0 + \tau v_0 + (s-\tau)v(\tau) \\ v(\tau) &= v_0 - \nu(v_0 - v')\cdot\nu \\ x_1(s) &= x_0 + \tau v_0 - \tau v' + sv' \end{aligned}$$

and therefore

$$x_0 + \tau v_0 + (s-\tau)v(\tau) + m = x_0 + \tau v_0 + (s-\tau)v'$$

giving $m = (s-\tau)(v' - v(\tau))$, and so

$$m\cdot\nu = (s-\tau)(v'-v(\tau))\cdot\nu = (s-\tau)(v'\cdot\nu - (v_0 - \nu\cdot(v_0-v')\nu)\cdot\nu) = 0.$$

Therefore the impact parameter vector ν must lie in a set of measure 0. Since, for fixed relative velocity, the map taking $(r,\zeta)\mapsto\nu(r,\zeta,v'-v_0)$ is an isometry, we must have that the existence of a recollision restricts onto a set of measure 0.

Now suppose that $j\geq 2$ and let $\Phi\in\mathcal{MT}_j$. If Φ is not recollision free then either two of the collisions correspond to the same background particle, or the tagged particle recollides with the same background particle at some time $s\in(\tau,T]$

In the first case, this means there exists $2\leq l\leq j$ and $1\leq k\leq l$ such that the kth and lth collisions correspond to the same background. Then $v_k = v_l$ and so the second is restricted to a set of measure 0.

In the second case, there exists $s\in(\tau,T]$ and $m\in\mathbb{Z}^3$ and $1\leq k\leq j$ such that $x(s)+m = x_k(s)$. However, similarly to before,

$$x(s) = x_0 + t_1 v_0 + (t_2-t_1)v(t_1) + \cdots + (t_j - t_{j-1})v(t_{j-1}) + (s-t)v(t_j)$$

and

$$x_k(s) = x(t_k) + (s-t_k)v_k$$

and therefore by combining the two we have

$$(t_k - t_{k-1})v(t_{k-1}) + \cdots + (t_j - t_{j-1})v(t_{j-1}) + (s-t_j)v(t_j) + m = (s-t_k)v_k.$$

Then by using the scattering operator σ^R in equation (2.1) and taking the dot product of the above with respect to ν_j gives

$$(t_k - t_{k-1})v(t_{k-1}) + \cdots + (t_j - t_{j-1})v(t_{j-1})\cdot\nu_j + m\cdot\nu_j - (s-t_k)v_k\cdot\nu_j = -(s-t_j)v_j\cdot\nu_j.$$

This thus implies that v_j lies in a set of zero measure.

For all other restrictions in $\mathcal{G}(0)$, it is clear that they have measure 0, and so

$$P_t^R(\mathcal{MT} \setminus \mathcal{G}(0)) = 0$$

and also that

$$P_t^R(\mathcal{G}(0)) = 1.$$

Since $\mathcal{G}(\varepsilon)$ is increasing as ε decreases, and $\lim_{\varepsilon\to 0} \mathcal{G}(\varepsilon) = \mathcal{G}(0)$, it follows by the dominated convergence theorem that

$$\lim_{\varepsilon\to 0} P_t^R(\mathcal{G}(\varepsilon)) = P_t^R(\mathcal{G}(0)) = 1$$

and therefore that

$$\lim_{\varepsilon\to 0} P_t^R(\mathcal{MT} \setminus \mathcal{G}(\varepsilon)) = 0$$

as required. □

In comparing the evolution equations for both $P^{\varepsilon,R}$ and for P^R, which are given in equations (3.2) and (3.3), we see that they are of the same form. The following estimate quantifies the difference due to loss of density at differing times.

Lemma 3.21. *Suppose that $\Phi \in \mathcal{G}(\varepsilon)$ with $R^2\,\varepsilon^2\,V(\varepsilon) \to 0$ and $\delta > R\,\varepsilon\,V_1(\varepsilon)$. Then for $\tau(\Phi) < t < \tau(\Phi) + \delta$ we have*

$$\begin{aligned} P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\,P_t^R(\Phi) = \big(P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\,P_\tau^R(\Phi)\big) \\ + \xi(\varepsilon,R)\,P_t^R(\Phi)\,\left(e^{(t-\tau)\,\mathcal{Q}_\tau^-(\Phi)} - 1\right). \end{aligned}$$

For $t > \tau(\Phi) + \delta$ we have

$$\begin{aligned} P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\,P_t^R(\Phi) \geq e^{-\int_\tau^t (1+\eta_s^{\varepsilon,R}(\Phi))\,\mathrm{d}s}\,\big(P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\,P_\tau^R(\Phi)\big) \\ + \xi(\varepsilon,R)P_t^R(\Phi)\,e^{-2\,(t-\tau)\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}_\tau^-(\Phi)}\,\left(e^{\delta\mathcal{Q}_\tau^-(\Phi)} - 1\right) \\ + \xi(\varepsilon,R)\,P_t^R(\Phi)\,\left(e^{-2\,(t-\tau)\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}_\tau^-} - 1\right) \end{aligned}$$

where we recall that

$$\eta_t^{\varepsilon,R}(\Phi) = \int_{\mathcal{U}} (1 - \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star))\,\mathcal{M}(v_\star)\,\mathrm{d}x_\star\,\mathrm{d}v_\star.$$

Before proving this, we recall the variation of constants formula, which is equation (3.6) for a one dimensional ode. If $y\colon [\tau,\infty)\to\mathbb{R}$ satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y(t) \geq a(t)\, y(t) + b(t) \\ y(\tau) = y_0 \end{cases}$$

then

$$y(t) \geq e^{\int_\tau^t a(s)\,\mathrm{d}s}\, y_0 + \int_\tau^t e^{\int_s^t a(\sigma)\,\mathrm{d}\sigma}\, b(s)\,\mathrm{d}s.$$

Proof: For $\tau(\Phi) < t < \tau(\Phi) + \delta$, we have

$$\partial_t P_t^{\varepsilon,R}(\Phi) = 0,$$
$$\partial_t P_t^R(\Phi) = -P_t^R(\Phi)\, \mathcal{Q}_\tau^-(\Phi)$$

and therefore

$$\partial_t \left(P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_t^R(\Phi) \right) = \xi(\varepsilon,R)\, P_t^R(\Phi)\, \mathcal{Q}_\tau^-(\Phi).$$

The variation of constants formula then gives

$$P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R) P_t^R(\Phi) = P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_\tau^R(\Phi) + \xi(\varepsilon,R) \int_\tau^t P_s^R(\Phi)\, \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s$$

and then observing that for all $t, s \geq \tau(\Phi)$ we have

$$P_s^R(\Phi) = P_t^R(\Phi)\, e^{(t-s)\mathcal{Q}_\tau^-(\Phi)}. \tag{3.9}$$

Thus by integrating explicitly the expression in the above line we get

$$\begin{aligned} \int_\tau^t P_s^R(\Phi)\, \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s &= \int_\tau^t P_t^R(\Phi)\, e^{(t-s)\mathcal{Q}_\tau^-(\Phi)} \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s \\ &= P_t^R(\Phi)\, \mathcal{Q}_\tau^-(\Phi) \left(\frac{e^{(t-s)\,\mathcal{Q}_\tau^-(\Phi)}}{-\mathcal{Q}_\tau^-(\Phi)} \right) \Bigg|_{s=\tau}^{s=t} \\ &= P_t^R(\Phi) \left(e^{(t-\tau)\,\mathcal{Q}_\tau^-(\Phi)} - 1 \right) \end{aligned} \tag{3.10}$$

and this then gives the desired formula for the difference at times $t \leq \tau + \delta$. For $t > \tau(\Phi) + \delta$ we observe that we can estimate the term in the denominator of the loss term in (3.2) using

$$
\begin{aligned}
\eta_t^{\varepsilon,R}(\Phi) &= \int_{\mathcal{U}} \mathcal{M}(v_\star)(1-\mathbb{1}_t^{\varepsilon^R}[\Phi])\,\mathrm{d}x_\star\,\mathrm{d}v_\star \\
&\le \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\left(\pi(R\varepsilon)^2\int_0^t |v^{\varepsilon,R}(t)-v_\star|\,\mathrm{d}s - c(\varepsilon,R)\right)\mathrm{d}v_\star \\
&\le \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\left(\pi\,(R\,\varepsilon)^2\,T(V(\varepsilon)+|v_\star|)-c(\varepsilon,R)\right)\mathrm{d}v_\star \\
&\le \pi\,(R\,\varepsilon)^2\,T\,\left((V(\varepsilon)+\left\|(1+|v_\star|^2)\,\mathcal{M}\right\|_{L^1}\right) \\
&\le \frac{1}{2}
\end{aligned}
$$

where the last line is by choice of ε, and the condition $(R\varepsilon)^2V(\varepsilon)\to 0$. Using $\frac{1}{1-z}\le 1+2z$ for $z<1/2$, we then have

$$
\frac{1}{1-\eta_t^{\varepsilon,R}(\Phi)} = \frac{1}{1-\int_{\mathcal{U}}(1-\mathbb{1}_t^{\varepsilon}[\Phi])\,\mathcal{M}\,\mathrm{d}x_\star\,\mathrm{d}v_\star} \le 1+2\,\eta_t^{\varepsilon,R}(\Phi)
$$

and so

$$
\begin{aligned}
\mathcal{Q}_t^{\varepsilon,R,-}(\Phi) &= (1-\gamma(t,\varepsilon))\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(t)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - c(\varepsilon)}{\int_{\mathcal{U}}\mathbb{1}_t^{\varepsilon,R}[\Phi]\,\mathcal{M}\,\mathrm{d}x_\star\,\mathrm{d}v_\star} \\
&\le \left(1+2\eta_t^{\varepsilon,R}(\Phi)\right)\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(t)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star.
\end{aligned}
$$

The condition on δ ensures that for $t>\tau+\delta$ the dynamics are post-collisional from Lemma 2.8, and so $v^{\varepsilon,R}(t)=v^R(t)=v^R(\tau)$. Inputting this into the above yields

$$
(1-\gamma(t,\varepsilon))\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(t)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - c(\varepsilon)}{\int_{\mathcal{U}}\mathbb{1}_t^{\varepsilon,R}[\Phi]\,\mathcal{M}\,\mathrm{d}x_\star\,\mathrm{d}v_\star} \le \left(1+2\eta_t^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi).
$$

Comparing the evolutions of P^R and $P^{\varepsilon,R}$ and inputting the above into this then allows us to obtain

$$
\begin{aligned}
\partial_t\Big(P_t^{\varepsilon,R}(\Phi)-\xi(\varepsilon,R)\,P_t^R(\Phi)\Big) &= \xi(\varepsilon,R)\,\mathcal{Q}_\tau^-(\Phi)\,P_t^R(\Phi) \\
&\quad -(1-\gamma(t,\varepsilon))\,P_t^{\varepsilon,R}(\Phi)\,\frac{\int_{\mathbb{R}^3}\int_{\mathcal{S}}\mathcal{M}(v_\star)\,|v^{\varepsilon,R}(t)-v_\star|\,\mathrm{d}S\,\mathrm{d}v_\star - c(\varepsilon)}{\int_{\mathcal{U}}\mathcal{M}(v_\star)\,\mathbb{1}_t^{\varepsilon,R}[\Phi]\,\mathrm{d}x_\star\,\mathrm{d}v_\star} \\
&\ge \xi(\varepsilon,R)\,\mathcal{Q}_\tau^-(\Phi)\,P_t^R(\Phi) - \left(1+2\eta_t^{\varepsilon,R}(\Phi)\right)\,P_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}_\tau^-(\Phi) \\
&= -\left(1+2\eta_t^{\varepsilon,R}(\Phi)\right)\,\mathcal{Q}_\tau^-(\Phi)\,\left(P_t^{\varepsilon,R}(\Phi)-\xi(\varepsilon,R)\,P_t^R(\Phi)\right) \\
&\qquad\qquad - 2\eta_t^{\varepsilon,R}(\Phi)\,\xi(\varepsilon,R)\,P_t^R(\Phi)\,\mathcal{Q}_\tau^-(\Phi).
\end{aligned}
$$

The variation of constants formula then allows us to write in this case

$$\begin{aligned} P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_t^R(\Phi) \geq{}& e^{-\int_{\tau+\delta}^t \left(1+2\eta_s^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s} \\ &\times \left(P_{\tau(\Phi)+\delta}^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_{\tau(\Phi)+\delta}^R(\Phi)\right) \\ &- \int_{\tau+\delta}^t e^{-\int_s^t \left(1+2\eta_\sigma^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}\sigma}\, \xi(\varepsilon,R)\, P_s^R(\Phi)\, 2\eta_s^\varepsilon(\Phi)\, \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s. \end{aligned}$$

We then input the formula for the time interval $[\tau, \tau+\delta]$ with $t = \tau+\delta$ to obtain

$$\begin{aligned} P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_t^R(\Phi) \geq{}& e^{-\int_{\tau+\delta}^t \left(1+2\eta_s^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s} \left(P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_\tau^R(\Phi)\right) \\ &+ e^{-\int_{\tau+\delta}^t (1+2\eta_s^\varepsilon(\Phi))\,\mathcal{Q}^-(\Phi)\,\mathrm{d}s}\xi(\varepsilon,R) P_{\tau+\delta}^R(\Phi) \left(e^{\delta\,\mathcal{Q}_\tau^-(\Phi)} - 1\right) \\ &- \int_{\tau+\delta}^t e^{-\int_s^t \left(1+2\eta_\sigma^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}\sigma}\, \xi(\varepsilon,R)\, P_s^R(\Phi) 2\eta_s^{\varepsilon,R}(\Phi)\, \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s. \end{aligned}$$

Using the fact that $\eta_t^{\varepsilon,R}$ is positive and increasing in time (which can easily be seen from the formula), as well as equation (3.9) we obtain

$$\begin{aligned} P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R) P_t^R(\Phi) \geq{}& e^{-\int_\tau^t \left(1+2\eta_s^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s} \left(P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_\tau^R(\Phi)\right) \\ &+ e^{-(t-(\tau+\delta))\left(1+2\eta_t^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)}\xi(\varepsilon,R) e^{(t-(\tau+\delta))\,\mathcal{Q}_\tau^-(\Phi)}\, P_t^R(\Phi) \left(e^{\delta\mathcal{Q}_\tau^-(\Phi)} - 1\right) \\ &- \int_{\tau+\delta}^t e^{-(t-s)\left(1+2\eta_t^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)}\, \xi(\varepsilon,R)\, e^{(t-s)\,\mathcal{Q}_\tau^-(\Phi)}\, P_t^R(\Phi) 2\eta_s^{\varepsilon,R}(\Phi)\, \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s \end{aligned}$$

and simplifying this results in

$$\begin{aligned} P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_t^R(\Phi) \geq{}& e^{-\int_\tau^t \left(1+2\eta_s^{\varepsilon,R}(\Phi)\right)\mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s} \left(P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_\tau^R(\Phi)\right) \\ &+ e^{-(t-(\tau+\delta))\, 2\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}_\tau^-(\Phi)}\, \xi(\varepsilon,R)\, P_t^R(\Phi) \left(e^{\delta\,\mathcal{Q}_\tau^-(\Phi)} - 1\right) \\ &- \int_{\tau+\delta}^t e^{-(t-s)2\eta_t^\varepsilon(\Phi)\mathcal{Q}_\tau^-(\Phi)}\, \xi(\varepsilon,R)\, P_t^R(\Phi)\, 2\eta_s^{\varepsilon,R}(\Phi)\, \mathcal{Q}_\tau^-(\Phi)\,\mathrm{d}s. \end{aligned}$$

Evaluating the final integral similarly to (3.10) obtains the desired result. ☐

The form however of the estimate in the previous lemma is unwieldy to work with, and so we observe that the terms of the form $e^x - 1$ in the inequality in this lemma we can bounded simply from below in the following manner.

Lemma 3.22. *Defining*

$$\mathcal{Q}_{\max}^-(\Phi) = \sup_{t\in[0,T]} \mathcal{Q}_t^-(\Phi)$$

and

$$\rho_t^{\varepsilon,R,0}(\Phi) := 2\,t\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_{\max}(\Phi)\left(1+\delta\,\mathcal{Q}^-_{\max}(\Phi)\right)$$

we have, for $t > \tau + \delta$,

$$e^{-2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\mathcal{Q}^-_\tau(\Phi)}\left(e^{\delta\mathcal{Q}^-_\tau(\Phi)}-1\right) \\ +\xi(\varepsilon,R)\,P_t(\Phi)\left(e^{-2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)}-1\right)\geq -\rho_t^{\varepsilon,R,0}(\Phi)$$

Proof: We consider the two terms on the left hand side of the inequality in Lemma 3.21 separately. Considering the first, since $e^y \geq y$ for all $y \in \mathbb{R}$ we have

$$e^{-2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\mathcal{Q}^-_\tau(\Phi)}\left(e^{\delta\,\mathcal{Q}^-_\tau(\Phi)}-1\right)\geq -2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)\left(e^{\delta\mathcal{Q}^-_\tau(\Phi)}-1\right)$$

and then since $e^y - 1 \geq y$ for all $y \in \mathbb{R}$ we have

$$-2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)\left(e^{\delta\,\mathcal{Q}^-_\tau(\Phi)}-1\right)\geq -2(t-\tau(\Phi))\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)\,\delta\,\mathcal{Q}^-_\tau(\Phi)$$

and we thus obtain

$$e^{-2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)}\left(e^{\delta\,\mathcal{Q}^-_\tau(\Phi)}-1\right)\geq -2t\,\eta_t^{\varepsilon,R}(\Phi)\,\delta\left(\mathcal{Q}^-_{\max}(\Phi)\right)^2.$$

For the second term, we use the fact that $e^y \geq 1 + y$ for $y \in \mathbb{R}$, and then proceed similarly to obtain

$$e^{-2(t-\tau(\Phi))\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)}-1\geq -2t\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_{\max}(\Phi).$$

Combining these gives the lemma. □

The combination of these two lemmas is sufficient for the purpose of providing an estimate on the evolution of the error for times $t > \tau$. We now turn to the second of the stated errors, and estimate the difference between the corresponding gain terms of the densities.

Lemma 3.23. *Suppose that $\Phi \in \mathcal{G}(\varepsilon)$ and that $t \in [0,T]$. Suppose further that*

$$R^2\,\varepsilon^2\,M(\varepsilon) \leq \varepsilon^{1/2}$$
$$V_2(\varepsilon)\,R^2\,\varepsilon^2 < \frac{1}{4\,\pi\,T}.$$

Then we have

$$1-\frac{\mathbb{1}_{t-\tau(\overline{\Phi})>\delta}(1-\gamma(\tau,\varepsilon))}{1-\eta_\tau^{\varepsilon,R}(\Phi)}\leq\varepsilon.$$

Proof: Firstly, using the estimate on η from before, we observe that we have

$$\eta_\tau^{\varepsilon,R}(\Phi) \leq T\pi R^2 \varepsilon^2 \left(V(\varepsilon) + \left\|(1+|\cdot|^2)\mathcal{M}\right\|_{L^1}\right)$$

and therefore

$$\frac{\mathbb{1}_{t-\tau(\overline{\Phi})>\delta}}{1-\eta_\tau^{\varepsilon,R}(\Phi)} \leq \frac{1}{1-\pi\, R^2\, \varepsilon^2\, T\, \left(V(\varepsilon) + \|(1+|\cdot|^2)\mathcal{M}\|_{L^1}\right)}.$$

We then have

$$\begin{aligned} 1 - \frac{\mathbb{1}_{t-\tau(\overline{\Phi})>\delta}(1-\gamma(\tau,\varepsilon))}{1-\eta_\tau^{\varepsilon,R}(\Phi)} &= \frac{\gamma(\tau,\varepsilon) - \eta_\tau^{\varepsilon,R}(\Phi)}{1-\eta_\tau^{\varepsilon,R}(\Phi)} \\ &\leq \frac{R^2\,\varepsilon^2\, n(\overline{\Phi})}{1-2\pi\, R^2\, \varepsilon^2\, T\, V_2(\varepsilon)} \\ &\leq \frac{R^2\,\varepsilon^2\, M(\varepsilon)}{1-2\pi\, R^2\, \varepsilon^2\, T\, V_2(\varepsilon)} \end{aligned}$$

and the estimates on M and V_2 ensure that

$$\frac{R^2\,\varepsilon^2\, M(\varepsilon)}{1-2\pi\, R^2\, \varepsilon^2\, T\, V_2(\varepsilon)} \leq 2\, R^2\, \varepsilon^2\, M(\varepsilon) \leq 2\varepsilon^{1/2}$$

and for ε sufficiently small this is less than ε as required. □

The previous three estimates can now be combined to show the following pointwise estimate.

Lemma 3.24. *For $\Phi \in \mathcal{G}(\varepsilon)$ and $t > \tau(\Phi)$ we have*

$$P_t^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_t^R(\Phi) \geq -\rho_t^{\varepsilon,n(\Phi)}(\Phi)\, \xi(\varepsilon,R)\, P_t^R(\Phi) \tag{3.11}$$

with

$$\rho_t^{\varepsilon,R,0}(\Phi) := 2\, t\, \eta_t^{\varepsilon,R}(\Phi)\, \mathcal{Q}^-_{\max}(\Phi)\left(1 + \delta\, \mathcal{Q}^-_{\max}(\Phi)\right) \tag{3.12}$$

as before, and for $k = 0, \ldots, n(\Phi)$ we have

$$\rho_t^{\varepsilon,k}(\Phi) := e^{\delta\, \mathcal{Q}^-_{\max}(\Phi)}\left(\varepsilon + (1-\varepsilon)\,\rho_t^{\varepsilon,k-1}(\Phi)\right) + \rho_t^{\varepsilon,0}(\Phi)$$

We remark that with this recursion, we can write

$$\rho_t^{\varepsilon,k}(\Phi) := e^{k\,\delta\,\mathcal{Q}^-_{\max}(\Phi)}(1-\varepsilon)^k\,\rho_t^{\varepsilon,0}(\Phi)+ \\ +\left(\rho_t^{\varepsilon,0}(\Phi)+e^{\delta\,\mathcal{Q}^-_{\max}(\Phi)}\,\varepsilon\right)\sum_{j=1}^{k} e^{(k-j)\,\delta\,\mathcal{Q}^-_{\max}(\Phi)}(1-\varepsilon)^{k-j}.$$

The proof is similar to [39] and is via induction on $n(\Phi)$. It is included mainly for completeness sake.

It will become clear in the proof that the addition of the $e^{\delta\mathcal{Q}^-_{\max}(\Phi)}$ term in the definition of $\rho_t^{\varepsilon,k}(\Phi)$ is added to ensure that the estimate (3.11) is valid for all $t>\tau$. Without this, the inequality would only hold for $t>\tau+\delta$.

Proof: Suppose firstly that $\Phi\in\mathcal{MT}_0\cap\mathcal{G}(\varepsilon)$. Then by definition we have $\tau(\Phi)=0$ and so

$$P_\tau^{\varepsilon,R}(\Phi)=P_0^{\varepsilon,R}(\Phi)=\xi(\varepsilon,R)\,f_0(x_0,v_0)=\xi(\varepsilon,R)\,P_0^R(\Phi)=\xi(\varepsilon,R)\,P_\tau^R(\Phi).$$

and this then satisfies the inequality.

For $\tau<t<\tau+\delta$ we have

$$P_t^{\varepsilon,R}(\Phi)-\xi(\varepsilon,R)\,P_t^R(\Phi)=\xi(\varepsilon,R)\,P_t^R(\Phi)\,(e^{(t-\tau)\,\mathcal{Q}^-_\tau(\Phi)}-1)$$

and we observe that

$$e^{(t-\tau)\,\mathcal{Q}^-_\tau(\Phi)}-1\geq(t-\tau)\,\mathcal{Q}^-_\tau(\Phi)\geq 0\geq-\rho_t^{\varepsilon,R,0}(\Phi).$$

For $t>\tau+\delta$ we have, by Lemma 3.21 that

$$P_t^{\varepsilon,R}(\Phi)-\xi(\varepsilon,R)\,P_t^R(\Phi)\geq\xi(\varepsilon,R)\,P_t^R(\Phi)\left(e^{-2(t-\tau)\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)}-1\right) \\ +\xi(\varepsilon,R)\,P_t^R(\Phi)\,e^{-2(t-\tau)\,\eta_t^{\varepsilon,R}(\Phi)\,\mathcal{Q}^-_\tau(\Phi)}\left(e^{\delta\,\mathcal{Q}^-_\tau(\Phi)}-1\right)$$

and using Lemma 3.22 we obtain

$$P_t^{\varepsilon,R}(\Phi)-\xi(\varepsilon,R)\,P_t^R(\Phi)\geq-\xi(\varepsilon,R)\,P_t^R(\Phi)\,\rho_t^{\varepsilon,R,0}(\Phi)$$

which justifies the base case.

We now suppose that (3.11) holds for all trees in $\mathcal{MT}_{k-1}\cap\mathcal{G}(\varepsilon)$, and suppose that $\Phi\in\mathcal{MT}_k\cap\mathcal{G}(\varepsilon)$. Clearly for $t<\tau$ this holds trivially, as both densities are 0.

We first aim to approximate $P_\tau^{\varepsilon,R}(\Phi)-\xi(\varepsilon,R)\,P_\tau^R(\Phi)$. Since $\Phi\in\mathcal{G}(\varepsilon)$ we know that $\tau(\Phi) - \tau(\overline{\Phi}_\tau) > \delta$, and so recalling (3.2) and (3.3) we obtain

$$\begin{aligned} P_\tau^{\varepsilon,R}(\Phi) - \xi(\varepsilon,R)\, P_\tau^R(\Phi) \\ &= a(\varepsilon,R)\,\bar r\,|\bar v - v^{\varepsilon,R}(\tau)|\,\mathcal{M}(\bar v) P_\tau^{\varepsilon,R}(\overline{\Phi}) \\ &\qquad - \bar r\,|\bar v - v^R(\tau)|\,\mathcal{M}(\bar v)\,\xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi}) \\ &= \bar r\,|\bar v - v^R(\tau)|\,\mathcal{M}(\bar v)\,\big(a(\varepsilon,R) P_\tau^{\varepsilon,R}(\overline{\Phi}) - \xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi})\big) \\ &\geq \bar r\,|\bar v - v^R(\tau)|\mathcal{M}(\bar v)\,\big((1-\varepsilon)\,P_\tau^{\varepsilon,R}(\overline{\Phi}) - \xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi})\big) \end{aligned} \tag{3.13}$$

where in the last line we used Lemma 3.23. Then using the induction hypothesis we can rearrange this by

$$\begin{aligned} (1-\varepsilon)\,P_\tau^{\varepsilon,R}(\overline{\Phi}) - \xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi}) &\geq (1-\varepsilon)\,\xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi})(1-\rho_\tau^{\varepsilon,k-1}(\overline{\Phi})) \\ &\qquad - \xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi}) \\ &= \xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi})\,\Big((1-\varepsilon)(1-\rho_\tau^{\varepsilon,k-1}(\overline{\Phi})) - 1\Big) \\ &= -\xi(\varepsilon,R)\,P_\tau^R(\overline{\Phi})(\varepsilon + (1-\varepsilon)\rho_\tau^{\varepsilon,k-1}(\overline{\Phi})) \end{aligned}$$

and we aim to rewrite this in terms of Φ. We first remark that by definition

$$\bar r\,|\bar v - v^R(\tau)|\,\mathcal{M}(\bar v)\,P_\tau^R(\overline{\Phi}) = P_\tau^R(\Phi), \tag{3.14}$$

and so we are left to consider the $\rho_\tau^{\varepsilon,k-1}(\Phi)$ term. We have

$$\begin{aligned} \eta_\tau^{\varepsilon,R}(\overline{\Phi}) &= \int_{\mathcal{U}} \mathcal{M}(v_\star)(1 - \mathbb{1}_\tau^{\varepsilon,R}[\overline{\Phi}](x_\star,v_\star))\,\mathrm{d}x_\star\,\mathrm{d}v_\star \\ &= \int_{\mathcal{U}} \mathcal{M}(v_\star)(1 - \mathbb{1}_\tau^{\varepsilon,R}[\Phi](x_\star,v_\star))\,\mathrm{d}x_\star\,\mathrm{d}v_\star \\ &= \eta_\tau^{\varepsilon,R}(\Phi) \end{aligned}$$

since the addition of the final collision in Φ does not change the position of the tagged particle at any time $t \leq \tau$.

Furthermore, since for any $t \geq \tau$ we have $\mathbb{1}_\tau^{\varepsilon,R}[\Phi] \geq \mathbb{1}_t^{\varepsilon,R}[\Phi]$, we conclude that

$$\eta_\tau^{\varepsilon,R}(\overline{\Phi}) \leq \eta_t^{\varepsilon,R}(\Phi).$$

We can also show that

$$\begin{aligned}
\mathcal{Q}^-_{\max}(\overline{\Phi}) &= \sup_t \int\int_{\mathcal{S}} \mathcal{M}(v_\star)|v^{\varepsilon,R}_{\overline{\Phi}}(t) - v_\star| \,\mathrm{d}S \,\mathrm{d}v_\star \\
&\le \sup_t \int\int_{\mathcal{S}} \mathcal{M}(v_\star)|v^{\varepsilon,R}_{\Phi}(t) - v_\star| \,\mathrm{d}S \,\mathrm{d}v_\star \\
&= \mathcal{Q}^-_{\max}(\Phi)
\end{aligned}$$

since we change the velocity only for times after $\tau(\Phi)$, and this can only make the term larger. These two combine to give, for $t \ge \tau$,

$$\begin{aligned}
\rho^{\varepsilon,0}_\tau(\overline{\Phi}) &= 2\tau\, \eta^\varepsilon_\tau(\overline{\Phi}), \mathcal{Q}^-_{\max}(\overline{\Phi}) \\
&\le 2t\, \eta^{\varepsilon,R}_t(\Phi)\, \mathcal{Q}^-_{\max}(\Phi) \\
&= \rho^{\varepsilon,0}_t(\Phi).
\end{aligned}$$

The definition of $\rho^{\varepsilon,n(\Phi)}_t$ then gives that

$$\rho^{\varepsilon,n(\overline{\Phi})}_\tau(\overline{\Phi}) \le \rho^{\varepsilon,n(\overline{\Phi})}_t(\Phi) = \rho^{\varepsilon,n(\Phi)-1}_t(\Phi) \tag{3.15}$$

and therefore for any $t > \tau$ we have, by combining (3.14) and (3.15) into (3.13),

$$\begin{aligned}
P^{\varepsilon,R}_\tau(\Phi) - \xi(\varepsilon,R)\, P^R_\tau(\Phi) &\ge -\xi(\varepsilon,R)\, P^R_\tau(\Phi) \left(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\overline{\Phi})}_\tau(\overline{\Phi})\right) \\
&\ge -\xi(\varepsilon,R)\, P^R_\tau(\Phi) \left(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\right).
\end{aligned}$$

We now strive to use this for arbitrary times greater than τ. We first consider $\tau < t < \tau + \delta$. Recalling Lemma 3.21, we have

$$\begin{aligned}
P^{\varepsilon,R}_t(\Phi) - \xi(\varepsilon,R)\, P^R_t(\Phi) &= P^{\varepsilon,R}_\tau(\Phi) - \xi(\varepsilon,R)\, P^R_\tau(\Phi) \\
&\qquad + \xi(\varepsilon,R)\, P^R_t(\Phi) \left(e^{\delta\, \mathcal{Q}^-_\tau(\Phi)} - 1\right) \\
&\ge -\xi(\varepsilon,R)\, P^R_\tau(\Phi) \left(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\right) \\
&\qquad - \xi(\varepsilon,R)\, P^R_t(\Phi)\, \rho^{\varepsilon,0}_t(\Phi)
\end{aligned}$$

and then since $P^R_\tau(\Phi) = e^{(t-\tau)\,\mathcal{Q}^-_\tau(\Phi)}\,P^R_t(\Phi)$ we obtain

$$\begin{aligned}
P^{\varepsilon,R}_t(\Phi) - \xi(\varepsilon,R)\,P^R_t(\Phi) &\geq -\xi(\varepsilon,R)\,e^{(t-\tau)\mathcal{Q}^-_\tau(\Phi)}\,P^R_t(\Phi)\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big) \\
&\qquad - \xi(\varepsilon,R)\,P^R_t(\Phi)\rho^{\varepsilon,0}_t(\Phi) \\
&\geq -\xi(\varepsilon,R)\,P^R_t(\Phi)\,\Big(e^{\delta\,\mathcal{Q}^-_{\max}(\Phi)}\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big) + \rho^{\varepsilon,0}_t(\Phi)\Big) \\
&= -\xi(\varepsilon,R)\,P^R_t(\Phi)\rho^{\varepsilon,n(\Phi)}_t(\Phi)
\end{aligned}$$

as required. For $t \geq \tau + \delta$ we first consider

$$\begin{aligned}
&e^{-\int_\tau^t (1+\eta^{\varepsilon,R}_s(\Phi))\,\mathrm{d}s}(P^{\varepsilon,R}_\tau(\Phi) - \xi(\varepsilon,R)\,P^R_\tau(\Phi)) \\
&\qquad\geq -e^{-\int_\tau^t (1+\eta^{\varepsilon,R}_s(\Phi))\,\mathrm{d}s}\,\xi(\varepsilon,R)\,P^R_\tau(\Phi)\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big) \\
&\qquad\geq -e^{-\int_\tau^t (\eta^{\varepsilon,R}_s(\Phi))\,\mathrm{d}s}\,\xi(\varepsilon,R)\,P^R_t(\Phi)\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big) \\
&\qquad\geq -\xi(\varepsilon,R)\,P^R_t(\Phi)\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big) \\
&\qquad\geq -\xi(\varepsilon,R)\,P^R_t(\Phi)\,e^{\delta\,\mathcal{Q}^-_{\max}(\Phi)}\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big).
\end{aligned}$$

Inputting this into the inequality in Lemma 3.21 we obtain

$$\begin{aligned}
P^{\varepsilon,R}_t(\Phi) - \xi(\varepsilon,R)\,P^R_t(\Phi) &\geq e^{-\int_\tau^t (1+\eta^{\varepsilon,R}_s(\Phi))\,\mathrm{d}s}\,\big(P^{\varepsilon,R}_\tau(\Phi) - \xi(\varepsilon,R)\,P^R_\tau(\Phi)\big) \\
&\qquad - \xi(\varepsilon,R)\,P^R_t(\Phi)\,\rho^{\varepsilon,0}_t(\Phi) \\
&\geq -\xi(\varepsilon,R)\,P^R_t(\Phi)\,\Big(\varepsilon + (1-\varepsilon)\rho^{\varepsilon,n(\Phi)-1}_t(\Phi)\Big) - \xi(\varepsilon,R)\,P^R_t(\Phi)\,\rho^{\varepsilon,0}_t(\Phi) \\
&= -\xi(\varepsilon,R)\,P^R_t(\Phi)\,\rho^{\varepsilon,n(\Phi)}_t(\Phi)
\end{aligned}$$

as required. □

We now estimate the size of the function $\rho^{\varepsilon,R,0}_t(\Phi)$. The estimates themselves are independent of the tree Φ but depend upon the space $\mathcal{G}(\varepsilon)$.

Lemma 3.25. *For $\Phi \in \mathcal{G}(\varepsilon)$, we have*

$$\mathcal{Q}^-_{\max}(\Phi) \leq 2\,\pi\,\frac{R^2}{2}\,\Big(V_2(\varepsilon) + \big\|(1+|v|^2)\,\mathcal{M}\big\|_{L^1}\Big)$$

and

$$\eta^{\varepsilon,R}_t(\Phi) \leq R^2\varepsilon^2 T\Big(V_2(\varepsilon) + \big\|(1+|v|^2)\,\mathcal{M}\big\|_{L^1}\Big).$$

We also have

$$\rho_t^{\varepsilon,R,0}(\Phi) \leq C\,T^2\,R^4\,\varepsilon^2 \left(V_2(\varepsilon) + \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}\right)^2 \times \left(1 + \delta\,2\,\pi\frac{R^2}{2}(\left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1} + V_2(\varepsilon))\right).$$

Proof: For the first inequality, observe that for $\Phi \in \mathcal{G}(\varepsilon)$,

$$\begin{aligned}
\mathcal{Q}_t^-(\Phi) &= \int_{\mathbb{R}^3}\int_{\mathcal{S}} \mathcal{M}(v_\star)|v_\star - v^R(\tau)|\,\mathrm{d}S\,\mathrm{d}v_\star \\
&\leq \int_0^{2\pi}\int_0^R \mathrm{d}r\,\mathrm{d}\zeta \int_{\mathbb{R}^3} \mathcal{M}(v_\star)(|v_\star| + |v^R(\tau)|)\,\mathrm{d}v_\star \\
&\leq 2\pi\frac{R^2}{2}\int \mathcal{M}(v_\star)(|v_\star| + V_2(\varepsilon))\,\mathrm{d}v_\star \\
&\leq \pi\,R^2\left(\left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1} + V_2(\varepsilon)\right)
\end{aligned}$$

and then taking a supremum over $[0,T]$ on both sides gives the required inequality.

For the second inequality, for $\Phi \in \mathcal{G}(\varepsilon)$,

$$\begin{aligned}
\eta_t^{\varepsilon,R}(\Phi) &= \int_{\mathcal{U}} \mathcal{M}(v_\star)(1 - \mathbb{1}_t^{\varepsilon,R}[\Phi](x_\star, v_\star))\,\mathrm{d}x_\star\,\mathrm{d}v_\star \\
&\leq \int_{\mathbb{R}^3} \mathcal{M}(v_\star)\,\pi\,R^2\,\varepsilon^2 \int_0^t |v^{\varepsilon,R}(s) - v_\star|\,\mathrm{d}s\,\mathrm{d}x_\star\,\mathrm{d}v_\star \\
&\leq \pi\,R^2\,\varepsilon^2 T \int_{\mathbb{R}^3} \mathcal{M}(v_\star)(V_2(\varepsilon) + |v_\star|)\,\mathrm{d}x_\star\,\mathrm{d}v_\star \\
&= \pi\,R^2\,\varepsilon^2\,T\,\left(V_2(\varepsilon) + \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}\right).
\end{aligned}$$

as required.

The definition of ρ_t in (3.12) is a product of the two terms estimated above, and so these give

$$\rho_t^{\varepsilon,R,0}(\Phi) \leq 2\,T^2\,\pi^2\,R^4\,\varepsilon^2\left(V_2(\varepsilon) + \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}\right)^2 \times \left(1 + \delta\,2\,\pi\frac{R^2}{2}(\left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1} + V_2(\varepsilon))\right)$$

and then combining the constants which do not depend on ε or R gives the desired form. □

Lemma 3.26. *For $\Phi \in \mathcal{G}(\varepsilon)$, and for the parameters*

$$\begin{aligned} V_1(\varepsilon) &= \frac{1}{|\log \varepsilon|} \\ V_2(\varepsilon) &= |\log \varepsilon| \\ M(\varepsilon) &= |\log \varepsilon| \\ \delta(\varepsilon) &= \varepsilon^{2/3} \end{aligned}$$

as stated in Theorem 2, we have that

$$\rho_t^{\varepsilon, n(\Phi)}(\Phi) \to 0$$

Proof: The formula for $\rho_t^{\varepsilon,n(\Phi)}(\Phi)$ in a previous lemma gives, for $\mu = e^{\delta\, \mathcal{Q}^-_{\max}(\Phi)}$, that

$$\begin{aligned} \rho_t^{\varepsilon,n(\Phi)}(\Phi) :&= \mu^{n(\Phi)}\,(1-\varepsilon)^{n(\Phi)}\,\rho_t^{\varepsilon,R,0}(\Phi) \\ &\qquad + (\rho_t^{\varepsilon,R,0}(\Phi) + \mu\,\varepsilon) \sum_{j=1}^{n(\Phi)} \mu^{n(\Phi)-j}\,(1-\varepsilon)^{n(\Phi)-j} \\ &\le \mu^{M(\varepsilon)}\,(1-\varepsilon)^{M(\varepsilon)}\,\rho_t^{\varepsilon,R,0}(\Phi) \\ &\qquad + (\rho_t^{\varepsilon,R,0}(\Phi) + \mu\,\varepsilon) \sum_{j=1}^{M(\varepsilon)} \mu^{n(\Phi)-j}\,(1-\varepsilon)^{n(\Phi)-j} \\ &\le \mu^{M(\varepsilon)}\,\rho_t^{\varepsilon,R,0}(\Phi) + (\rho_t^{\varepsilon,R,0}(\Phi) + \mu\,\varepsilon) \sum_{j=1}^{M(\varepsilon)} \mu^{M(\varepsilon)} \\ &\le \mu^{M(\varepsilon)}\,\rho_t^{\varepsilon,R,0}(\Phi) + \rho_t^{\varepsilon,R,0}(\Phi)\,M(\varepsilon) + M(\varepsilon)\,\varepsilon\,\mu^{M(\varepsilon)} \end{aligned}$$

and we analyse each term.

First notice that

$$\mu^{M(\varepsilon)} = e^{M(\varepsilon)\,\delta\,\mathcal{Q}^-_{\max}(\Phi)} \le e^{C\varepsilon^{2/3}\,M(\varepsilon)\,V_2(\varepsilon)\,R^2} = e^{C\varepsilon^{2/3}|\log\varepsilon|^2\,R^2} \to 1$$

since the logarithm term is dominated by the polynomial.

We also have that

$$M(\varepsilon)\varepsilon \le \varepsilon \log \varepsilon \to 0.$$

To analyse the term $\rho_t^{\varepsilon,R,0}(\Phi)$ we remark that, for ε sufficiently small, we have $\rho_t^{\varepsilon,R,0}(\Phi) \le \rho_t^{\varepsilon,R,0}(\Phi) M(\varepsilon)$. Thus it is enough to consider the latter term. To that end, using Lemma 3.25 we observe that

$$\begin{aligned}
M(\varepsilon)\,\rho_t^{\varepsilon,R,0}(\Phi) &\le CM(\varepsilon)R^4\varepsilon^2\left(V_2(\varepsilon)+\left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}\right)^2 \\
&\qquad\qquad \times\left(1+\delta\,R^2\,\left(\left\|(1+|v|^2\,\mathcal{M}\right\|_{L^1}+V_2(\varepsilon)\right)\right) \\
&= C\,M(\varepsilon)\,R^4\,\varepsilon^2\,\left(V_2(\varepsilon)+\left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}\right)^2 \\
&\qquad\qquad + C\,M(\varepsilon)\,\delta\,R^6\,\varepsilon^2\,\left(V_2(\varepsilon)+\left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}\right)^3 \\
&=: I + II
\end{aligned}$$

and we consider these two terms separately.

The dominant term in I is, up to a constant,

$$M(\varepsilon)\,R^4\,\varepsilon^2\,V_2(\varepsilon)^2 \le |\log\varepsilon|^3\,\varepsilon^2\,R^4 \le R^4\,\varepsilon^{3/2}$$

which tends to 0 as $\varepsilon \to 0$.

The dominant term in II is, up to a constant,

$$M(\varepsilon)\,\delta\,R^6\,\varepsilon^2\,V_2(\varepsilon)^3 \le |\log\varepsilon|^4\,\varepsilon^{2/3}\,R^6\,\varepsilon^2 \le R^6\,\varepsilon^2$$

which again tends to 0 as $\varepsilon \to 0$, as required. □

We now have proved enough machinery to show Theorem 2.

Proof: (of Theorem 2) We compare $|P_t^R(S) - P_t^{\varepsilon,R}(S)|$ for $S \subset \mathcal{MT}$.

Since we have shown in Lemma 3.20 that $P_t^R(\mathcal{MT}\setminus\mathcal{G}(\varepsilon)) \to 0$ we have for $\alpha > 0$ and for sufficiently small ε that

$$P_t^R(\mathcal{MT}\setminus\mathcal{G}(\varepsilon)) \le \alpha$$

and so we have

$$\begin{aligned}
P_t^R(S) - P_t^{\varepsilon,R}(S) &\le P_t^R(S\cap\mathcal{G}(\varepsilon)) + P_t^R(S\setminus\mathcal{G}(\varepsilon)) - P_t^{\varepsilon,R}(S\cap G(\varepsilon)) \\
&< P_t^R(S\cap\mathcal{G}(\varepsilon)) - P_t^{\varepsilon,R}(S\cap\mathcal{G}(\varepsilon)) + \alpha \\
&= P_t^R(S\cap\mathcal{G}(\varepsilon)) - \xi(\varepsilon,R)\,P_t^R(S\cap\mathcal{G}(\varepsilon)) \\
&\qquad\qquad - (P_t^{\varepsilon,R}(S\cap\mathcal{G}(\varepsilon) - \xi(\varepsilon,R)\,P_t^R(S\cap\mathcal{G}(\varepsilon)) + \alpha.
\end{aligned}$$

We are thus left with comparing the initial conditions $1-\xi(\varepsilon,R)$ and the densities $P_t^R(S\cap\mathcal{G}(\varepsilon))$ and $P_t^{\varepsilon,R}(S\cap\mathcal{G}(\varepsilon))$.

We observe that

$$1-\xi(\varepsilon,R) = 1-\left(1-\frac{4}{3}\,\pi\,R^2\,\varepsilon^3\right)^N \le 1-1-\frac{4}{3}\,\pi\,N\,R^3\,\varepsilon^3 \le \frac{4}{3}\,\pi\,R\,\varepsilon$$

and so we have

$$P_t^R(S \cap \mathcal{G}(\varepsilon)) - \xi(\varepsilon, R)\, P_t^R(S \cap \mathcal{G}(\varepsilon)) \leq \frac{4}{3}\,\pi\, R\, \varepsilon\, P_t^R(S \cap \mathcal{G}(\varepsilon)) \leq \frac{4}{3}\,\pi\, R\, \varepsilon$$

since P^R is a probability measure. For ε small enough we have $4\pi\, R\, \varepsilon/3 \leq \alpha/3$, and so this deals with term for the difference of initial conditions.

To compare the difference in densities, we recall Lemma 3.24 which gives $\rho_t^{\varepsilon,k}(\Phi)\, P_t^R(\Phi)$ as a pointwise lower bound on this difference to obtain

$$P_t^R(\Phi) - P_t^{\varepsilon,R}(\Phi) \leq P_t^R(\Phi) - \xi(\varepsilon, R)\, P_t^R(\Phi) + \rho_t^{\varepsilon,n(\Phi)}(\Phi)\, \xi(\varepsilon, R) P_t^R(\Phi)$$

and then by the estimate on the initial conditions, and the estimate on $\rho_t^{\varepsilon,k}$ in Lemma 3.26 we obtain

$$\begin{aligned} P_t^R(\Phi)\,(1 - \xi(\varepsilon, R)) + \rho_t^{\varepsilon,n(\Phi)}(\Phi)\, \xi(\varepsilon, R)\, P_t^R(\Phi) &\leq P_t^R(\Phi)\,(1 - \xi(\varepsilon, R)) + \frac{\alpha}{3}\, P_t^R(\Phi) \\ &\leq P_t^R(\Phi)\, \frac{2\alpha}{3} \end{aligned}$$

and therefore

$$P_t^R(S) - P_t^{\varepsilon,R}(S) \leq \int_S \Big(P_t^R(\Phi) - P_t^{\varepsilon,R}(\Phi) \Big)\, \mathrm{d}\Phi = \int_S P_t^R(\Phi) \frac{2\alpha}{3}\, \mathrm{d}\Phi \leq \frac{2\alpha}{3}$$

as required. One concludes total variation convergence by observing that

$$\begin{aligned} P_t^{\varepsilon,R}(S) - P_t^R(S) &= (1 - P_t^{\varepsilon,R}(\mathcal{MT} \setminus S)) - (1 - P_t^R(\mathcal{MT} \setminus S)) \\ &= P_t^R(\mathcal{MT} \setminus S) - P_t^{\varepsilon,R}(\mathcal{MT} \setminus S) \end{aligned}$$

and then by applying the above we get convergence in modulus. □

Chapter 4

Comparison of Phase Space Densities for differing Particle Dynamics

The previous chapter analysed the relationship between the density for short range particle dynamics and the density for the linear Boltzmann equation with associated short range potential, and showed that the density converges as the spatial scale $\varepsilon \to 0$ in the Boltzmann Grad limit. To continue the proof of Theorem 1, we now analyse the impact of the long range part of the potential on the associated short range particle evolution. The main aim for this chapter is to show that the difference between the densities f^{ε} for long range particle dynamics, and the density $P^{\varepsilon,R}$ for short range particle dynamics, converges to 0 as $\varepsilon \to 0$, where the cut off parameter R is taken as a function of ε.

For convenience of the reader, recall that we have defined f^{ε} as the tagged particle density on $\mathcal{U}$ for long range dynamics under the equations

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = -\frac{1}{\varepsilon}\sum_{i=1}^{N} \nabla\phi\left(\frac{x(t)-x_i(t)}{\varepsilon}\right)$$
$$\dot{x}_i(t) = v_i(t), \qquad \dot{v}_i(t) = 0.$$

and we have defined $P_t^{\varepsilon,R}$ as the tagged particle density on $\mathcal{MT}$ corresponding to particle dynamics with short range potential ϕ^R, as in equation (3.1), which relates

$$\int_{\Omega} f_t^{\varepsilon,R}(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\,\mathrm{d}\Phi$$

for the phase space density $f^{\varepsilon,R}$ for evolution with potential ϕ^R.

Furthermore, we derived an effective evolution equation for $P^{\varepsilon,R}$ on the space of good trees $\mathcal{G}(\varepsilon)$. One major difference in this section is that this evolution equation is not useful for comparing with the long range density, since the long range evolution is not Markovian. It is useful however is providing L^∞ estimates on the density $P^{\varepsilon,R}$.

There are furthermore more fundamental problems. We cannot even describe the long range dynamics on $\mathcal{MT}$, as these dynamics do not even have a well defined notion of collision. We are thus recourse to using the density f^ε for the long range evolution, and so must compare as one would compare Lagrangian and Eulerian densities, although here it is somewhat more involved.

No longer having a well defined notion of collision has the extra implication that we cannot identify a subset of the N background particles through which we can effectively restrict the dynamics. Every background particle alters the trajectory of the tagged particle. This thus has the implication that, when given a tree Φ and a short range evolution on this tree, we cannot identify a deterministic long range evolution with the same initial background and scatterers, as the remaining $N-n(\Phi)$ background particles affect the long range evolution.

This issue is countered by considering the long range evolution as a random variable on each tree Φ, with randomness given by the position of the remaining background particles. This then enables us to identify the corresponding long range dynamics for the given short range dynamics, and to prove the following theorem.

Theorem 3. *Let ϕ be an admissible long range potential with a $\rho_2 > 0$ and a $\gamma > 0$ such that for all $\rho > \rho_2$ we have*

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho} \psi(\rho) \right| \leq C e^{-C\rho^{\frac{3}{2}+\gamma}}. \tag{4.1}$$

Furthermore, let

$$\begin{aligned} R(\varepsilon) &= \varepsilon^{-1/(3+\gamma)}, \\ M(\varepsilon) &= |\log \varepsilon|, \\ V_1(\varepsilon) &= \frac{1}{|\log \varepsilon|}, \\ V_2(\varepsilon) &= |\log \varepsilon|. \end{aligned}$$

Let f^ε be the tagged particle density for ϕ, and let $P^{\varepsilon,R}$ be the probability density on $\mathcal{MT}$ for short range potential ϕ^R.

Then for any $h \in C_b([0,T] \times \mathcal{U})$ we have

$$\int_\Omega f_t^\varepsilon \, h_t \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R} \, h_t \, \mathrm{d}\Phi \to 0,$$

where

$$S_t^{\varepsilon,R}(\Omega) = \Big\{ \Phi \in \mathcal{G}(\varepsilon) : (x^{\varepsilon,R}(t), v^{\varepsilon,R}(t)) \in \Omega \Big\}.$$

Remark 4.1. *This theorem is the stopping point for improving the decay assumption in* (4.1)*. Indeed, the proof provided does not allow for potentials with slower decay.*

The idea behind the proof is the following. Given a tree $\Phi \in \mathcal{G}(\varepsilon)$ we have, as in Section 3.1, deterministic evolutions $(x^{\varepsilon,R}, v^{\varepsilon,R})$ for the particle dynamics with short range potential ϕ^R with $n(\Phi)$ background given by the node labels of Φ. We also introduce $(x^\varepsilon, v^\varepsilon)$ as random variables on the tree Φ corresponding to solutions of the equations

$$\begin{aligned} \dot{x}(t) &= v(t), \qquad & \dot{v}(t) &= -\frac{1}{\varepsilon} \sum_{i=1}^{N} \nabla\phi\left(\frac{x(t) - x_i(t)}{\varepsilon}\right) \\ \dot{x}_i(t) &= v_i(t), \qquad & \dot{v}_i(t) &= 0. \end{aligned}$$

where the first $n(\Phi)$ background are distributed as in Φ, and where the remaining $N - n(\Phi)$ are independently and identically distributed according to $\mathcal{M}$ in velocity and uniformly in space. We emphasise here that this apparent increase in randomness is not because the system is any more random, more so that this interpretation is a convenient way to represent the system.

We compare the difference between short and long range dynamics in two differing ways.

(1) For a subset of evolutions where both the long and short range trajectories encounter the same background particles in near collisions, we can explicitly estimate the difference between evolutions with potential ϕ and with potential ϕ^R.

(2) We then estimate the size of the set of background particles where the tagged particles for long range and short range evolutions do not exhibit the same collisional structure, and show that the measure of this set tends to 0 as $\varepsilon \to 0$.

The first estimate is carried out in Section 4.1, and the second is carried out in Section 4.2.

We then proceed to compare the densities $P^{\varepsilon,R}$ and f^{ε} as follows. Firstly we remove a set of background scatterers so that with large probability, $(x^{\varepsilon}, v^{\varepsilon})$ encounters the same background particles within a distance at most $R\varepsilon$ that the evolution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ collides with. We then show that the set of background scatterers we have removed has probability zero in the limit $\varepsilon \to 0$.

Secondly, on those trajectories where both evolutions encounter the same collisions, we can estimate the deviation between these trajectories. This deviation is then used to quantify the spread in density of $P^{\varepsilon,R}$ with respect to f^{ε} and vice versa. The size of this spread is then used to compare the densities on these trajectories with the same collisions directly.

4.1 Preliminary Estimates on Particle Evolutions

We start this analysis by calculating estimates on the deviation of particle dynamics. The aim of this section is, for a tree $\Phi \in \mathcal{G}(\varepsilon)$, to specify the error between the short range evolution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ associated to this tree with the corresponding long range evolution under the assumption that they encounter the same near collisions.

The reader should be reminded of Lemma 2.11, which gave an estimate between solutions of long and short range dynamics with the same number of background particles, and the spirit of this lemma is used throughout this section. We do however use notation based upon the space of marked trees, and so the statements are much cleaner.

Lemma 4.2. *Let ϕ be an admissible potential with decay as in* (1.7)*, and let $k \in \mathbb{N}$ with $k < N$. Let $\Phi \in \mathcal{G}(\varepsilon) \cap \mathcal{MT}_k$, and let $(\bar{x}^{\varepsilon}, \bar{v}^{\varepsilon})$ solve the equations*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^{\varepsilon}(t) &= \bar{v}^{\varepsilon}(t), & \frac{\mathrm{d}}{\mathrm{d}t}\bar{v}^{\varepsilon}(t) &= -\frac{1}{\varepsilon}\sum_{i=1}^{k} \nabla\phi\left(\frac{\bar{x}^{\varepsilon}(t) - x_i(t)}{\varepsilon}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_i(t) &= v_i(t), & \frac{\mathrm{d}}{\mathrm{d}t}v_i(t) &= 0. \end{aligned}$$

with initial conditions and background as given by Φ. Then, for $(x^{\varepsilon,R}, v^{\varepsilon,R})$ the evolution on Φ under short range potential ϕ^R and for $t \in [0,T]$, we have

$$|x^{\varepsilon,R}(t) - \bar{x}^{\varepsilon}(t)| + |v^{\varepsilon,R}(t) - \bar{v}^{\varepsilon}(t)| \leq Ck\frac{e^{CR\,V_1(\varepsilon)^{-1}\,k}}{\varepsilon^k}e^{-CR^{\frac{3}{2}+\gamma}}$$

where recall that V_1 is the minimum separation of pre-collisional velocities.

We remark that condition (6) in Definition 3.3 of good trees ensures that the dynamics described in Φ encounter exactly k collisions, as we do not have recollisions present.

Proof: We proceed by induction on the number of collisions already encountered. We first consider the base case.

If the short range tagged particle has encountered no collisions, then since all the background particles are at least $R\varepsilon$ from it, by directly estimating the error on the right hand side of the ODEs we obtain

$$|v^{\varepsilon,R} - v^{\varepsilon}| \leq k\,T\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}$$

and by integrating the above we furthermore obtain

$$|x^{\varepsilon,R} - x^{\varepsilon}| \leq k\,T^2\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}.$$

If the short range tagged particle then encounters a collision, at the time of collision, these errors can then be used to estimate the difference of initial conditions in Lemma 2.10 and so one obtains an error of

$$|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \leq C\frac{e^{CR/\eta}}{\varepsilon}k\,T^2\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}$$

up to the end of the first collision. This concludes the base case of the argument.

Suppose now for the inductive hypothesis that the tagged particles have encountered $k-1$ collisions and that the error is bounded by

$$|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \leq C\frac{e^{C\,R\,(k-1)/\eta}}{\varepsilon^{k-1}}k\,T^2\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}.$$

Then, since the short range evolution proceeds through free flow, we have, after the $k-1$th collision, that

$$|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \leq C\frac{e^{C\,R\,(k-1)/\eta}}{\varepsilon^{k-1}}k\,T^2\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty} + k\,T\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty}.$$

Then another application of Lemma 2.10 gives the error during the kth collision as

$$\begin{aligned}|x^{\varepsilon,R} - x^{\varepsilon}| + |v^{\varepsilon,R} - v^{\varepsilon}| \leq C\frac{e^{C\,R\,k/\eta}}{\varepsilon^{k}}k\,T^2\, &\left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty} \\ &+ C\frac{e^{CR/\eta}}{\varepsilon}k\,T\, \left\|(1-\Lambda^R)\nabla\phi\right\|_{\infty},\end{aligned}$$

which concludes the proof of the lemma. □

This lemma provides an estimate between deterministic evolutions given a known number of background scatterers positions. We now estimate the maximum deviation between long range evolutions with k and with N background particles.

Lemma 4.3. *Let ϕ be an admissible long range potential with decay given by* (1.7)*, and for $k < N$ let $\Phi \in \mathcal{G}(\varepsilon) \cap \mathcal{MT}_k$. Let $(\bar{x}^\varepsilon, \bar{v}^\varepsilon)$ solve in some interval $[0, T]$ with $T < \infty$, the equations*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^\varepsilon &= \bar{v}^\varepsilon \\ \frac{\mathrm{d}}{\mathrm{d}t}\bar{v}^\varepsilon &= -\frac{1}{\varepsilon}\sum_{i=1}^{k} \nabla\phi\left(\frac{\bar{x}^\varepsilon(t)-x_i}{\varepsilon}\right) \end{cases}$$

and suppose that $(x^\varepsilon, v^\varepsilon)$ is a solution to the equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x^\varepsilon &= v^\varepsilon \\ \frac{\mathrm{d}}{\mathrm{d}t}v^\varepsilon &= -\frac{1}{\varepsilon}\sum_{i=1}^{N} \nabla\phi\left(\frac{x^\varepsilon(t)-x_i}{\varepsilon}\right) \end{cases}$$

where the background particles $1, \ldots, k$ are given as in tree Φ, and the remaining $k+1, \ldots, N$ are distributed uniformly in the region $\mathbb{T}^3$ such that

$$|x^\varepsilon(t) - x_i| > R\varepsilon$$
$$|\bar{x}^\varepsilon(t) - x_i| > R\varepsilon$$

with velocities distributed independently and identically according to $\mathcal{M}$. Then the difference

$$|\bar{x}^\varepsilon(t) - x^\varepsilon(t)| + |\bar{v}^\varepsilon(t) - v^\varepsilon(t)| \leq C\frac{e^{C\sqrt{k/\varepsilon}}N}{\sqrt{\varepsilon}}\left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}$$

where $C > 0$ is a constant dependent upon the potential ϕ and on T.

Proof: Let $z = x^\varepsilon - \bar{x}^\varepsilon$ and $w = v^\varepsilon - \bar{v}^\varepsilon$. Then (z, w) solves

$$\begin{cases} \dot{z} = w \\ \dot{w} = -\frac{1}{\varepsilon}\sum_{i=1}^{k}\left(\nabla\phi\left(\frac{x^\varepsilon(t)-x_i}{\varepsilon}\right) - \nabla\phi\left(\frac{\bar{x}^\varepsilon(t)-x_i}{\varepsilon}\right)\right) - \frac{1}{\varepsilon}\sum_{i=k+1}^{N}\nabla\phi\left(\frac{x^\varepsilon(t)-x_i}{\varepsilon}\right) \\ z(0) = 0 \\ w(0) = 0. \end{cases}$$

Using the Lipschitz nature of $\nabla\phi$ and the fact that $|x^\varepsilon(t) - x_i| > R\varepsilon$ results in the pair $(|z|_1, |w|_1)$ solving the equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}|z|_1 = |w|_1 \\ \frac{\mathrm{d}}{\mathrm{d}t}|w|_1 \leq \frac{1}{\varepsilon} C\, k \frac{|z|_1}{\varepsilon} + \frac{N-k}{\varepsilon} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}. \end{cases}$$

Performing the change of coordinates to $(\hat{z}, \hat{w})$ of

$$\hat{z} = \sqrt{C\, k/\varepsilon}|z|_1/2 + |w|_1/2,$$
$$\hat{w} = -\sqrt{C\, k/\varepsilon}|z|_1/2 + |w|_1/2,$$

with $\hat{z}(0) = 0 = \hat{w}$, this then decouples these equations and we obtain that $(\hat{z}, \hat{w})$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z} \leq \sqrt{C\, k/\varepsilon}\, \hat{z} + \frac{1}{2}\frac{(N-k)}{\varepsilon} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{w} \leq -\sqrt{C\, k/\varepsilon}\, \hat{w} + \frac{1}{2}\frac{(N-k)}{\varepsilon} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}.$$

Then using the variation of constants formula (3.6) one obtains

$$\hat{z} \leq \int_0^t e^{\int_s^t \sqrt{C\, k/\varepsilon}\,\mathrm{d}\sigma} \frac{(N-k)}{2\varepsilon} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty} \mathrm{d}s$$
$$\hat{w} \leq \int_0^t e^{-\int_s^t \sqrt{C\, k/\varepsilon}\,\mathrm{d}\sigma} \frac{(N-k)}{2\varepsilon} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty} \mathrm{d}s$$

and performing the inverse transformation, one obtains

$$|z|_1 \leq \frac{(N-k)\left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}}{2\varepsilon\sqrt{C\, k/\varepsilon}} \left(\int_0^t e^{\sqrt{C\, k/\varepsilon}(t-s)} - e^{-\sqrt{C\, k/\varepsilon}\,(t-s)}\,\mathrm{d}s \right)$$
$$|w|_1 \leq \frac{(N-k)}{2\,\varepsilon} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty} \left(\int_0^t e^{\sqrt{C\, k/\varepsilon}\,(t-s)} - e^{-\sqrt{C\, k/\varepsilon}\,(t-s)}\,\mathrm{d}s \right)$$

and simplifying this becomes

$$|z|_1 \leq C\frac{e^{\sqrt{C\, k/\varepsilon}\, t}(N-k)}{k} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty} \left(1 - e^{-\sqrt{C\, k/\varepsilon}\, t}\right)$$
$$|w|_1 \leq C\frac{e^{\sqrt{C\, k/\varepsilon}\, t}(N-k)}{2\sqrt{\varepsilon}} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty} \left(1 - e^{-\sqrt{C\, k/\varepsilon}\, t}\right)$$

and using the equivalence of norms on $\mathbb{R}^3$ we can consider the Euclidean distance on the left hand side. Then by taking the supremum over all k we obtain that

$$|x^\varepsilon(t) - \bar{x}^\varepsilon(t)| \le Ce^{C\sqrt{N/\varepsilon}}N \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}$$
$$|v^\varepsilon(t) - \bar{v}^\varepsilon(t)| \le C\frac{e^{C\sqrt{N/\varepsilon}}N}{\sqrt{\varepsilon}} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}$$

and furthermore, since $\varepsilon \ll 1$, the second of these two is much larger. Thus

$$|x^\varepsilon(t) - \bar{x}^\varepsilon(t)| \le C\frac{e^{C\sqrt{N/\varepsilon}}N}{2\sqrt{\varepsilon}} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}$$
$$|v^\varepsilon(t) - \bar{v}^\varepsilon(t)| \le C\frac{e^{C\sqrt{N/\varepsilon}}N}{2\sqrt{\varepsilon}} \left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty}.$$

This concludes the proof of the lemma. □

We now aim to combine these previous two estimates to be able to compare the maximum difference between the deterministic evolution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ and the random evolution $(x^\varepsilon, v^\varepsilon)$. This demonstrates the main point of using the deterministic evolution $(\bar{x}^\varepsilon, \bar{v}^\varepsilon)$ as we use it solely to relate the short range and random long range evolutions. This comparison is however far from optimal. We have the following.

Lemma 4.4. *Let ϕ be an admissible potential with decay as in* (1.7)*, and let $k \in \mathbb{N}$ with $k \le M(\varepsilon)$, and let $R = \varepsilon^{-1/(3+\gamma)}$. Let $(x^{\varepsilon,R}, v^{\varepsilon,R})$ be the evolution for tree $\Phi \in \mathcal{G}(\varepsilon) \cap \mathcal{MT}_k$, and let $(x^\varepsilon, v^\varepsilon)$ solve, for $t \in [0,T]$, the system*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x = v \\ \frac{\mathrm{d}}{\mathrm{d}t}v = -\frac{1}{\varepsilon}\sum_{i=1}^N \nabla\phi(\frac{x-x_i}{\varepsilon}) \end{cases}$$

with the same initial conditions and background as in Φ, and assume that the remaining $N-k$ background particles are distributed such that for all $t \in [0,T]$ we have

$$|x^\varepsilon(t) - x_i| > R\varepsilon, \; |x^{\varepsilon,R}(t) - x_i| > R\varepsilon.$$

Furthermore, suppose that there are times such that $|x^\varepsilon(\cdot) - x_i(\cdot)| \le R\varepsilon$. Then there exists $C > 0$ depending on ϕ and T such that, for all $t \in [0,T]$ we have,

$$|x^{\varepsilon,R}(t) - x^\varepsilon(t)| + |v^{\varepsilon,R}(t) - v^\varepsilon(t)| \le b(\varepsilon)$$

where

$$b(\varepsilon) = Ce^{-C(1/\varepsilon)^{\gamma/(3+\gamma)}} \tag{4.2}$$

where γ comes from the exponent in the decay (1.7) *of the potential ϕ.*

Proof: An immediate application of Lemmas 4.2 and 4.3 results in

$$|x^{\varepsilon,R}(t) - x^{\varepsilon}(t)| + |v^{\varepsilon,R}(t) - v^{\varepsilon}(t)| \leq C \left(\frac{e^{C\sqrt{k/\varepsilon}}N}{\sqrt{\varepsilon}} + \frac{k\, e^{CRV_1(\varepsilon)^{-1}}}{\varepsilon^k} \right) e^{-CR^{\frac{3}{2}+\gamma}}$$

and then plugging in the explicit forms of the parameters in this equation results in

$$|x^{\varepsilon,R}(t) - x^{\varepsilon}(t)| + |v^{\varepsilon,R}(t) - v^{\varepsilon}(t)| \leq C \left(\frac{e^{C\sqrt{|\log\varepsilon|/\varepsilon}}}{\varepsilon^{5/2}} + \frac{|\log\varepsilon|\, e^{CR|\log\varepsilon|^2}}{\varepsilon^{|\log\varepsilon|}} \right) e^{-CR^{\frac{3}{2}+\gamma}}$$

from which the statement follows. □

4.2 Removal of Bad Particle Evolutions

We now aim to address the second point on page 94. In the previous section we made certain assumptions on the background scatterers so that we could easily compare the long range and short range evolutions. We now want to characterise conditions so that these assumptions hold true for a large subset of dynamics, and show that these conditions restrict onto a set of measure 0 in the Boltzmann Grad limit.

We must ensure that two events pertaining to the tagged particle happen. Firstly the tagged particle for long range and short range evolutions must encounter the same background particles in near collisions, and secondly the remaining background must graze both the short range and long range evolutions. We first deal with the former, and to do so we define the following subset of $\mathcal{MT}$.

Definition 4.5. *We define the set $\mathcal{R}(\varepsilon)$ to be those trees $\Phi \in \mathcal{G}(\varepsilon)$ such that all impact parameter node labels are bounded by*

$$0 \leq r_i \leq R - \frac{b(\varepsilon)}{\varepsilon}\left(1 + \frac{1}{V_1(\varepsilon)}\right).$$

where $b(\varepsilon)$ is defined in equation (4.2)*, and V_1 in Definition 3.3.*

The motivation for this set is as follows. By removing a region of impact parameters near to the range of the support of the potential ϕ that is larger than the possible distance between the positions of the tagged particle under short range and long range evolutions, we ensure that the tagged particle under long range potential does indeed collide sufficiently closely with this background.

In order to demonstrate the usefulness of this set, we now must prove two properties. Firstly we must show that it has small measure, and that the measure decays to 0 as $\varepsilon \to 0$. Secondly we must prove that by removing this set, the dynamics $(x^{\varepsilon,R}, v^{\varepsilon,R})$ and $(x^{\varepsilon}, v^{\varepsilon})$ exhibit the same collisions with the same background.

We start with the former consideration.

Lemma 4.6. *Suppose that ϕ is an admissible long range potential with decay as in (1.7), which we recall means that there is a $\rho_2 > 0$ and a $\gamma > 0$ such that for all $\rho > \rho_2$ we have*

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) \leq Ce^{-C\rho^{\frac{3}{2}+\gamma}}$$

and recall the sets $\mathcal{G}(\varepsilon)$ and $\mathcal{R}(\varepsilon)$ in Definitions 3.3 and 4.5, and the parameters

$$\begin{aligned} R(\varepsilon) &= \varepsilon^{-1/(3+\gamma)}, \\ M(\varepsilon) &= |\log \varepsilon|, \\ V_1(\varepsilon) &= \frac{1}{|\log \varepsilon|}, \\ V_2(\varepsilon) &= |\log \varepsilon|. \end{aligned}$$

Then for $P^{\varepsilon,R}$ the short range tagged particle density on $\mathcal{G}(\varepsilon)$ we have

$$P_t^{\varepsilon,R}(\mathcal{G}(\varepsilon \setminus \mathcal{R}(\varepsilon)) \to 0$$

as $\varepsilon \to 0$.

Proof: Recall that λ is the Lebesgue measure on $\mathcal{MT}$. We start by observing that we have Lebesgue measure of the time label and velocity label of $TV_2(\varepsilon)^3$ since there is no restriction on those values. Therefore by using the asymptotics of the parameters, we have

$$\begin{aligned} \lambda\left(\mathcal{G}(\varepsilon) \setminus \mathcal{R}(\varepsilon)\right) &= V_2(\varepsilon)^3 \sum_{k=1}^{M(\varepsilon)} \left(T\, V_2(\varepsilon)^3 \, \frac{b(\varepsilon)}{\varepsilon} \left(1 + \frac{1}{V_1(\varepsilon)} \right) \right)^k \\ &\leq |\log \varepsilon|^3 \sum_{k=1}^{|\log \varepsilon|} \left(T |\log \varepsilon|^3 \, \frac{b(\varepsilon)}{\varepsilon} \, (1 + |\log \varepsilon|) \right)^k \\ &\leq |\log \varepsilon|^3 \sum_{k=1}^{|\log \varepsilon|} \left(T\, b(\varepsilon)\, (1 + |\log \varepsilon|)^4 \right)^k \\ &\leq T \, \frac{b(\varepsilon)}{\varepsilon} \, (1 + |\log \varepsilon|)^7 \sum_{k=0}^{\infty} \left(T \, \frac{b(\varepsilon)}{\varepsilon} \, (1 + |\log \varepsilon|)^4 \right)^k . \end{aligned}$$

Then, due to the form of $b(\varepsilon)$, there is an ε' such that for all $\varepsilon < \varepsilon'$ we have $T\,\frac{b(\varepsilon)}{\varepsilon}\,(1+|\log\varepsilon|)^4 < 1$, and so the sum is finite. Since the multiplying factor tends to 0 as $\varepsilon \to 0$, we have that

$$\lambda\left(\mathcal{G}(\varepsilon)\setminus\mathcal{R}(\varepsilon)\right) \to 0$$

as well.

Since $P_t^{\varepsilon,R}$ is absolutely continuous with respect to the Lebesgue measure, we also have

$$P_t^{\varepsilon,R}(\mathcal{G}(\varepsilon)\setminus\mathcal{R}(\varepsilon)) \to 0$$

as $\varepsilon \to 0$. □

We now turn to show the other requirement, that by restricting the dynamics, we encounter the same near collisions. To do this, we first prove a geometric result on the minimum radius of a collision.

Lemma 4.7. *Suppose that for spatial scale $\varepsilon > 0$ in a binary collision under potential ϕ^R, the impact parameter r and relative velocity w are bounded by*

$$0 \le r \le R - \frac{b(\varepsilon)}{\varepsilon} - \frac{b(\varepsilon)}{\varepsilon|w|},$$
$$|w| \ge \frac{1}{|\log\varepsilon|}.$$

Then for ε sufficiently small the minimum radius is bounded by

$$\rho_\star \le R\varepsilon - b(\varepsilon).$$

Proof: The minimum radius satisfies the equation

$$1 = \frac{r^2}{\rho_\star^2} + \frac{\frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)}{|w|^2}$$

from conservation of angular momentum. Rearranging this, we obtain

$$\rho_\star^2 = r^2 + \frac{\rho_\star^2\frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)}{|w|^2},$$

and inputting the constraint on r into this equation results in

$$\rho_\star^2 = r^2 + \frac{\rho_\star^2\frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)}{|w|^2} \le (R\varepsilon - b)^2 - \frac{2b}{|w|}(R\varepsilon - b) + \frac{b^2}{|w|^2} + \frac{\rho_\star^2\frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)}{|w|^2},$$

and to conclude we must show the final three terms on the right hand side of this are negative. For ε sufficiently small, we have

$$\frac{1}{|\log\varepsilon|} \geq \frac{b^2 + \rho_\star^2 \frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)}{2b(R\varepsilon - b)}$$

due to the specific form of b. Therefore

$$\frac{1}{|w|} \leq \frac{2b(R\varepsilon - b)}{b^2 + \rho_\star^2 \frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)}$$

and so

$$\frac{1}{|w|}\left(\left(b^2 + \rho_\star^2 \frac{1}{\varepsilon}\phi^R\left(\frac{\rho_\star}{\varepsilon}\right)\right)\frac{1}{|w|} - 2b(R\varepsilon - b)\right) \leq 0$$

as required. □

This estimate is then used to prove that the removal of the impact parameters in Definition 4.5 ensures that the short and long range evolutions exhibit the same collisional structure with high probability. We recall from Chapter 3 the notation of $\omega = \{x_1, v_1, \dots, x_N, v_N\}$ to be the initial positions and velocities of the background particles. The initial conditions of the ith background particle are then denoted by ω_i.

Lemma 4.8. *Suppose that $\Phi \in \mathcal{R}(\varepsilon)$ with $\varepsilon > 0$ sufficiently small. Furthermore suppose that $R = \varepsilon^{-\frac{1}{3+\gamma}}$, then we have*

$$\mathbb{P}\Big[x^{\varepsilon,R} \text{ and } x^{\varepsilon} \text{ have same collisions} \\ \Big|\omega_{k+1}, \dots, \omega_N, \text{ s.t. } \forall s \in [0,T],\ |x^{\varepsilon,R} - (x_i + sv_i)| > R\varepsilon + 2b(\varepsilon)\Big] = 1.$$

Proof: We aim to show that by restricting the impact parameters using the set $\mathcal{R}(\varepsilon)$ we ensure that the evolutions $x^{\varepsilon,R}$ and x^{ε} encounter the same background. We prove by induction on the number of collisions already encountered.

If one has encountered no collisions, then under the constraint that the background particles are at least $R\varepsilon + 2b(\varepsilon)$ from $x^{\varepsilon,R}$, by integrating the equations (1.1) we have

$$|x^{\varepsilon,R}(t) - x^{\varepsilon}(t)| \leq N\,t\,\left\|(1-\Lambda^R)\nabla\phi\right\|_{L^\infty} \leq b(\varepsilon)$$

and so the long range evolution does not encounter a near collision with any of the $N - n(\Phi)$ background particles not described in the tree Φ.

Now suppose that the short range evolution collides at time t_1. Again by

Lemma 2.11, we know that

$$|x^{\varepsilon,R}(t) - x^{\varepsilon}(t)| \le b(\varepsilon)$$

and we must ensure that the long range tagged particle also encounters a collision with this background. Since $\Phi \in \mathcal{R}(\varepsilon)$, the impact parameter of the collision is thus smaller than $R - b(\varepsilon)(1 + 1/V_1(\varepsilon))/\varepsilon$ and so by an application of Lemma 4.7, we know that the minimum radius of the collision is smaller than $R\varepsilon - b(\varepsilon)$ thus ensuring the long range evolution has a near collision with this background particle.

This then concludes the base case of the inductive argument. The remainder of the argument is identical to the base case. We use Lemma 2.11 to estimate the error between the long and short range evolutions, before using Lemma 4.7 to ensure that the long range evolution encounters the same near collision. □

It should be clear that the conditioning on the background particles in the previous lemma has probability 0 in the limit $\varepsilon \to 0$. Indeed, the conditioning forces

$$\inf_{t\in[0,T]} |x^{\varepsilon,R} - x_s| \notin [R\varepsilon - b(\varepsilon)(1 + 1/V_1(\varepsilon)),\, R\varepsilon + 2b(\varepsilon)],$$

for all time $t \in [0,T]$. This then forces the initial positions and velocities of the background particles to lie outside a cylinder of size $\left(CT\,V_2(\varepsilon)\,b(\varepsilon))^2\right)^{N-n(\Phi)}$, which we observe tends to 0 as $\varepsilon \to 0$.

4.3 Comparison of Densities

We now aim to use the estimates in the previous two sections to compare the tagged particle densities and to show that the difference between them tends to 0 as $\varepsilon \to 0$. We aim to exploit the structure of the dynamics that we have described in the previous two sections.

We first compare for those trees in the space $\mathcal{R}(\varepsilon)$ defined in the previous section. We start with a comparison of the densities for evolutions when the dynamics encounter the same collisions when we test with indicator functions. This result is then used to prove convergence for bounded continuous functions.

Lemma 4.9. *For ϕ an admissible long range potential with decay as in* (4.1) *meaning that there is a $\rho_2 > 0$ and a $\gamma > 0$ such that for all $\rho > \rho_2$ we have*

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) - \le Ce^{-C\rho^{\frac{3}{2}+\gamma}}$$

and for

$$\begin{aligned} R(\varepsilon) &= \varepsilon^{-1/(3+\gamma)}, \\ M(\varepsilon) &= |\log \varepsilon|, \\ V_1(\varepsilon) &= \frac{1}{|\log \varepsilon|}, \\ V_2(\varepsilon) &= |\log \varepsilon|, \end{aligned}$$

we have, for $\Omega \subset \mathcal{U}$, the relation

$$\int_\Omega f_t^\varepsilon \, \mathbb{P}[A] \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R} \, \mathbb{1}_{\mathcal{R}(\varepsilon)} \, \mathrm{d}\Phi \to 0,$$

where the set

$$A := \left\{\omega : \ |x^{\varepsilon,R} - x_s| \notin [R\varepsilon - b(\varepsilon)(1 + 1/V_1(\varepsilon)), \ R\varepsilon + 2b(\varepsilon)]\right\}.$$

The proof of this lemma aims to combine the results of the previous two sections. We use the set $\mathcal{R}(\varepsilon)$ and the comparable set A of phase space points to restrict to dynamics with the desirable collisional structure where one can identify the long and short range evolutions. Using these assumptions on the dynamics, we can then use the estimates in Section 4.1 to describe the spread of the densities f^ε and $P^{\varepsilon,R}$, which then allows us to quantify the difference between them, and show that it tends to 0 as $\varepsilon \to 0$.

Proof: We first observe that, by Lemma 4.4, there is some radius $b > 0$ dependent upon ε so that the evolution $(x^{\varepsilon,R}, v^{\varepsilon,R})$ for tree $\Phi \in \mathcal{R}(\varepsilon)$ and the evolution ending at (x, v) with N background particles, lie within b of each other. Thus for $\Omega \subset \mathcal{U}$, we obtain

$$\int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi) \mathbb{1}_{\mathcal{R}(\varepsilon)} \, \mathrm{d}\Phi \leq \int_{\Omega_b} f^\varepsilon(t, x, v) \mathbb{P}[A] \, \mathrm{d}x \, \mathrm{d}v.$$

where

$$\Omega_b = \{(x, v) \in \mathcal{U} : \ \exists (y, w) \in \Omega \text{ such that } |x - y| < b, \, |v - w| < b\}$$

is the set of points within b of the set Ω.

Furthermore, the symmetry of Lemma 4.4 enables one to conclude that

$$\int_\Omega f^\varepsilon(t, x, v) \, \mathbb{P}[A] \, \mathrm{d}x \, \mathrm{d}v \leq \int_{S_t^{\varepsilon,R}(\Omega_b)} P_t^{\varepsilon,R}(\Phi) \, \mathbb{1}_{\mathcal{R}(\varepsilon)} \, \mathrm{d}\Phi$$

We therefore use these two relations to estimate the difference $\int f^\varepsilon - \int P_t^{\varepsilon,R}$, from above in the following manner

$$\int_\Omega f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v - \int_{S_t(\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi$$
$$\leq \int_{S_t(\Omega_b)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi - \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi$$
$$= \int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi,$$

and secondly we can also use them to bound the difference $\int f^\varepsilon - \int P_t^{\varepsilon,R}$ from below by

$$\int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi - \int_\Omega f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v$$
$$\leq \int_{\Omega_b} f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v - \int_\Omega f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v$$
$$= \int_{\Omega_b\setminus\Omega} f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v.$$

Therefore,

$$-\int_{\Omega_b\setminus\Omega} f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v$$
$$\leq \int_\Omega f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi$$
$$\leq \int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi. \quad (4.3)$$

We then compute the outside integrals of this expression. Firstly, observe that from the evolution equation for $P_t^{\varepsilon,R}$ in equation (3.2), we can estimate, for $\Phi \in \mathcal{G}(\varepsilon)$, with ε sufficiently small,

$$P_t^{\varepsilon,R}(\Phi) \leq (4R\,V_2(\varepsilon))^{M(\varepsilon)}$$

by estimating the maximum of the gain term of the density. Therefore $P_t^{\varepsilon,R} \in L^\infty(\mathcal{G}(\varepsilon))$.

We thus have

$$\begin{aligned}
\int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi &\leq \left\|P_t^{\varepsilon,R}\right\|_{L^\infty}\int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)}\mathrm{d}\Phi \\
&= \left\|P_t^{\varepsilon,R}\right\|_{L^\infty}\sum_{k=0}^{M(\varepsilon)}\int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)\cap\mathcal{MT}_k}\mathrm{d}\Phi \\
&\leq (4R\,V_2(\varepsilon))^{M(\varepsilon)}\sum_{k=0}^{M(\varepsilon)}\lambda(S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)\cap\mathcal{MT}_k)
\end{aligned}$$

We then calculate the size of these sets. The velocity constraint in $S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)$ enforces the initial velocity of the tagged particle to lie in a region of size at most

$$\mathrm{diam}(\Omega)^2\,b,$$

and the impact parameters and velocities lie in sets of size at most $C\,R\,V_2(\varepsilon)^3$. The times lie in $[0,T)$, and therefore

$$\lambda(S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)\cap\mathcal{MT}_k)\leq C^k\,T^k\,R^k\,V_2(\varepsilon)^{3k+2}\,b(\varepsilon)$$

We then have

$$\begin{aligned}
\sum_{k=0}^{M(\varepsilon)}\lambda(S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)\cap\mathcal{MT}_k) &\leq \sum_{k=0}^{M(\varepsilon)}C^k\,T^k\,R^k\,V_2(\varepsilon)^{3k+2}\,b(\varepsilon) \\
&\leq V_2(\varepsilon)^2\,b(\varepsilon)\sum_{k=0}^{M(\varepsilon)}C^k\,T^k\,R^k\,V_2(\varepsilon)^{3k} \\
&\leq V_2(\varepsilon)^2\,b(\varepsilon)\frac{\left(C\,T\,R\,V_2(\varepsilon)^3\right)^{M(\varepsilon)+1}-1}{C\,T\,R\,V_2(\varepsilon)^3-1}
\end{aligned}$$

Therefore

$$\int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi \leq (4R\,V_2(\varepsilon))^{M(\varepsilon)}\,V_2(\varepsilon)^2\,b(\varepsilon)\frac{\left(C\,R\,V_2(\varepsilon)^3\right)^{M(\varepsilon)+1}-1}{C\,R\,V_2(\varepsilon)^3-1}$$

For the other side of (4.3) we first must show that f^ε is in L^∞. With T_k^{-t} the solution operator for (1.4) with k background particles at $(x_i,\, v_i)$, we have

$$\begin{aligned} f^\varepsilon(t,x,v) &= \int \prod_{i=1}^{N} \mathcal{M}(v_i) f_0(T_N^{-t}(x,v))\,\mathrm{d}v_1 \ldots \mathrm{d}v_N \\ &\leq \|f_0\|_{L^\infty} \int \prod_{i=1}^{N} \mathcal{M}(v_i)\,\mathrm{d}v_1 \ldots \mathrm{d}v_N \\ &\leq \|f_0\|_{L^\infty} \end{aligned}$$

and since we assume $f_0 \in L^\infty$, by taking the supremum over x, v we have $f^\varepsilon \in L^\infty$.

For the other side of the inequality (4.3) we can then estimate to obtain

$$\begin{aligned} \int_{\Omega_b \setminus \Omega} f^\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v &\leq \|f^\varepsilon\|_{L^\infty} \int_{\Omega_b \setminus \Omega} \mathrm{d}x\,\mathrm{d}v \\ &\leq \|f_0\|_{L^\infty}\; b\,C\mathrm{diam}(\Omega)^2 \\ &\leq C\,\|f_0\|_{L^\infty}\; b\,V_2(\varepsilon)^2. \end{aligned}$$

Then, since

$$\begin{aligned} &\left| \int_\Omega f^\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi \right| \\ &\qquad \leq \max\left\{ \int_{S_t^{\varepsilon,R}(\Omega_b\setminus\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi,\, \int_{\Omega_b\setminus\Omega} f^\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \right\} \end{aligned}$$

we obtain

$$\begin{aligned} &\left| \int_\Omega f^\varepsilon(t,x,v)\mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega)} P_t^{\varepsilon,R}(\Phi)\mathbb{1}_{\mathcal{R}(\varepsilon)}\,\mathrm{d}\Phi \right| \\ &\qquad \leq (4R\,V_2(\varepsilon))^{M(\varepsilon)}\,V_2(\varepsilon)^2\,b(\varepsilon)\frac{\left(C\,R\,V_2(\varepsilon)^3\right)^{M(\varepsilon)+1} - 1}{C\,R\,V_2(\varepsilon)^3 - 1} \end{aligned}$$

since the maximum is bounded by the former of the two bounds.

Since $b(\varepsilon)$ tends to 0 exponentially fast, and all other terms diverge at most algebraically, this term tends to 0 as $\varepsilon \to 0$ as required. □

Before proving the main theorem of the chapter, we must first address those evolutions which are not described by the sets $\mathcal{R}(\varepsilon)$ and A. We have the following.

Lemma 4.10.

$$\int_{\mathcal{G}(\varepsilon)\setminus\mathcal{R}(\varepsilon)} h(\Phi)P_t^{\varepsilon,R}(\Phi)\,\mathrm{d}\Phi \to 0$$

and

$$\mathbb{P}[A] \to 1$$

as $\varepsilon \to 0$.

Proof: We observe that, since $h \in L^\infty$ the first term tends to 0 by an application of Lemma 4.6.

For the second term, we estimate the probability of the set A by estimating the size of the cylinder one must remove for each background particle to lie outside of A. This thus results in

$$\begin{aligned}\mathbb{P}[A^C] &\leq CV_2(\varepsilon)^{M(\varepsilon)} \left((R\varepsilon + 2b(\varepsilon))^2 - (R\varepsilon - b(\varepsilon)(1 + 1/V_1(\varepsilon)))^2\right)^{M(\varepsilon)} \\ &\leq CV_2(\varepsilon)^{M(\varepsilon)}\, b(\varepsilon)^{2M(\varepsilon)}\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$. □

We are now able to conclude the chapter with a proof of Theorem 3.

Proof: (of Theorem 3)

Suppose that $h \in C_b([0,T) \times \mathcal{U})$. We then write

$$\begin{aligned}\int_{\mathcal{U}} h f_t^\varepsilon \,\mathrm{d}x\,\mathrm{d}v - \int_{\mathcal{MT}} h\, P_t^{\varepsilon,R}\,\mathrm{d}\Phi \leq{}& \int_{\mathcal{U}} h\, f_t^\varepsilon (1 - \mathbb{P}[A])\,\mathrm{d}x\,\mathrm{d}v \\ &+ \int_{\mathcal{U}} h\, f_t^\varepsilon \mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathcal{R}(\varepsilon)} h\, P_t^{\varepsilon,R}\,\mathrm{d}\Phi \\ &- \int_{\mathcal{MT}\setminus\mathcal{R}(\varepsilon)} h\, P_t^{\varepsilon,R}\,\mathrm{d}\Phi.\end{aligned}$$

Then by Lemma 4.10 we have the first and final terms of this expression tending to zero. We are thus left to analyse the middle term. We use a bootstrapping type argument.

We observe that for a test function $\sum_{i=1}^m \mathbb{1}_{\Omega_i}$, we have,

$$\begin{aligned}&\left|\int_{\mathcal{U}} f_t^\varepsilon\, \mathbb{P}[A] \sum_{i=1}^m \mathbb{1}_{\Omega_i}\,\mathrm{d}x\,\mathrm{d}v - \int_{S_t^{\varepsilon,R}(\mathcal{U})} P_t^{\varepsilon,R}\, \mathbb{1}_{\mathcal{R}(\varepsilon)} \sum_{i=1}^m \mathbb{1}_{\Omega_i}\,\mathrm{d}\Phi\right| \\ &\qquad\leq \sum_{i=1}^m \left|\int_{\Omega_i} f_t^\varepsilon\, \mathbb{P}[A]\,\mathrm{d}x\,\mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega_i)} \mathbb{1}_{\mathcal{R}(\varepsilon)}\, P_t^{\varepsilon,R}\,\mathrm{d}\Phi\right|\end{aligned}$$

and thus the right hand side tends to 0 as an immediate conclusion of Lemma 4.9.

We now suppose that $h \in C_b$. A standard result of measure theory states that there exists an increasing sequence of simple functions that converge uniformly to h. We let $\alpha > 0$ be arbitrary and choose a simple function h_k such that

$$\sup_{x,v} |h_k - h| \leq \frac{\alpha}{4},$$

and we write

$$h_k = \sum_{i=1}^{m} c_i \mathbb{1}_{\Omega_i}.$$

Then we have

$$\begin{aligned}
&\left| \int_{\mathcal{U}} f_t^\varepsilon \, \mathbb{P}[A] \, h \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\mathcal{U})} P_t^{\varepsilon,R} \, \mathbb{1}_{\mathcal{R}(\varepsilon)} \, h \, \mathrm{d}\Phi \right| \\
&\qquad = \left| \int_{\mathcal{U}} f_t^\varepsilon \, \mathbb{P}[A] \sum_{i=1}^{m} \mathbb{1}_{\Omega_i} \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\mathcal{U})} P_t^{\varepsilon,R} \, \mathbb{1}_{\mathcal{R}(\varepsilon)} \sum_{i=1}^{m} \mathbb{1}_{\Omega_i} \, \mathrm{d}\Phi \right| \\
&\qquad\qquad + \frac{\alpha}{4} \left| \int_{\mathcal{U}} f_t^\varepsilon \, \mathbb{P}[A] \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\mathcal{U})} P_t^{\varepsilon,R} \, \mathbb{1}_{\mathcal{R}(\varepsilon)} \, \mathrm{d}\Phi \right| \\
&\qquad \leq \sum_{i=1}^{m} c_i \left| \int_{\Omega_i} f_t^\varepsilon \, \mathbb{P}[A] \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega_i)} \mathbb{1}_{\mathcal{R}(\varepsilon)} \, P_t^{\varepsilon,R} \, \mathrm{d}\Phi \right| + \frac{\alpha}{2}.
\end{aligned}$$

We then choose ε sufficiently small so that

$$(4R \, V_2(\varepsilon))^{M(\varepsilon)} \, V_2(\varepsilon)^2 \, b(\varepsilon) \frac{\left(C \, R \, V_2(\varepsilon)^3\right)^{M(\varepsilon)+1} - 1}{C \, R \, V_2(\varepsilon)^3 - 1} \leq \frac{\alpha}{2 \sum_{i=1}^{m} c_i},$$

which, by Lemma 4.9, results in

$$\left| \int_{\Omega_i} f_t^\varepsilon \, \mathbb{P}[A] \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\Omega_i)} \mathbb{1}_{\mathcal{R}(\varepsilon)} \, P_t^{\varepsilon,R} \, \mathrm{d}\Phi \right| \leq \frac{\alpha}{2 \sum_{i=1}^{m} c_i}$$

and therefore

$$\left| \int_{\mathcal{U}} f_t^\varepsilon \, \mathbb{P}[A] \, h \, \mathrm{d}x \, \mathrm{d}v - \int_{S_t^{\varepsilon,R}(\mathcal{U})} P_t^{\varepsilon,R} \, \mathbb{1}_{\mathcal{R}(\varepsilon)} \, h \, \mathrm{d}\Phi \right| \leq \alpha$$

as required.

For arbitrary h, we split into the positive and negative parts and then apply the previous rationale to the separate functions to conclude. This thus concludes the proof of Theorem 3. □

Chapter 5

Comparison of Solutions of Related Boltzmann Equations

We now aim to provide an analysis of the contribution of grazing collisions on solutions to the linear Boltzmann equation. The argument uses the estimates in Chapter 2 to compare the collision operators for long and short range dynamics, as well as a simple compactness argument to extract a solution of the long range Boltzmann equation.

The argument we use is of a similar flavour to [24] and [4], although both are different. Both arguments differ in the manner by which one compares the collision operators. The former is applicable only to inverse power law potentials, and the comparison of collision operators proceeds by comparing the Boltzmann kernels for cut-off and long range interactions. These arguments are in a similar vein to the proof of Lemma 2.5. Ayi [4] on the other hand states that to compare L and L^R it is enough to compare L^R and L^{2R} and one performs this by analysing solutions of an ODE to compare the post collisional velocities of the scattering by ϕ^R and ϕ^{2R} which they then input back into the difference $L^R - L^{2R}$.

The argument we take proceeds as follows. We use condition (2) of Definition 1.5 to bound the solutions f^R of the linear Boltzmann equation for ϕ^R independently of R. This then enables us to extract a convergent subsequence, and we are thus required to show that this limit does indeed satisfy the linear Boltzmann equation for ϕ.

At this point, our argument differs from [24, 4] because we instead use the estimate in Lemma 2.6 on the difference between post collisional velocities with scattering under ϕ^R and ϕ to produce an estimate on the difference between the linear collision operators L^R and L. It is in observing the estimates between L^R and

L that one realises why we worked so hard in the proof of Lemma 2.6 to produce such a bizarre looking estimate.

Firstly, let us recall that a solution f of the linear Boltzmann equation for admissible long range potential ϕ satisfies, from Definition 1.6, for $h \in C_c^\infty([0,T) \times \mathcal{U})$, the equation

$$-\int_0^T \int_{\mathcal{U}} (\partial_t h + v \cdot \nabla_x h)\; f \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t - \int_{\mathcal{U}} f_0\, h(0)\,\mathrm{d}x\,\mathrm{d}v = \int_0^T \langle L(f),\, h\rangle\,\mathrm{d}t$$

where

$$\langle L(f),\, h\rangle := \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{\mathcal{S}} (h(v') - h(v))\, f(v)\, \mathcal{M}(v_\star)\, |v_\star - v|\,\mathrm{d}S\; \mathrm{d}v_\star\; \mathrm{d}v\; \mathrm{d}x$$

for v' the pre-collisional velocity as in (2.1) for the potential ϕ.

Furthermore recall that weak solutions of the linear Boltzmann equation for ϕ^R satisfy, for $h \in C_c^\infty([0,T) \times \mathcal{U})$, the equation

$$-\int_0^T \int_{\mathcal{U}} (\partial_t h + v \cdot \nabla_x h)\; f^R \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t - \int_{\mathcal{U}} f_0\, h(0)\,\mathrm{d}x\,\mathrm{d}v = \int_0^T \langle L^R(f^R),\, h\rangle\,\mathrm{d}t \quad (5.1)$$

where

$$\langle L^R(f^R),\, h\rangle := \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{B_R} (h(v'^{,R}) - h(v))\, f^R(v)\, \mathcal{M}(v_\star)\, |v_\star - v|\,\mathrm{d}S\; \mathrm{d}v_\star\; \mathrm{d}v\; \mathrm{d}x,$$

where $v'^{,R}$ is the pre-collisional velocity of the tagged particle for ϕ^R.

Finally recall that both solutions are required to have the regularity of

$$\int_{\mathcal{U}} (1 + |v|^2)\, f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty$$

for $t \in [0,T]$.

In this chapter we prove the following result.

Theorem 4. *Suppose that ϕ is an admissible long range potential such that there is a $\rho_2 > 0$ and $s > 2$ with*

$$\psi(\rho) \leq \rho^{-s}$$

for all $\rho > \rho_2$. Suppose that f_0 the initial density satisfies definition 1.5.

Then f^R the weak solution of (5.1) *converges as $R \to \infty$ weakly-$\star$ in L^∞ to f a weak solution of the linear Boltzmann equation* (1.5).

Remark 5.1. *This is the only section where all conditions on the initial density* *are required. We can however replace condition (2) with any condition that enables uniform in R estimates on solutions to the linear Boltzmann equations for ϕ^R. This is removed in the paper [25].*

To simplify the estimates between the linear collision operators L^R and L we use as an intermediary the long range collision operator with Grad's angular cut off applied. For $h \in C_c^\infty([0,T] \times \mathcal{U})$ this is given by

$$\langle \widetilde{L}^R(f),\, h\rangle := \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{B_R} (h(v') - h(v))\, f(v)\, \mathcal{M}(v_\star)\, |v_\star - v| \,\mathrm{d}S \;\mathrm{d}v_\star \,\mathrm{d}x \,\mathrm{d}v.$$

This cut off is a formal removal of grazing collisions by restricting the domain of the impact parameter to exclude these types of collisions, as introduced in [29]. It should be noted that this cut off is completely unphysical, and so it never appeared in the analysis of the particle dynamics.

The proof proceeds in three steps:

(1) Firstly we use the maximum principle for the solutions f^R for short range potential ϕ^R to extract a convergent subsequence.

(2) Secondly we use Lemma 2.6 to compute estimates on the collision operators.

(3) Finally we combine the two previous steps to show that the limit of the subsequence is a solution of the linear Boltzmann equation associated to ϕ.

5.1 Maximum Principle

Lemma 5.2. *Suppose that f^R is a weak solution of the linear Boltzmann equation associated to ϕ^R, such that $f_0 \geq 0$.*

Then for all $t > 0$ we have $f^R(t, x, v) \geq 0$ for almost all $(x, v) \in \mathcal{U}$.

Remark 5.3. *A necessary condition for a maximum principle for the linear Boltzmann equation to hold is that the gain part of the collision operator can be uniformly bounded in v, namely*

$$\sup_v L_+(f) = \sup_v \int_{\mathbb{R}^3} \int_{\mathcal{S}} f'\, \mathcal{M}'_\star\, |v_\star - v| \,\mathrm{d}S \,\mathrm{d}v_\star < \infty$$

We only require the maximum principle for solutions associated to ϕ^R, and so we state only for those.

Proof: By Lemma 2.13 it suffices to show the result for the mild solution.

We claim that the result follows if we can show that the gain operator $L^R_+ \colon L^1(\mathcal{U}) \to L^1(\mathcal{U})$ is a positive operator. Indeed, by formula [6, Theorem 4.9], we can write a mild solution to equation (5.1) as

$$f^R(t,x,v) = f_0(x - tv, v) e^{-\int_0^t L^R_- \, \mathrm{d}s} + \int_0^t L^R_+(f^R) e^{-\int_s^t L^R_- \, \mathrm{d}r} \, \mathrm{d}s,$$

and then one observes that if L^R_+ preserves the sign of its argument then all terms on the right hand side are positive, and so f^R is positive for all time.

To show that L^R_+ is a positive operator, observe that, if $f \geq 0$ then

$$L^R_+(f) = \int_{\mathbb{R}^3} \int_{B_R} f'^{,R} \, \mathcal{M}'^{,R}_\star \, |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star \geq 0$$

since all terms on the right hand side are positive. This concludes the proof. □

The outcome of using the maximum principle here is that, when combined with point (2) of Definition 1.5, one obtains uniform in R estimates on weak solutions to the linear Boltzmann equation for ϕ^R.

Lemma 5.4. *Suppose that, for all $R > 0$, we have f^R is a weak solution to the linear Boltzmann equation for ϕ^R as in* (5.1)*, all with initial density f_0 satisfying Definition 1.5, and in particular that*

$$0 \leq f_0 \leq C\mathcal{M}$$

for Maxwellian $\mathcal{M}$.

Then there exists a function f such that $f^R \to f$ weak-$\star$ in L^∞, up to a subsequence.

Proof: Firstly, we have that $\mathcal{M}'^{,R}\mathcal{M}'^{,R}_\star = \mathcal{M}_\star\mathcal{M}$ for all $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$, (see for instance [20]) and therefore we have that $L^R(\mathcal{M}) = 0$ for all $R > 0$.

Therefore, for each $R > 0$ we can apply the maximum principle to the function $F(t,x,v) = C\mathcal{M}(x - tv, v) - f^R(t,x,v)$. Since $C\mathcal{M}(x,v) - f_0(x,v) \geq 0$, we have

$$0 \leq C\mathcal{M}(x - tv, v) - f^R(t,x,v)$$

for all R and for all $(t,x,v) \in [0,T] \times \mathcal{U}$. Using the maximum principle on f^R again then results in

$$0 \leq f^R(t,x,v).$$

Combining these two we see that the sequence is uniformly bounded in L^∞.

Endowing L^∞ with the weak-$\star$ topology, Banach Alaoglu then gives the existence of a convergent subsequence. $\square$

5.2 Comparison of the Collision Operators

Before showing that f is a weak solution of the linear Boltzmann equation, we first compare the collision operators L and L^R. This comparison will then be used directly to show that f is a weak solution of the linear Boltzmann equation.

The estimate on the difference between the deviation angles for short and long range interactions given in Lemma 2.6, which we recall is

$$|\theta^R(r,w) - \theta(r,w)| \leq \begin{cases} \frac{C}{1+\eta^2 r^s} & r > R - 1 - 1/\eta \\ \\ \frac{C\,\kappa(r,R)}{\eta^2} & \text{otherwise} \end{cases}$$

is the main tool we use in order to compare the weak formulations of the linear collision operators.

The argument is a simple application of this estimate to show that $|v' - v'^{,R}|$ is small. It is however complicated because Lemma 2.6 is only valid for relative velocities bounded away from zero. The estimate for large relative velocities is therefore simple, but for small relative velocities we instead use a similar estimate to Lemma 2.6 but with an added dependency on the relative velocity. These considerations give us the following.

Lemma 5.5. *Let $R > 0$, and suppose that ϕ is an admissible long range potential with a $\rho_2 > 0$ and $s > 2$ such that*

$$\psi(\rho) \leq \rho^{-s}$$

for $\rho > \rho_2$. Then for all $f \in L^1(\mathbb{R}^3, (1+|v|^2)\,\mathrm{d}v)$ and for all test functions $h \in C_c^\infty([0,T)\times\mathcal{U})$ we have

$$|\langle L^R(f),\, h\rangle - \langle \widetilde{L}^R(f),\, h\rangle| \leq C\,\|\nabla h\|_{L^\infty} \left(\int_0^R r\,(\log^2 R)\,\kappa(r,R)\,\mathrm{d}r + \frac{C\,\kappa_2(r,R)}{\log^3 R} \right) \\ \times \left\|(1+|v^2|)\,f\right\|_{L^1} \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}.$$

Remark 5.6. *The proof is very simple, essentially several applications of the triangle inequality, but is important as it demonstrates how the estimates in Lemma 2.6* *are used to compare the collision operators.*

Proof: We have

$$
\begin{aligned}
&|\langle L^R(f),\, h\rangle - \langle \widetilde{L}^R(f),\, h\rangle| \\
&\qquad = \bigg| \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{B_R} (h'^R - h)\, f\, \mathcal{M}_\star\, |v_\star - v| \,\mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v \\
&\qquad\qquad - \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{B_R} (h' - h)\, f\, \mathcal{M}_\star\, |v_\star - v| \,\mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v \bigg|
\end{aligned}
$$

and rearranging, this becomes

$$
\begin{aligned}
&|\langle L^R(f),\, h\rangle - \langle \widetilde{L}^R(f),\, h\rangle| \\
&\qquad = \bigg| \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{B_R} (h'^R - h')\, f\, \mathcal{M}_\star\, |v_\star - v| \;\mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v \bigg| \\
&\qquad \le \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{B_R} |h'^R - h'|\, f\, \mathcal{M}_\star\, |v_\star - v| \,\mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v
\end{aligned}
$$

and then, since h is C^∞ it is also Lipschitz and so we estimate it by

$$
|h'^R - h'| \le C\, \|\nabla h\|_{L^\infty}\, |v'^R - v'|
$$

and by analysing this difference in velocities, we obtain

$$
\begin{aligned}
|v'^R - v'| &\le \big(|\cos\theta^R - \cos\theta| + |\sin\theta^R - \sin\theta|\big)\, |v_\star - v| \\
&\le C\, |\theta^R - \theta|\, |v_\star - v|.
\end{aligned}
$$

For $|v_\star - v| > \eta := \frac{1}{\log R}$, we use Lemma 2.6 to estimate this difference in deviation angles by $R^{-s}\, \kappa(r, R)\, \eta^{-2}$. Inputting this into the above and evaluating the angular integral in $\mathcal{S}$ enables one to write

$$
\begin{aligned}
&\int_{\mathcal{U}} \int_{\mathbb{R}^3 \setminus B_\eta(v)} \int_{B_R} |h'^R - h'|\, f\, \mathcal{M}_\star\, |v_\star - v| \,\mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v \\
&\qquad \le C\, \|\nabla h\|_{L^\infty} \int_{\mathcal{U}} \int_{\mathbb{R}^3 \setminus B_\eta(v)} \int_0^R r\, R^{-s}\, \kappa(r, R)\, \eta^{-2}\, |v_\star - v|^2 \\
&\qquad\qquad \times \mathcal{M}_\star\, f\, \mathrm{d}v_\star\, \mathrm{d}r\, \mathrm{d}x\, \mathrm{d}v.
\end{aligned}
$$

Estimating

$$|v_\star - v|^2 \le |v|^2 + |v_\star|^2 + 2\,|v|\,|v_\star| \\ \le (1+|v|^2)\,(1+|v_\star|^2) + 2\,(1+|v|)\,(1+|v_\star|)$$

and then since

$$\int_{\mathbb{R}^3} (1+|v|)\, f(v)\,\mathrm{d}v \le \int_{\mathbb{R}^3} (2+|v|^2)\, f(v)\,\mathrm{d}v$$

we obtain

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v_\star - v|^2\, \mathcal{M}_\star\, f\,\mathrm{d}v_\star\,\mathrm{d}v \le 3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (1+|v|^2)\, f\,(1+|v_\star|^2)\,\mathcal{M}_\star\,\mathrm{d}v_\star\,\mathrm{d}v$$

and this gives the first part of the estimate.

We are now left to deal with

$$\int_{\mathcal{U}}\int_{B_\eta(v)}\int_{B_R} |h'^R - h'|\, f\,\mathcal{M}_\star\,|v_\star - v|\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}v \\ \le C\,\|\nabla h\|_{L^\infty}\int_{\mathcal{U}}\int_{B_\eta(v)}\int_{B_R} |\theta^R - \theta|\, f\,\mathcal{M}_\star\,|v_\star - v|^2\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}v.$$

On $B_\eta(v)$, following the proof of Lemma 2.6 without using the bound $|v_\star - v| \ge \eta$ in the form of the inequality, one can estimate

$$|\theta^R(r, v_\star - v) - \theta(r, v_\star - v)| \le \begin{cases} \frac{C}{1+|v_\star - v|^2 r^s} & r > R - 1 - 1/|v_\star - v| \\ \frac{C}{|v_\star - v|^2}\,\kappa(r,R) & \text{else} \end{cases}$$

and using this estimate we obtain

$$\int_{\mathcal{U}}\int_{B_\eta(v)}\int_{B_R} |\theta^R - \theta|\, f\,\mathcal{M}_\star\,|v_\star - v|^2\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}v \\ \le \int_{\mathcal{U}}\int_{B_\eta(v)} \left(\int_0^{R-1-1/|v_\star - v|} C\,r\,\kappa(r,R)\,\mathrm{d}r + \int_{R-1-1/|v_\star - v|}^{R} \frac{C\,r\,|v_\star - v|^2}{1+|v_\star - v|^2 r^s}\,\mathrm{d}r \right) \\ \times f\,\mathcal{M}_\star\,\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}v.$$

Since $\mathcal{M} \in L^\infty$, and $\frac{r\,|v_\star - v|^2}{1+|v_\star - v|^2 r^s} \leq \frac{C\,r}{1+r^s}$ we obtain

$$\int_{\mathcal{U}}\int_{B_\eta(v)} \left(\int_0^{R-1-1/|v_\star - v|} C\,r\,\kappa(r,R)\,\mathrm{d}r + \int_{R-1-1/|v_\star - v|}^{R} \frac{C\,r\,|v_\star - v|^2}{1+|v_\star - v|^2 r^s}\,\mathrm{d}r \right) \times f\,\mathcal{M}_\star\,\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}v$$

$$\leq C\eta^3\,\|f\|_{L^1}\,\|\mathcal{M}\|_{L^\infty} \left(\int_0^{R-1-1/|v_\star - v|} C\,r\,\kappa(r,R)\,\mathrm{d}r + \int_{R-1-1/|v_\star - v|}^{R} \frac{C\,r}{1+r^s}\,\mathrm{d}r \right)$$

and by using the form of κ, and by defining

$$\kappa_2(r,R) := \int_0^{R-1} \frac{C\,r\,\mathrm{d}r}{R^s\left(1 - \frac{r^2}{R^2}\right)} + \frac{1}{\log^{7/2} R R^{s-1/2}} + \int_{R-1-\log R}^{R} \frac{C\,r}{1+r^s}\,\mathrm{d}r$$

we obtain the result. □

One of the aims of this section was to compare the collision operators L and L^R. The previous lemma compared L^R and $\widetilde{L}^R$, and so we are now left to compare $\widetilde{L}^R$ and L. The reader should think of this second comparison as an analysis of the contribution of grazing collisions on the operator L. The proof is in the same spirit as the proof of the previous lemma, and so minimal details are given.

Lemma 5.7. *Let $R > 0$, and suppose that ϕ is an admissible long range potential with a $\rho_2 > 0$ and $s > 2$ such that*

$$\psi(\rho) \leq \rho^{-s}$$

for $\rho > \rho_2$. Then for all $f \in L^1(\mathbb{R}^3, (1+|v|)\,\mathrm{d}v)$ and for $h \in C_c^\infty([0,T) \times \mathcal{U})$

$$|\langle \widetilde{L}^R(f), h\rangle - \langle L(f), h\rangle| \leq C\,\|\nabla h\|_{L^\infty} \int_R^\infty \frac{r}{1+r^s}\,\mathrm{d}r \times \times \left\|(1+|v|^2)\,f\right\|_{L^1} \left\|(1+|v|^2)\,\mathcal{M}\right\|_{L^1}$$

Proof: We again compare the two operators, and observe that

$$|\langle \widetilde{L}^R(f),\, h\rangle - \langle L(f), h\rangle| = \left| \int_{\mathcal{U}}\int_{\mathbb{R}^3}\int_0^{2\pi}\int_R^\infty (h' - h)\,f\,\mathcal{M}_\star\,r\,|v_\star - v|\,\mathrm{d}r\,\mathrm{d}\zeta\,\mathrm{d}v_\star\,\mathrm{d}v\,\mathrm{d}x \right|$$

Again using the differentiability of h, and by bounding

$$|v' - v| \leq \frac{1}{2}\theta(r, v_\star - v)|v_\star - v|,$$

we can use Lemma 2.5 on the scattering angle θ. As before we need to split the $v_\star$ integration into the sets B_1 and $\mathbb{R}^3 \setminus B_1$. The terms can be bounded similarly to before. $\square$

The above enables us to compare the collision operators L and L^R, at least for near collisions. Before we compare the grazing part of the long range collision operator, we first prove a continuity type estimate on the short range collision operator. We use the estimate on the deviation angle for ϕ^R proved before in Lemma 2.5, which we recall is

$$\theta^R(r,w) \leq \frac{C}{1+|w|^2\, r^s}.$$

We use this in the following lemma.

Lemma 5.8. *If ϕ is an admissible long range potential such that there is a $\rho_2 > 0$ and $s > 2$ with*

$$\psi(\rho) \leq \rho^{-s}$$

for $\rho > \rho_2$, and if $F \in D(L^R)$, and $\int (1+|v|^2)\, F < \infty$, and $h \in C_c^\infty([0,T)\times \mathcal{U})$, we have

$$|\langle L^R(F), h\rangle| \leq C\, \|\nabla h\|_{L^\infty} \int_0^R \frac{r}{1+\frac{r^s}{\log^2 R}}\, \mathrm{d}r\, \left\|(1+|v|^2)\, F\right\|_{L^1} \left\|(1+|v|^2)\, \mathcal{M}\right\|_{L^1}$$
$$+ C\, \|\nabla h\|_{L^\infty}\, \|F\|_{L^1}\, \|\mathcal{M}\|_{L^\infty} \frac{1}{\log^5 R} \int_0^R \frac{C\, r\, \mathrm{d}r}{1+r^s}$$

Proof: We have

$$\begin{aligned} |\langle L^R(F),\, h\rangle| &= \left| \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{\mathcal{S}} (h'^R - h)\, F\, \mathcal{M}_\star\, |v_\star - v|\, \mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v \right| \\ &\leq \int_{\mathcal{U}} \int_{\mathbb{R}^3} \int_{\mathcal{S}} |h'^R - h|\, |F|\, \mathcal{M}_\star\, |v_\star - v|\, \mathrm{d}S\, \mathrm{d}v_\star\, \mathrm{d}x\, \mathrm{d}v. \end{aligned}$$

Using the differentiability of h allows one to write

$$|h'^R - h| \leq C\, \|\nabla h\|_{L^\infty}\, |v'^R - v|$$

and using the form of $v^{\prime,R}$ in (2.1) enables one to write

$$\begin{aligned} |v'^R - v| &\leq \sin\left(\frac{1}{2}\theta^R(r, v_\star - v)\right)\, |v_\star - v| \\ &\leq \frac{1}{2}\theta^R(r, v_\star - v)\, |v_\star - v|. \end{aligned}$$

Splitting the integration in $v_\star$ into the regions $B_\eta(v)$ and $\mathbb{R}^3 \setminus B_\eta(v)$ for $\eta = \frac{1}{\log R}$, we use Lemma 2.5 to provide the estimate on the latter region as

$$\begin{aligned}\int_{\mathcal{U}} \int_{\mathbb{R}^3\setminus B_\eta(v)} \int_{\mathcal{S}} |h'^R - h|\, |F|\, \mathcal{M}_\star\, |v_\star - v| \,\mathrm{d}S \,\mathrm{d}v_\star \,\mathrm{d}x \,\mathrm{d}v \\ \leq \int_{\mathcal{U}} \int_{\mathbb{R}^3\setminus B_\eta(v)} C \,\|\nabla h\|_{L^\infty} \int_0^R \frac{r}{1+\eta^2\, r^s} \,\mathrm{d}r\, |F|\, \mathcal{M}_\star |v_\star - v|^2 \,\mathrm{d}v_\star \,\mathrm{d}x \,\mathrm{d}v.\end{aligned}$$

Similar rearranging to before enables one to say

$$\int_{\mathcal{U}} \int_{\mathbb{R}^3} f \mathcal{M}_\star |v_\star - v|^2 \;\mathrm{d}v_\star \,\mathrm{d}v \,\mathrm{d}x \leq \left\|(1+|v|^2)\, f\right\|_{L^1} \left\|(1+|v|^2)\, \mathcal{M}\right\|_{L^1}$$

which gives the first term in the statement of the lemma.

On $B_\eta(v)$ we obtain from the same estimates on the collision angle

$$\begin{aligned}\int_{\mathcal{U}} \int_{B_\eta(v)} \int_{\mathcal{S}} |h'^R - h|\, |F|\, \mathcal{M}_\star\, |v_\star - v| \,\mathrm{d}S \,\mathrm{d}v_\star \,\mathrm{d}x \,\mathrm{d}v \\ \leq \int_{\mathcal{U}} \int_{B_\eta(v)} C \,\|\nabla h\|_{L^\infty} \int_0^R \frac{r}{1+|v_\star - v|^2\, r^s} \,\mathrm{d}r\, |F|\, \mathcal{M}_\star |v_\star - v|^2 \,\mathrm{d}v_\star \,\mathrm{d}x \,\mathrm{d}v\end{aligned}$$

and then by estimating $|v_\star - v| \leq \eta$, and by changing constants so that $\frac{r}{1+|v_\star - v|^2\, r^s} \leq \frac{C\, r}{1+r^s}$, we obtain

$$\begin{aligned}\int_{\mathcal{U}} \int_{B_\eta(v)} C \,\|\nabla h\|_{L^\infty} \int_0^R \frac{r}{1+|v_\star - v|^2\, r^s} \,\mathrm{d}r\, |F|\, \mathcal{M}_\star |v_\star - v|^2 \,\mathrm{d}v_\star \,\mathrm{d}x \,\mathrm{d}v \\ \leq C \,\|\nabla h\|_{L^\infty} \,\|F\|_{L^1} \,\|\mathcal{M}\|_{L^\infty} \,|B_\eta| \int_0^R \frac{C\, r\, \eta^2}{1+r^s} \,\mathrm{d}r\end{aligned}$$

which concludes the proof □

5.3 Conclusion of convergence

We now conclude this chapter with a proof of Theorem 4. This combines the results proved before in this chapter.

Section 5.1 showed that $f^R \to f$ weak-$\star$ in L^∞ up to a subsequence. We now show that f is a weak solution of equation (1.5).

Firstly, we observe that since we have (2) in Definition 1.5, we then have, for all R, that

$$\int (1+|v|^2)\, f^R \leq C \int (1+|v|^2)\, \mathcal{M} < \infty$$

and we can thus pass to the limit in the term on the left hand side of this inequality to obtain that

$$\int (1+|v|^2) f = \int (1+|v|^2) \lim_{R\to\infty} f^R = \lim_{R\to\infty} \int (1+|v|^2)\, f^R < \infty$$

which proves that the limit function f does have the required regularity for a weak solution.

We now need to show that f satisfies equation (1.5) for any suitable test function h, namely we must show that

$$-\int_0^T \int_{\mathcal{U}} (\partial_t h + v \cdot \nabla_x h)\, f \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t - \int_{\mathcal{U}} f_0\, h(0) \,\mathrm{d}x \,\mathrm{d}v = \int_0^T \langle L(f), h\rangle \,\mathrm{d}t.$$

Since f^R is a weak solution of the linear Boltzmann equation for ϕ^R, we know that equation (5.1) holds. We can pass to the limit in the left hand side of this equation to obtain

$$\int_0^T \int_{\mathcal{U}} (\partial_t h + v \cdot \nabla_x h)\ f^R \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t \to \int_0^T \int_{\mathcal{U}} (\partial_t h + v \cdot \nabla_x h)\ f \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t.$$

We then observe that

$$\begin{aligned}\int_0^T \langle L^R(f^R), h\rangle - \langle L(f), h\rangle \,\mathrm{d}t &= \int_0^T \langle L^R(f^R - f), h\rangle \,\mathrm{d}t + \int_0^T \langle L^R(f) - L(f), h\rangle \,\mathrm{d}t \\ &= \int_0^T \langle L^R(f^R - f), h\rangle \,\mathrm{d}t + \int_0^T \langle L^R(f) - \widetilde{L}^R(f), h\rangle \,\mathrm{d}t \\ &\qquad + \int_0^T \langle \widetilde{L}^R(f) - L(f), h\rangle \,\mathrm{d}t \\ &\leq I + II + III\end{aligned}$$

where the terms I, II and III come from Lemmas 5.8, 5.5, and 5.7 respectively.

The decay of the potential assumed in Theorem 4 ensures that these three terms tend to 0 as $R \to \infty$, thus showing f is indeed a weak solution of equation (1.5).

Chapter 6

Concluding Remarks

We conclude by using the results of the previous chapters in a proof of the main result of this thesis, , before describing suggested future work which can extend this theorem.

6.1 Proof of Theorem 1

We now demonstrate how we prove the main theorem. We first recall that this states.

Theorem. 1. *Let f^{ε} be the phase space density for a tagged particle evolving according to* (1.4) *with initial density given by f_0 satisfying Definition 1.5, with an admissible potential ϕ as in Definition 1.4 such that there is a $\rho_2 > 0$ and $\gamma > 0$ with*

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\psi(\rho) \leq e^{-C\rho^{\frac{3}{2}+\gamma}}$$

for all $\rho > \rho_2$. Then as $\varepsilon \to 0$ with $N\varepsilon^2 = 1$, we have f^{ε} converges weak-$\star$ in L^{∞} to f a weak solution of the linear Boltzmann equation associated to ϕ, as given by Definition 1.6.

Proof: We compare as follows. We firstly let $R = \varepsilon^{-\frac{1}{3+\gamma}}$ and then let $P_t^{\varepsilon,R}$ and P_t^R be probability measures on $\mathcal{MT}$ as defined in the previous chapters, as well as f^R a solution of the linear Boltzmann equation with short range potential ϕ^R.

We furthermore define the parameters for the subset $\mathcal{G}(\varepsilon)$ as in Definition 3.3

to be

$$\begin{aligned}
V_1(\varepsilon) &= \frac{1}{|\log \varepsilon|} \\
V_2(\varepsilon) &= |\log \varepsilon| \\
M(\varepsilon) &= |\log \varepsilon| \\
\delta(\varepsilon) &= \sqrt{\varepsilon},
\end{aligned}$$

and we remark that for any probability measure on $\mathcal{MT}$ absolutely continuous with respect to the Lebesgue measure, we have $P(\mathcal{MT} \setminus \mathcal{G}(\varepsilon)) \to 0$ as $\varepsilon \to 0$ under these parameters.

We then write, for $h \in L^\infty(\mathcal{U})$ a test function,

$$\begin{aligned}
\int_{\mathcal{U}} (f^\varepsilon - f)\, h(x,v)\, \mathrm{d}x\, \mathrm{d}v \leq \left| \int_{\mathcal{U}} f^\varepsilon(x,v)\, h(x,v)\, \mathrm{d}x\, \mathrm{d}v - \int_{\mathcal{MT}} h(\Phi)\, P_t^{\varepsilon,R}(\Phi)\, \mathrm{d}\Phi \right| \\
+ \left| \int_{\mathcal{MT}} h(\Phi)\, P_t^{\varepsilon,R}(\Phi)\, \mathrm{d}\Phi - \int_{\mathcal{MT}} h(\Phi)\, P_t^{R}(\Phi)\, \mathrm{d}\Phi \right| \\
+ \left| \int_{\mathcal{U}} (f^R - f)\, h(x,v)\, \mathrm{d}x\, \mathrm{d}v \right|,
\end{aligned}$$

and we analyse each of these terms in the limit $\varepsilon \to 0$.

Firstly, Theorem 3 on page 93 ensures that, with the choices of parameters specified above, the first term converges to 0 as $\varepsilon \to 0$. The choice of $R = \varepsilon^{-\frac{1}{3+\gamma}}$ ensures that as $R \to \infty$ we must have $\varepsilon \to 0$, and so this term tends to 0 in the limit $\varepsilon \to 0$. In particular, this result required the specification of $R \to \infty$ algebraically in ε to ensure convergence of the error terms.

Secondly, the choice of R then ensures that as $\varepsilon \to 0$ we have $R \to \infty$ and so we can apply Theorem 4 on page 112 that shows that in the limit $R \to \infty$ we have $f^R \to f$ weak -$\star$ in L^∞, and so the final term tends to 0 as $R \to \infty$.

For fixed $R > 0$, Theorem 2 on page 75 proves convergence in total variation of $P^{\varepsilon,R}$ to P^R, and so this implies weak-$\star$ convergence of the two probability densities. However, this in itself is not sufficient to ensure that we have convergence when R diverges with $\varepsilon \to 0$.

By considering the estimates of the error terms here in Lemmas 3.23, 3.25 and 3.26 we observe that the error terms decay to 0 as $\varepsilon \to 0$ even where $R \to \infty$ where R diverges slower than $\varepsilon^{-\frac{1}{3}}$. The choice of R above ensures that this is the case, and so we have convergence in total variation for the second term. This thus concludes the proof of the theorem. $\square$

One should see that for the purposes of the proof we have considered a fixed interval of time over which one looks for solutions. Indeed, the estimates we provide on the particle dynamics are not valid if we instead have the interval $[0, \infty)$ as the constants depend upon time in such a manner that they tend to ∞ as the end of the interval tends to ∞. These estimates cannot be improved and so we could never have a statement for the interval $[0, \infty)$.

6.2 Topics for Further Analysis

We now highlight some potential directions to improve the result of the thesis, and give brief insights into the manner in which one would extend. Several of these are only briefly commented upon, while one has been considered significantly, but insufficiently to include throughout the thesis.

6.2.1 Weakening of Decay on Potential

While Theorem 1 gives convergence of the density of a particle system to the linear Boltzmann equation for a relatively wide class of potentials (compared to [4]), there is still room for improvement.

The major limitation in improving the decay is the use of a Gronwall argument for comparing solutions to the particle dynamics for short and long range potentials.

This estimate is rough because it assumes with every collision that the worst case scenario is being enacted. By replacing this L^∞ type estimate for an L^∞_{loc} estimate, which instead then takes into account more of the physics of the collision, one may well be able to weaken the decay assumption.

Another possibility is to perform estimates for the potential in L^p for some $p \in [1, \infty)$ as opposed to in L^∞.

6.2.2 Extension to General Background

While this thesis gives a proof of justification of the linear Boltzmann equation for background given by a Maxwellian, one would ideally desire the distribution of the background velocities to be given by a function $g \in L^1(\mathbb{R}^3, (1 + |v|^2)\,\mathrm{d}v)$. This has now been done and can be found in [25]. The main issue that was resolved was using a different method to the maximum principle to extract uniform in R estimates on the solutions f^R.

This extension does pose an interesting question pertaining to properties of the linear collision operator

$$L_g(f) = \int_{\mathbb{R}^3} \int_{\mathcal{S}} (f' \, g'_\star - f \, g_\star) \, |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star. \tag{6.1}$$

The question is whether one has a stationary distribution of the linear Boltzmann equation, as given in (2) in Definition 1.5. Such a question has been addressed in [31], and geometric conditions are described on the underlying dynamics to ensure that a stationary distribution exists. However, these are made under the assumption that one can split the collision operator into a gain and loss term, which we cannot do.

As can be seen, the existence of such a function is not at all obvious. However, the advantage of assuming the background was Maxwellian meant that this was an immediate consequence from the well known fact that $\mathcal{M}' \, \mathcal{M}'_\star = \mathcal{M} \, \mathcal{M}_\star$, which then can easily be used to show that that $\mathcal{M}$ is a stationary solution of the linear Boltzmann equation.

One potential way to show existence of a stationary distribution would be to show the existence of an ergodic invariant measure for an associated Markov process for the operator L_g. Harris' theorem, stated in [30], gives conditions on the Markov transition kernel so that the Markov process has such an invariant measure.

The first issue one has in this setting is that Harris' theorem is valid for discrete time Markov processes, and to interpret the linear Boltzmann equation as a generator for such a process, one is required for the associated Markov process to have finitely many jumps in a finite time interval. This is not possible for the collision operator as defined in (6.1), because the integral over the associated Lévy measure is infinite.

In order to be able to consider the Markov process as a discrete time process, one thus must introduce a regularisation parameter R and truncate the integration over $\mathcal{S}$ into integration over $B_R(0)$ to ensure that in any given time interval one has finitely many jumps. Then, up to a rescaling of time, one would have a discrete time Markov process, and could look to apply Harris' theorem.

To then find an ergodic measure for L_g one would expect that the grazing collisions one has removed with the regularisation do not affect the shape of the stationary distribution to a great extent, and so one should be able to find bounds on these ergodic measures independent of R.

One can easily specify the Markov transition function for this process by

$$\mathcal{P}(v, v') = \frac{1}{\int_{\mathbb{R}^3} \int_{\mathcal{S}} g(v_\star) |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star} \int_{\mathbb{R}^3} \int_{\mathcal{S}} g(v_\star) \mathbb{1}_{v' = \sigma_1(v, v_\star)} |v_\star - v| \, \mathrm{d}S \, \mathrm{d}v_\star$$

although this relationship is not particularly pretty, and we recall that σ is the scattering operator for the potential ϕ. This states the natural fact that to jump from v to v' one must encounter a background particle $v_\star$ and relevant geometric parameters so that v' is the post collisional velocity of the particle with pre collisional velocity v. We have assumed that $\mathcal{P}(v,\cdot)$ is absolutely continuous with respect to the Lebesgue measure here as well, which is a natural assumption.

To be able to apply Harris' theorem, one then must satisfy the following two conditions. Firstly one requires a function $V\colon \mathbb{R}^3 \to [0,\infty)$ and constants $K \geq 0$, and $\gamma \in (0,1)$ such that

$$\mathcal{P}(V)(v) \leq \gamma V(v) + K$$

for all $v \in \mathbb{R}^3$, where $\mathcal{P}$ is the transition kernel of the Markov process.

Secondly, one requires, for every $H > 0$, the existence of a constant $\alpha > 0$ such that

$$|\mathcal{P}(f)(v) - \mathcal{P}(f)(w))| \leq 2(1-\alpha)$$

for all v, w such that $V(v) + V(w) \leq H$.

Firstly, we remark that the form of $\mathcal{P}(v,v')$ given above has many dependencies. Removing these by using the Carleman representation [17] we can rewrite this, up to renormalisation, by

$$\mathcal{P}(v,v') = \int_{\bar{v}\cdot(v'-v)=0} g(v'+\bar{v})|v'-v+\bar{v}|\,\mathrm{d}\bar{v}.$$

To satisfy the first condition, the natural candidate function for the Lyapunov function V is

$$V(v) = -g(v)\left(\log g(v) \wedge 0\right),$$

since this is the equivalent of the entropy for the non-linear Boltzmann equation. This then leads to the expression for $\mathcal{P}(V)$ of

$$\mathcal{P}(V) = -\int_{\mathbb{R}^3} \frac{g(v')\left(\log g(v') \wedge 0\right)}{\int_{\mathbb{R}^3}\int_{\mathcal{S}} g(v_\star)|v_\star - v|\,\mathrm{d}S\,\mathrm{d}v_\star} \int_{\mathbb{R}^3}\int_{\mathcal{S}} g(v_\star)\mathbb{1}_{v'=\sigma_1(v,v_\star)}|v_\star - v|\,\mathrm{d}S\,\mathrm{d}v_\star\,\mathrm{d}v'$$

which has the issue of a combination of integrations pre and post collisional. To my knowledge I know of no formulae that combine such integrations.

The second condition is possibly somewhat more straightforward to check. It requires a careful consideration of those velocities that are obtainable as post-collisional velocities from two different pre collisional velocities, and an analysis of the probability of such sets.

6.2.3 Different Collisional Structures

A future aim would be to allow for a more general collisional structure. There are two natural extensions.

(1) Firstly, one would like to allow the background to interact in a more sophisticated manner. Recent work in [38] allows for a background which is no longer spatially homogeneous. This could be interpreted eventually as the background colliding with each other in such a manner that the collisions do not preserve spatial invariance. This paper furthermore establishes the semi-group theoretic arguments to be used for a spatially inhomogeneous background. This together with modifications of the long range estimates could be used to analyse the long range interactions in this case.

(2) Secondly, one would ultimately aim to show convergence for the fully non-linear Boltzmann equation. For sufficiently weak interactions, one may well be able to postulate that the solution of the non-linear Boltzmann equation should appear as a product over solutions to suitable linear Boltzmann equations, and this may well be a potential avenue for future analysis.

Appendix A

Ancillary Results

We now describe the existence result used in chapter 2 which is taken from [6, Ch.10].

Suppose that the linear Boltzmann equation has an extra force term, and is written as

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = -L^-(f) + L^+(f) \tag{A.1}$$

and we have

$$L^-(f) = \nu(x, v) f(v)$$

where ν is called the collision frequency.

For this equation one then has the following existence result.

Theorem 5. *Suppose that the following conditions for equation* (A.1) *are satisfied.*

(A_1) The field $F : \mathcal{U} \to \mathbb{R}^3$ is independent of time and is Lipschitz continuous.

(A_2) The field F is divergence free, meaning

$$\nabla_v \cdot F = 0.$$

(A_3) The collision frequency $\nu : \mathcal{U} \to \mathbb{R}$ satisfies $0 \geq \nu \in L^1_{\text{loc}}(\mathcal{U})$

(A_4) There exists a positive constant C such that for any $(x, v) \in \mathcal{U}$ we have

$$F(x, v) \cdot v \leq C|v|$$

(A_5) For any $V > 0$ there is a $M < \infty$ such that for almost all $x \in \mathbb{T}^3$ and $|v| \leq V$ we have

$$\nu(x, v) \leq M$$

(A_6) The operator L^+ is an integral operator, meaning

$$L^+(f)(x,v) = \int_{\mathbb{R}^3} k(x,v,v')\, f(x,v')\,\mathrm{d}v'$$

where k is measurable and a non-negative real valued function defined on $\mathcal{U}\times\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} k(x,v',v)\,\mathrm{d}v' = \nu(x,v)$$

(A_7) There exists a $C > 0$ such that for any fixed $V > 0$ we have

$$\int_{|v'|\geq V} k(x,v',v)\,\mathrm{d}v' \leq C$$

for almost all $x \in \mathbb{T}^3$ and $|v| \leq V$.

Then the operator $L^+ - L^- - v\cdot\nabla_x$ generates a sub-stochastic honest semi-group, and in particular, for any $f_0 \in D(L^+ - L^- - v\cdot\nabla_x)$ we have existence of solutions, and the solutions do not blow up in finite time.

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