Homological Algebra

Admin info

- No lecture next week.
- Lectures 2 hours (w. 5min break)

Assesment: No exam,
but coursework sheets.
— I'll mark + eturn what you submit (even if not taking course for credit.)

- Course web page

Books etc

- Weibel: very readable, covers everything + more
  but uses homology (not cohomology) notation.
- Gelfand-Manin (chaps II & III).
- McCleary for spectral seqs

1. Category Theory

1.1 Categories

Def^n A category $\mathcal{C}$ is the following:

- a class $Ob(\mathcal{C})$ of "objects"
- for each $X, Y \in Ob(\mathcal{C})$, a set $Hom_{\mathcal{C}}(X, Y)$ of "arrows" or "morphisms"
- for all $X, Y, Z \in Ob(\mathcal{C})$, a binary operation

$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \to Hom(X,Z)$$
$$f \quad , \quad g \quad \mapsto g \circ f$$

Such that:

- morphisms have unique source & target: $Hom(X,Y) \cap Hom(X',Y') = \emptyset$ unless $X' = X$, $Y' = Y$
- composition of morphisms is associative

$$f \circ (g \circ h) = (f \circ g) \circ h$$

- for every X $\exists$ a morphism $Id_X \in Hom(X,X)$ neutral for composition of morphisms on either side (necessarily unique).

Remarks

(i) Objects have "no structure"
— emphasis on morphisms.

(ii) Objects of a cat aren't necessarily a set – e.g. "set of all sets" doesn't exist.
but $\mathrm{Hom}(X,Y)$ is a set ("locally small")
If $\mathcal{C}$ is st. $\mathrm{Ob}(\mathcal{C})$ is a set, say $\mathcal{C}$ is small.

E.g. · category **Set** of sets + functions.
$\mathrm{Hom}_{\mathbf{Set}}(X,Y) = \{\text{fcns } X \to Y\}$.

· **Grp** (groups + group homs)

· **Ab** (abelian grps + grp homs)

· **R-Mod**, R some ring – (left) R-modules
Mod-R right modules.

· Given any set X, $\exists$ cat. with objects X and all morphisms identity.

· Given any group G, $\exists$ cat. with 1 obj. + $\mathrm{Hom}(\bullet,\bullet) = G$.
(works for any monoid – needn't be a group)

· For any partially ordered set $(X, \geq)$, $\exists$ cat with objects X and $\mathrm{Hom}(x,y) = \begin{cases} \emptyset & \text{if } x \not\geq y \\ \text{singleton} & \text{if } x \geq y \end{cases}$
("poset category").

Notations A morphism $f: X \to Y$ in $\mathcal{C}$ is an isomorphism if $\exists\, g: Y \to X$ st $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.
(g is unique if it exists — why?

g, g' two inverses

$$\underbrace{(g' \circ f)}_{\substack{\| \\ g'}} \circ g = g$$)

Def $f: X \to Y$ is a monomorphism in $\mathcal{C}$ if for any Z and homs $g_1, g_2 : Z \to X$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.
— note that monos in Set are precisely injections.

Dually an epimorphism is $f: X \to Y$ st for any $g_1, g_2 : Y \to Z$ st $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$. Epis in Set = surjections.

Initial + final objs

Def For a cat $\mathcal{C}$, say $X \in Ob(\mathcal{C})$ is <u>initial</u> if $\forall\, Y \in Ob(\mathcal{C})$, $\exists!$ hom $X \to Y$

<u>terminal</u> if $\exists!\ Y \to X$.

Eg: $\emptyset$ is initial in <u>Set</u>

terminal objs are <u>singletons</u>.

So initial/term objs aren't quite unique, but <u>unique up to unique isomorphism</u>.

Ex: Does <u>Ring</u> have initial/terminal objs?

$\mathbb{Z}$ is initial   terminal: zero ring.

1.2 Functors

<u>Def^n</u> Let $\mathcal{C}, \mathcal{D}$ categories.

A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ consists of

- for every $X \in \mathrm{Ob}(\mathcal{C})$, an obj $FX \in \mathrm{Ob}(\mathcal{D})$
- for every $f : X \to Y$ in $\mathcal{C}$, a morphism
  $Ff : FX \to FY$ in $\mathcal{D}$

respecting identity morphisms + composition.

Clearly we can compose functors, & any cat has an identity functor.

$\exists$ category <u>Cat</u> whose objs are <u>small</u> cats + whose morphisms are functors.

<u>Boring examples</u>

- "Forgetful" functors $\mathbf{Grp} \to \mathbf{Set}$,
  <u>Ring</u> $\to$ AbGrp $(R \mapsto (R,+))$
  <u>Grp</u> $(R \mapsto (R^\times, \times))$
- <u>Top</u> cat of top spaces + cts maps
  $H_i(-)$: <u>Top</u> $\to$ <u>Ab</u> (singular) homology.
- "Free" functors eg <u>Set</u> $\to$ <u>AbGrp</u>
  $X \mapsto \{$formal $\mathbb{Z}$-linear combs of elts of $X\}$.

$\underline{\text{Def}^n}$ A functor $F : \mathcal{C} \to \mathcal{D}$ is

- $\underline{\text{faithful}}$ if it's injective on hom sets $F : \text{Hom}(X,Y) \hookrightarrow \text{Hom}(FX,FY)$
- $\underline{\text{full}}$ if it's surj on homsets.

E.g. subcategories: if $\mathcal{C}$ cat, $\mathcal{D}$ subcat (ie. $\text{Ob}(\mathcal{D}) \subseteq \text{Ob}(\mathcal{C})$ and $\text{Hom}_\mathcal{D}(-,-) \subseteq \text{Hom}_\mathcal{C}(-,-)$) then inclusion $\mathcal{D} \to \mathcal{C}$ is a faithful functor but not full in gen^l. If full, say $\mathcal{D}$ is a $\underline{\text{full subcat}}$.

E.g. $\underline{\text{AbGrp}}$ is a full subcat of $\underline{\text{Grp}}$

$\underline{\text{Def}^n}$ for $\mathcal{C}$ any cat, $\mathcal{C}^{op}$ = cat with same objs and $\text{Hom}_{\mathcal{C}^{op}}(X,Y) = \text{Hom}_\mathcal{C}(Y,X)$

A $\underline{\text{contravariant functor}}$ $\mathcal{C} \to \mathcal{D}$ is just a functor $\mathcal{C} \to \mathcal{D}^{op}$ (eqvt^{ly} $\mathcal{C}^{op} \to \mathcal{D}$)

For clarity write $\underline{\text{covariant functor}}$ for "usual" functor.

E.g. $\underline{\text{Top}} \to \underline{\mathbb{R}\text{-Vect}}$ contravariant.

$$X \mapsto \{\text{fns } X \xrightarrow{\text{cts}} \mathbb{R}\}$$

$$f : X \to Y \mapsto (g \mapsto g \circ f)$$

Note that $\text{Hom}_\mathcal{C}(X,-)$ covariant $\mathcal{C} \to \text{Set}$

$\text{Hom}_\mathcal{C}(-,X)$ contravar $\mathcal{C} \to \text{Set}$

for any fixed X.

Example 2 $\underline{\text{Man}}$ cat of smooth manifolds + smooth maps

$H^i : \underline{\text{Man}}^{op} \longrightarrow \mathbb{R}\text{-VS}$ (singular) cohomology

$H^i_{dR} : \cdots \longrightarrow \cdots$ de Rham cohomology

$\exists$ nat^l transfmns $H^i \Rightarrow H^i_{dR}$

and $H^i_{dR} \Rightarrow H^i$.

(inverse to each other).

1.4 Equivalence of categories

$\exists$ obvious notion of isomorphism of categories

$C \equiv D$: functors $C \to D$, $D \to C$ whose compositions are Id_C, Id_D.

Never use this.

Better: if $F : C \to D$, say F is an <u>equivalence</u> if $\exists$ $G : D \to C$ + nat^l isomorphisms

$F \circ G \Rightarrow Id_D$

$G \circ F \Rightarrow Id_C$.

(G is a <u>quasi-inverse</u> of F). Note G is <u>not unique</u>.

1.3 Natural transformations

Def^n $\mathcal{C}, \mathcal{D}$ cats, F, G functors $\mathcal{C} \to \mathcal{D}$

A nat^l transformation $T : F \to G$ is,
for every $X \in Ob(\mathcal{C})$, a $T(X) \in Hom(FX, GX)$
st $\forall$ morphisms $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "TX"'] & FY \arrow[d, "TY"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd}$$

commutes

write

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow T"] & \mathcal{D} \end{tikzcd}$$

Eilenberg: Categories were invented not to study objects + functors, but to study nat^l transfmns.

Example 1

$F : \underline{Ab} \to \underline{Set}$
$G \mapsto \{ g \in G : 6g = 0 \}$

$H : \underline{Ab} \to \underline{Set}$
$G \mapsto \{ g : 3g = 0 \}$

$\exists$ nat^l transfmn $F \Rightarrow H$
given by $g \in F(G)$

$$\begin{array}{c} g \in F(G) \\ \downarrow \times 2 \\ 2g \in H(G). \end{array}$$

Example $\mathcal{D} = \mathbb{R}\text{-Vect}^{fd}$ real vector spaces of finite dim^n

$\mathcal{C}$ = the cat with objs $\mathbb{R}^n$ $\forall n \in \mathbb{N}$.

$\mathcal{C}$ is a full subcat of $\mathcal{D}$ but <u>much</u> smaller.

$\mathcal{C} \hookrightarrow \mathcal{D}$ is an equiv of cats.

Thm A functor $F: \mathcal{C} \to \mathcal{D}$ is an equiv of cats iff:

- F is full and faithful
- every obj of $\mathcal{D}$ is isomorphic to FX for some $X \in \text{Ob}\,\mathcal{C}$.

Won't prove this here.

Next time: <u>additive categories</u>