Extreme Events of Lagrangian Model of Passive Scalar Turbulence via Large Deviation Theory

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A. Large Deviations Theory

A large deviation principle (LDP) : Let P^ϵ be a family of probability measures on a suitable measurable space $(\mathcal{X}, \Sigma)$, then it satisfies an LDP with a rate function $I : \mathcal{X} \to \mathbb{R}$ if for all subsets $\Omega \subset \Sigma$, we have [4, 5],

$$P^\epsilon(\Omega) \asymp \exp\left(\epsilon^{-1} \inf_{z\in\Omega} I(z) \right), \qquad (1)$$

where $\asymp$ denotes log-asymptotic equivalence in the limit $\epsilon \to 0$.

The Gärtner-Ellis theorem : If the limiting behavior of a scaled CGF,

$$G(\lambda) \equiv \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}[e^{\epsilon^{-1}\langle\lambda, z^\epsilon\rangle}]. \qquad (2)$$

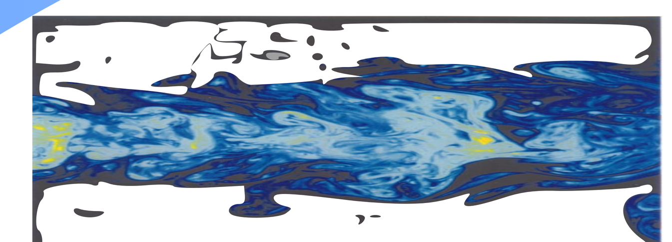
exists for each λ, then its Legendre-Fenchel (LF) transform is the rate function of the LDP of the process z^ϵ, i.e.

$$I(z) = \sup_{\lambda\in\mathbb{R}^n} (\langle\lambda, z\rangle - G(\lambda)), \qquad z = \lim_{\epsilon\to 0} z^\epsilon,$$

A geometric interpretation of the LF transform (2) is that the argument of one of $G(\lambda)$ and $I(z)$ is the slope of the other [7], i.e.,

$$\nabla I(z) = \lambda, \quad \nabla G(\lambda) = z,$$

when $G(\lambda)$ is a ***finite*** and a ***differentiable*** function, and the rate function $I(z)$ is ***strictly convex***.



Dye in a turbulent jet (Source: Shraiman and Siggia. Scalar turbulence. Nature, 2000).

I. Motivation \ Introduction

*) In turbulence theory, one often Studies statistical and probabilistic properties in fluid dynamics.

*) In this poster, we investigate the probability of extreme gradients of a passive scalar in medium with a variety of Reynolds numbers R_e, using LDT (section A).

*) The numerical algorithm for finding rare events is based on changing probability measures, and instanton equations (sections B & C).

*) For a turbulent flow with heavy-tailed distributions, the standard method fails (section C). Therefore, here we use a ***nonlinear reparametrization*** (section D) which is a newly proposed in [1] (or you can scan the QR code).

II. Recent Fluid Deformation (RFD) Model of Passive Scalar Turbulence

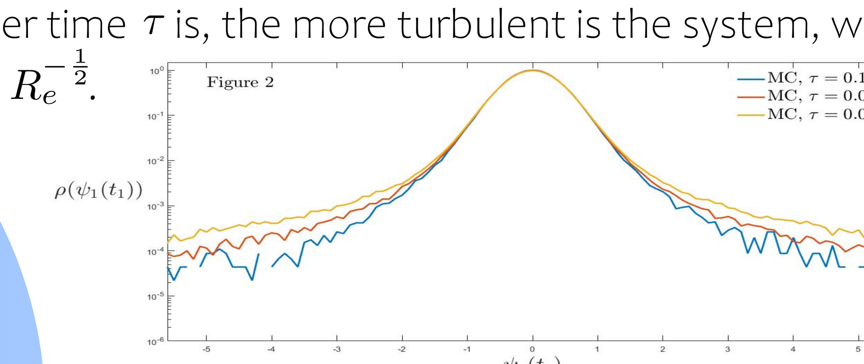
This statistical model of the gradient of a passive scalar, $\psi = \nabla\theta \in \mathbb{R}^3$, is given by [2,3]:

$$d\psi(t) = b(\psi, \mathbf{A})\, dt + \sqrt{\epsilon}\, dM(t),$$
$$d\mathbf{A}(t) = V(\mathbf{A})\, dt + \sqrt{\epsilon}\, d\mathbf{W}(t),$$

where the drift terms are:

$$b(\psi, \mathbf{A}) = -\mathbf{A}^T(t)\,\psi(t) - \frac{\mathrm{tr}(\mathbf{C}_\tau^{-1})}{3\,T_\theta}\,\psi(t),$$
$$V(\mathbf{A}) = \mathbf{A}^2 + \frac{\mathrm{tr}(\mathbf{A}^2)}{\mathrm{tr}(\mathbf{C}_\tau^{-1})}\,\mathbf{C}_\tau^{-1} \quad \frac{\mathrm{tr}(\mathbf{C}_\tau^{-1})}{3\,T_A}\,\mathbf{A}.$$

And $\mathbf{C}_\tau(t) \approx e^{\tau\mathbf{A}}\, e^{\tau\mathbf{A}^T}$ is a stationary Cauchy-Green tensor, and $\mathbf{A}(t) \in \mathbb{R}^{3\times 3}$ is the ***velocity gradient***. The small parameter of the system is ϵ. Setting the integral times T_A and T_θ to unity, the only remaining temporal scale is τ, which is the ***decorrelation time scale*** after which any correlation of $\mathbf{A}$ is neglected. ***Figure (2)*** shows that the shorter time τ is, the more turbulent is the system, where $\tau \sim R_e^{-\frac{1}{2}}$.



B. Lagrange Multiplier and Exponentially Tilted Measures

The problem of finding the rare event probability is now reduced to an optimization problem, i.e. $\inf_{z\in\Omega} I(z)$, where the minimizer is called ***instanton***. This constrained optimization problem can be transformed into an unconstrained one using a ***Lagrange multiplier*** [8], i.e.,

$$\inf_{z\in\Omega} I(z) \Leftrightarrow \inf_{z\in\mathbb{R}^n} (I(z) - \langle\lambda, z\rangle). \qquad (3)$$

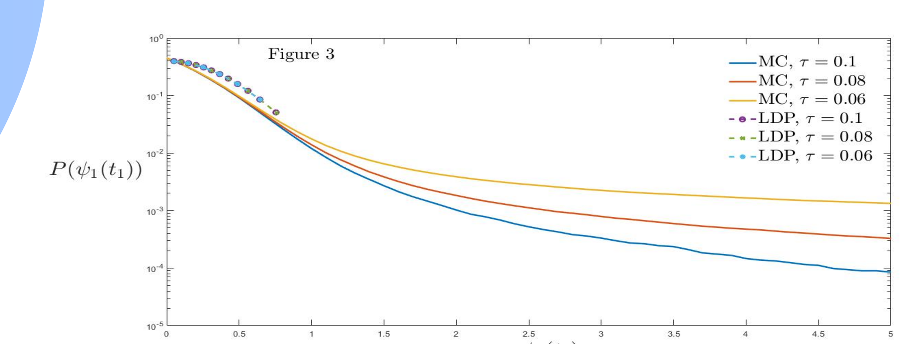
A probabilistic interpretation of Lagrange multiplier λ is in the form of the ***exponentially tilted measure*** [9]. If

$$\mathbb{E}_p\left[\exp\left(\epsilon^{-1}\langle\lambda, z\rangle\right)\right] < \infty$$

then the exponentially tilted measure is defined as,

$$\underbrace{p_\lambda(z)}_{\text{Tilted measure}} = \underbrace{\frac{\exp\left(\epsilon^{-1}\langle\lambda, z\rangle\right)}{\mathbb{E}_p[\exp\left(\epsilon^{-1}\langle\lambda, z\rangle\right)]}}_{\text{Tilting factor}} \underbrace{p(z)}_{\text{Original measure}}. \qquad (4)$$

In order to use the described ***methodology to find the instanton*** (minimizer) for a rare outcome z, we must demand that the ***the mapping*** $z \to \lambda(z)$ ***is a bijection:*** For every outcome z there must be a unique tilt λ, which holds only when the rate function I is ***strictly convex.***



III. Instantons of Extreme Gradients of a Passive Scalar

Finding the maximum likelihood pathways, instantons, of achieving ***extreme final configurations*** $z := \psi_i(t_1)$ can be done via instanton equations (8), (section C). ***Figure (3)*** shows the distribution of end points of instantonic trajectories (dashed lines) against MC results (solid lines), for a range of τ that exhibits ***heavy-tailed distributions***. It displays an excellent agreement between the two. However, instantons are limited to small values of z due to the ***nonconvexity*** of the rate function (see section B), which has been overcome using the non-linear reparameterizations (section D).

C. Instanton Equations

Consider a stochastic system,

$$dX_t^\epsilon = b(X_t^\epsilon)\, dt + \sqrt{\epsilon}\,\sigma dW_t\,,\; X_{t_0}^\epsilon = x_0. \qquad (5)$$

The noise covariance $\chi = \sigma\sigma^T \in \mathbb{R}^{n\times n}$ is assumed to be invertible. We are interested in the chance of trajectories X_t^ϵ departing from an asymptotically stable fixed point $\bar{x}$, and eventually leaving $D \subset \mathbb{R}^n$ that is attracted to $\bar{x}$. These trajectories belong to the set:

$$A_z := \{\varphi \in \mathbf{C}_{t_0 t_1}(\mathbb{R}^n) | \varphi(t_0) = \bar{x}, \varphi(t_1) = z \notin D\}. \qquad (6)$$

From LDP, we have,

$$\lim_{\epsilon\to 0} \epsilon \log p(z) = -\inf_{z\in A_z} I(z) = -S(\varphi^*), \qquad (7)$$

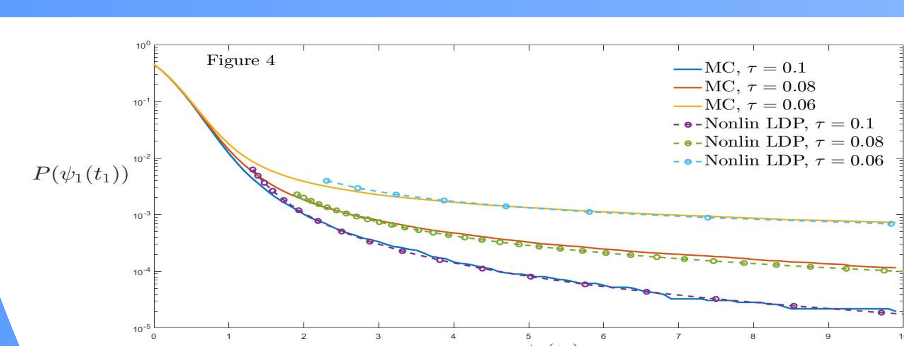
where φ^* is the minimizer (instanton), and

$$S(\varphi) = \frac{1}{2}\int_{t_0}^{t_1} \|\dot{\varphi}_t - b(\varphi(t))\|_\chi^2\, dt$$ is ***Freidlin-Wentzell rate function*** of system (5). The integrand of S can be understood as a ***Lagrangian***.

In ***Hamiltonian formulation***, $H(\varphi, \vartheta) = \sup_{\dot{\varphi}} (\langle\vartheta, \dot{\varphi}\rangle - L(\varphi, \dot{\varphi}))$, where $\vartheta = \partial L/\partial\dot{\varphi}$ is the conjugate momentum of φ [10]. Now, the minimizer φ^* can also be expressed as the solution of Hamilton's equations,

$$\dot{\varphi} = \partial_\vartheta H(\varphi, \vartheta) = b(\varphi) + \chi\,\vartheta,$$
$$\dot{\vartheta} = -\partial_\varphi H(\varphi, \vartheta) = -\left(\nabla_\varphi b(\varphi)\right)^T \vartheta, \qquad (8)$$

with boundary conditions, $\varphi(t_0) = \bar{x}$, $\vartheta(t_1) = \lambda$. These quations are often termed ***instanton equations***.



IV. Modification: Nonlinearly Tilted Instantons

Optimizations algorithm is now done via instanton equations with ***nonlinear final time tilting***, $F(z) = \mathrm{sign}(z)\log\log|z|$, $z \in \mathbb{R} \setminus [-1, 1]$, for which the reparametrized expectation remains bounded. Other choices of F would be possible. ***Figure (4)*** compares the nonlinearly tilted instantons (dashed lines), with MC results (solid lines). It shows that the nonlinear instantons correctly predicts the far tail probilities, since reparametrizing the observable via F convexifies its rate function.

D. Instantons of Nonlinear Reparametrizations

Every situation where the tails of $p(z)$ are ***fat*** correspond with non-convex rate functions, and will break the above assumption of the mapping $z \to \lambda(z)$ being a bijection.

The ***main contribution*** of [1] is the realization that the introduction of ***a nonlinear map*** $F : \mathbb{R}^n \to \mathbb{R}^n$ allows us to loosen the restriction of the convexity of $I(z)$. In analogy to (4) and the description in section B, we can now define the ***nonlinearly tilted measure***;

$$p_\lambda^F(z) = \frac{\exp(\epsilon^{-1}\langle\lambda, F(z)\rangle)}{\mathbb{E}_p[\exp(\epsilon^{-1}\langle\lambda, F(z)\rangle)]}\; p(z)$$
$$= \exp\left(\epsilon^{-1}(\langle\lambda, F(z)\rangle - G_F(\lambda))\right) p(z),$$

where the ***nonlinearly tilted CGF*** is given by

$$G_F(\lambda) = \sup_{F^{-1}(y)\in\mathbb{R}^n} \left(\langle\lambda, y\rangle - I \circ F^{-1}(y)\right), \quad y = F(z),$$

and is being bounded and differentiable. At the same time, the ***effective rate function*** $I \circ F^{-1}(y)$ is strictly convex. The ***conditions*** of the nonlinear reparametrization are;

1- F is a diffeomorphism, and

2- $I \circ F^{-1}(y)$ is strictly convex, i.e. $\langle v\; \mathrm{Hess}(I \circ F^{-1})(y)\, v\rangle > 0 \;\forall\, v \in \mathbb{R}^n$.

References [1] Alqahtani and Grafke, Journal of Physics A: Mathematical and Theoretical, 2021. [2] Chevillard and Meneveau, Physical review letters, 2006. [3] Gonzalez, Physics of Fluids, 2009. [4] Dembo and Zeitouni, Springer-Verlag, 2010. [5] Freidlin and Wentzell, Springer, 2012. [6] Ellis et al.,The Annals of Probability, 1984. [7] Touchette, published online: www.maths.qmul.ac.uk, 2005.[8] Rindler, Springer, 2018. [9] Touchette, Physics Reports, 2009. [10] Deriglazov, Springer International Publishing, 2017.