HOMOLOGICAL ALG, LECTURE 2

- Limits + colimits
- Adjunctions
- Additive categories
- Kernels / cokernels
- Abelian cats

1.6 Limits + Colimits

Idea: given a bunch of objs + arrows in a cat. $\mathcal{C}$ find "best approximation" by a single obj.

<u>Def^n</u> $\mathcal{C}$ cat, $\mathcal{J}$ small cat ($\mathrm{Ob}(\mathcal{J})$ is a set)
A <u>$\mathcal{J}$-diagram</u> in $\mathcal{C}$ is a functor $\mathcal{J} \to \mathcal{C}$.

Eg if $\mathcal{J}$ has 3 objs & morphisms

$$\begin{tikzcd} \cdot \arrow[dr] & & \cdot \arrow[dl] \\ & \cdot & \end{tikzcd}$$

then a $\mathcal{J}$-diagram is a triple of objs X, Y, Z in $\mathcal{C}$ & arrows $X \to Y$, $Z \to Y$

$$\begin{tikzcd} X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

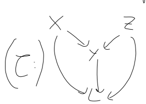
<u>Def^n</u> If D is a $\mathcal{J}$-diagram in $\mathcal{C}$, a <u>cone</u> of D is a single obj L of $\mathcal{C}$, and arrows $\varphi_X : L \to D(X)$ $\forall X \in \mathrm{Ob}\,\mathcal{J}$, compatible with composition.

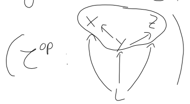
$$\begin{tikzcd} & L \arrow[dl] \arrow[d] \arrow[dr] & \\ X \arrow[r] & Y & Z \arrow[l] \end{tikzcd}$$

__Def^n__ A __limit__ of D is a "universal cone": a cone $(L, \{\varphi_x : x \in J\})$ st $\forall$ other cones (L', φ'), $\exists$ unique morphism $L' \xrightarrow{\psi} L$ st φ' given by compositions $\psi \circ \varphi$.

__Prop__ Limits are unique up to unique iso if they exist.

Dually __co-cone__, __colimit__ = (image in $\mathcal{C}$ of) cone, limit of opposite diagram $J^{op} \to \mathcal{C}^{op}$.





__Examples__ $J = (\cdot \to \cdot \to \cdot \to \cdot \to \cdot \to \dots)$

limit of a J-diagram = single obj universal wrt maps

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[r] & \cdots \\ X \arrow[u] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

and colimit =

$$\begin{tikzcd} X_1 \arrow[r] \arrow[dr] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[dl] \\ & X & \end{tikzcd}$$

So direct limits are colimits (in Ab or Set etc)
inverse limits are limits.

Special case: pairs of morphisms

$$J = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

so a J-diagram = a pair of obs with two morphisms.

Def^n If $f_1, f_2 : X \to Y$ are two morphisms, the equalizer is the limit of

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & Y. \end{tikzcd}$$

Concretely: it's an obj E and arrow $E \xrightarrow{e} X$ st $f_1 \circ e = f_2 \circ e$, + universal w.r.t. this.

Example In Set, the equalizer of f_1, f_2 is $E = \{x \in X : f_1(x) = f_2(x)\}$, e = inclusion map.

- any $Z \xrightarrow{g} X$ factoring via E satisfies $f_1 \circ g = f_2 \circ g$.

Dually <u>coequalizer</u>: colimit of
$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & Y \end{tikzcd}.$$

<u>Lemma</u>

A) Equalizers are monomorphisms, & coeqs epimorphisms.

B) If $g: Y \to Z$ mono, then equalizer of f_1, f_2 is the equalizer of $(g \circ f_1, g \circ f_2)$ (if they exist).

<u>Pf</u> (A) Let $E \xrightarrow{e} X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ equalizer.

and let $Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} E$ st $e \circ g_1 = e \circ g_2$.

WTS: $g_1 = g_2$.

Note that $h = e \circ g_1 = e \circ g_2$ satisfies:

$$Z \xrightarrow{h} X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$$ commutes, i.e. $f_1 \circ h = f_2 \circ h$ (as h factors thru e).

So $\exists!\ Z \xrightarrow{\gamma} E$ st $h = e \circ \gamma$.

But both g_1 and g_2 satisfy this $\Rightarrow g_1 = g_2$.

Reverse arrows $\Rightarrow$ statement for coeqs.

(B): Exercise.

1.7 Adjunctions

Recall examples of functors

$\underline{Ab} \to \underline{Set}$ forgetful

$\underline{Set} \to \underline{Ab}$ "free abgp" functor.

Fact For any set X & (ab) group G

$$\mathrm{Hom}_{\underline{Ab}}\left(\begin{matrix}\text{free abgp}\\ \text{on } X\end{matrix}, G\right) = \mathrm{Hom}_{\underline{Set}}\left(X, \begin{matrix}\text{underlying}\\ \text{set of } G\end{matrix}\right)$$

Def^n $\mathcal{C}, \mathcal{D}$ cats. An adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ is a pair of functors $L : \mathcal{C} \to \mathcal{D}$

$R : \mathcal{D} \to \mathcal{C}$

st $\mathrm{Hom}_{\mathcal{D}}(L(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, R(-))$

as functors $\mathcal{C}^{op} \times \mathcal{D} \to \underline{Set}$

((Say L is left adjt to R, R right adjt to L.))

E.g. "forgetful - free" adjunctions.

Exercise Show that $\underline{Top} \to \underline{Set}$ (forgetful) has both a left and a right adjt. (& they aren't the same).

§2. Additive + Abelian Cats

§2.1 An <u>additive cat</u> is a cat $\mathcal{C}$ together with binary operations "+" on each homset $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ making them into ab gps, st.

(A1) Composition distributes over addition:
$$f\circ(g+h) = f\circ g + f\circ h \quad + \text{ similarly}$$
$$(f+g)\circ h = f\circ h + g\circ h$$

(A2) $\exists$ zero obj $0_{\mathcal{C}}$ st $\mathrm{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, 0_{\mathcal{C}}) = 0$
(hence $\mathrm{Hom}(0,X) = \mathrm{Hom}(X,0) = 0 \ \forall X$)

(A3) For any $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, $\exists$ an obj Y with morphisms

$$\begin{tikzcd}
X_1 \arrow[dr, "i_1"] \arrow[dd, equal] & & X_2 \arrow[dl, "i_2"'] \arrow[dd, equal] \\
& {} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 & & X_2
\end{tikzcd}$$

st diag^l compositions are 0 and $\boxed{i_1 p_1 + i_2 p_2 = \mathrm{id}_Y}$

If the tuple (Y, p_1, p_2, i_1, i_2) exists it is unique up to unique iso + we write $X_1 \oplus X_2$ for Y.

Note $X_1 \oplus X_2$ is both limit of $X_1 \rightarrow 0 \leftarrow X_2$ (product)
+ colimit of $X_1 \leftarrow 0 \rightarrow X_2$ (coproduct)

E.g. · <u>R-Mod</u>, any ring R
· <u>Ab</u>
· <u>$\mathrm{Ban}_{\mathbb{C}}$</u> (Banach spaces / $\mathbb{C}$ & cts maps)

__Def__ If $\mathcal{C}, \mathcal{D}$ additive cats, a functor $\mathcal{C} \to \mathcal{D}$ is an __additive functor__ if it respects addition of morphisms ($\Rightarrow$ also respects zero obj. + direct sums)

Non-example: $\underline{k\text{-Vect}} \to \underline{k\text{-Vect}}$

$$V \mapsto V \otimes V.$$

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§2.2 Kernels & Cokernels

__Def^n__ In an add. cat, if $f: X \to Y$,

$\ker(f) =$ equalizer of f and 0
$\mathrm{coker}(f) =$ co "

} if they exist!

If A add. cat. st all kernels & cokernels exist, for $f: X \to Y$

define $\mathrm{coker}(\ker(f)) =: \mathrm{coim}(f)$

$\mathrm{im}(f) := \ker(\mathrm{coker} f)$

$$\boxed{\begin{tikzcd}
K \arrow[dr, "k"] & & \mathrm{Im}(f) \arrow[dr] & & \\
& X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dr] & \\
& & \mathrm{coim}\, f \arrow[uu, dotted] & & \mathrm{coker}\, f
\end{tikzcd}}$$

Lemma For such a cat A, $\exists!$ morphism $\mathrm{Coim}(f) \xrightarrow{\bar{f}} \mathrm{Im}(f)$ st

$$X \longrightarrow \mathrm{Coim}(f) \xrightarrow{\bar{f}} \mathrm{im}(f) \longrightarrow Y$$

is f.

Proof

$$\begin{tikzcd}
\mathrm{Ker}\, f \arrow[d] \arrow[dr, "0"] & \mathrm{Im}(f) \arrow[d] \\
X \arrow[d] \arrow[r, "f"] \arrow[ur, dashed, "1"] \arrow[dr, "0"'] & Y \arrow[d] \\
\mathrm{Coim}\, f \arrow[uur, dotted, "2"] & \mathrm{Coker}\, f
\end{tikzcd}$$

1 exists by uni- property of $\mathrm{Im}(f)$.

Composite $\ker(f) \longrightarrow X \longrightarrow \mathrm{Im} f$ is 0, so 2 exists by uni- property of $\mathrm{Coim}(f)$.

Define $\bar{f}$ as the arrow 2. Check: it has the right properties.

Def^n An <u>abelian cat</u> is an additive cat st all kers & cokers exist & $\bar{f}$ is an isomorphism $\forall f$.

(Think: we've made First Isomorphism Thm of elementary algebra into a definition)

Example $\text{Ban}_{\mathbb{C}}$ has kernels + cokernels but not abelian.

$\ker(f)$ = set-theoretic kernel with subspace topology

$\text{coker}(f) = Y / (\text{closure of set-theoretic image})$ with quotient norm

So if $f: X \to Y$ is injective with dense image, $\text{Ker}(f) = 0$ & $\text{coker}(f) = 0$, so $\text{coim}(f) = X$, $\text{im}(f) = Y$

but f may not be an iso. (find an example if you haven't seen this.)

So $\text{Ban}_{\mathbb{C}}$ not abelian.

Prop In an abelian cat,

$\ker \varphi = 0 \iff \varphi$ is a mono.

$\text{coker}\,\varphi = 0 \iff \varphi$ is an epi.

$\ker \varphi = \text{coker}\,\varphi = 0 \iff \varphi$ is an iso.

Pf If $\ker \varphi = 0$, $\text{coim}(\varphi) = X$

so $X \xrightarrow[\sim]{\bar{\varphi}} \text{coim}\,\varphi \to Y$ this is by def^n an equalizer, hence mono.

other cases similar.

§2.3 Exact sequences

Defⁿ A complex in a(n ab) category $\mathcal{A}$, indexed by some interval $I \subset \mathbb{Z}$ is a collection of objs X^i : $i \in I$ & $d^i : X^i \to X^{i+1}$ st

$$d^i \cdot d^{i-1} = 0 \text{ (wherever defined).}$$

This is sometimes called a cochain complex (indices as superscript). Chain complexes have the arrows reversed, $X_i \xrightarrow{d_i} X_{i-1}$ etc.

Defⁿ Say a complex $X^\bullet$ is exact at i if $\ker(d^i) = \text{image}(d^{i-1})$. If exact at i $\forall i \in I$ say it's an exact sequence.

Defⁿ For any complex $X^\bullet$, write

$$H^i(X^\bullet) = \text{coker}\left(\text{im}(d^{i-1}) \to \ker(d^i) \right).$$

Have $H^i(X^\bullet) = 0 \iff X^\bullet$ exact at all $i \in I$

Def^n An additive functor $F: \mathcal{C} \to \mathcal{D}$ between ab. cats. is:

- exact if $\forall$ exact seqs
  $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$,
  $0 \to FX \to FY \to FZ \to 0$ is exact in $\mathcal{D}$.
- left exact if $\forall$ $0 \to X \to Y \to Z \to 0$,
  $0 \to FX \to FY \to FZ$ is exact.
- right exact similarly
  $FX \to FY \to FZ \to 0$.

Fiddly Lemma

F is left-exact $\iff$ $F(\ker \varphi) = \ker(F\varphi)$ for all morphisms φ

right-exact $\iff$ $F(\operatorname{coker} \varphi) = \operatorname{coker}(F\varphi)$ $\forall \varphi$.