$$\begin{tikzcd}
\ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[dll, rounded corners, to path={-- ([xshift=4ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-4ex]\tikztotarget.west) \tikztonodes -- (\tikztotarget)}, "S"] & \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] \\
& \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma
\end{tikzcd}$$

Will prove this assuming $\mathcal{C} = R\text{-Mod}$ some R.

Existence of maps $\ker\alpha \to \ker\beta$. etc, is obvious. Need to check:

- exactness at $\ker\beta$ and $\operatorname{coker}\beta$
- existence of S
- exactness at $\ker\gamma$ and $\operatorname{coker}\alpha$.

<u>$\ker\beta$</u> let $b \in \ker(\ker\beta \to \ker\gamma)$

i.e. $b \in B$ st $\beta(b) = 0$ and $g(b) = 0$.

Then $\exists\ a \in A$ st $b = f(a)$

Want to show $a \in \ker(\alpha)$.

We know $f'(\alpha(a)) = \beta(f(a))$
$= \beta(b) = 0$

but f' is inj $\Rightarrow \alpha(a) = 0$.

("diagram chase").

cokernel(β) similar.

existence of S: see Hollywood clip
+ remaining checks: exercise.

For gen^l abelian categories, use:

<u>Freyd-Mitchell thm</u> For any ab. cat. $\mathcal{C}$, $\exists$ a ring R and a fully faithful functor $\mathcal{C} \to R\text{-Mod}$ which is exact.

(Warning: R is <u>horrible</u>! + doesn't send inj/proj objs to inj/proj).

Corollary $\mathcal{C}$ any ab cat,

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

SES in $Ch(\mathcal{C})$. Then $\exists$ long exact seq

$$\cdots \to H^iA^\bullet \to H^iB^\bullet \to H^iC^\bullet \xrightarrow{\delta} H^{i+1}A^\bullet \to H^{i+1}B^\bullet \to H^{i+1}C^\bullet \to \cdots$$

δ = "coboundary map"

Proof Firstly, apply snake to

$$\begin{tikzcd}
0 \arrow[r] & A^i \arrow[r] \arrow[d] & B^i \arrow[r] \arrow[d] & C^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0
\end{tikzcd}$$

to get

$$0 \to \ker(d_A^i) \to \ker(d_B^i) \to \ker(d_C^i) \to \operatorname{coker}(d_A^i) \to \operatorname{coker}(d_B^i) \to \operatorname{coker}(d_C^i) \to 0$$

Consider

$$\begin{tikzcd}
& H^i(A^\bullet) \arrow[r] \arrow[d] & H^i(B^\bullet) \arrow[r] \arrow[d] & H^i(C^\bullet) \arrow[d] & \\
& \operatorname{coker} d_A^i \arrow[r] \arrow[d] & \operatorname{coker} d_B^i \arrow[r] \arrow[d] & \operatorname{coker} d_C^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker d_A^{i+1} \arrow[r] \arrow[d] & \ker d_B^{i+1} \arrow[r] \arrow[d] & \ker d_C^{i+1} \arrow[d] & \\
& H^{i+1}(A^\bullet) \arrow[r] & H^{i+1}(B^\bullet) \arrow[r] & H^{i+1}(C^\bullet) &
\end{tikzcd}$$

kernels, resp. cokernels, of vertical maps are H^i, resp H^{i+1} $\square$

Remark Can show δ is natural, i.e. if we have two SESs in $Ch(\mathcal{C})$

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'^\bullet \arrow[r] & B'^\bullet \arrow[r] & C'^\bullet \arrow[r] & 0
\end{tikzcd}$$

then get morphisms of long exact seqs compatible with the δ's.

§ 3.2 Resolutions

Def. for $X \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}$ ab cat,
$[X]$ - complex with X in deg 0 & 0 elsewhere.

$$\cdots \quad 0 \to 0 \to X \to 0 \to \cdots$$

- A resolution of X is a quasi-iso $[X] \simeq$ (something).

Particular cases:
right resolution: complex $R^\cdot$ st $R^i = 0$ for $i < 0$,
st $H^i(R^\cdot) = 0 \ \forall i > 0$, & isomorphism $H^0(R^\cdot) \cong X$.
i.e. "augmented cplx"

$$\cdots 0 \to X \to R^0 \to R^1 \to \cdots \quad \text{is exact.}$$

left res^n: objects L_i in a (exact) seq

$$\cdots \quad L_2 \to L_1 \to L_0 \to X \to 0.$$

Key cases: injective right resolutions
projective left resolutions.

Prop If $\mathcal{C}$ has enough injs, respectively projs, then every obj has an inj, resp proj, resolution.

Proof STP result for injectives.
Have $0 \to X \to I^0$ exact, I^0 inj. (def^n of enough injs.)
Consider I^0/X: this also maps into some inj I^1

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 \\
 & X \arrow[dr] & & & & I^1/(I^0/X) \arrow[ur] \arrow[dr] & \\
 & & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[ur] \arrow[rr] & & I^2 \arrow[r] & {} \\
 & & & I^0/X \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

"splicing"

Then $0 \to X \to I^0 \to I^1 \to \cdots$ is exact. $\square$

Examples

- In $\underline{Ab}$, injective objs are divisible groups.

Inj res^n of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \quad \text{exact, so}$$

$$[\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}] \text{ inj res}^n \text{ of } \mathbb{Z}.$$

- Proj res^{n}s in $\underline{Ab}$: $\mathbb{Z}$ is free, hence proj.

$$0 \leftarrow \mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z} \xleftarrow{[2]} \mathbb{Z} \leftarrow 0 \ldots$$

$[\mathbb{Z} \xrightarrow{[2]} \mathbb{Z}]$ proj res^n of order 2 cyclic gp.

- in $\mathbb{Q}[X,Y]$-Mod:

take $\mathbb{Q}$ with X, Y acting trivially

$$\begin{tikzcd}
0 & \mathbb{Q} \arrow[l] & \mathbb{Q}[X,Y] \arrow[l] & \mathbb{Q}[X,Y]^2 \arrow[l] & \mathbb{Q}[X,Y] \arrow[l] & 0 \arrow[l] \\
 & 0 & x, y \arrow[l, mapsto] & f, g \arrow[ll, mapsto, start anchor=south west, end anchor=east, shift left=0] & f \arrow[l, mapsto] & \\
 & & Xf + Yg & (Yf, -Xf) & &
\end{tikzcd}$$

$0 \mapsfrom x, y$; $\quad Xf+Yg \mapsfrom f, g$; $\quad (Yf, -Xf) \mapsfrom f$

free, hence proj, resolution.

§3.3 Chain Homotopies

Def^n $\mathcal{C}$ ab cat, $A^\cdot$, $B^\cdot$ obj's of $Ch(\mathcal{C})$

A cochain map $f^\cdot : A^\cdot \to B^\cdot$
is null-homotopic if $\exists$ collection of morphisms
$s^i : A^i \to B^{i-1}$ st

$$f^i = d_B^{i-1} \cdot s^i + s^{i+1} \cdot d_A^i \qquad (\text{"} f = ds + sd \text{"})$$

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & A^i \arrow[r] \arrow[d] \arrow[dl] & A^{i+1} \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

motivation from topology (Weibel)

or: let $Hom(A^\cdot, B^\cdot)$ cplx with i^{th} term

$Hom(..) \in Ch(Ab)$ $\qquad \prod_j Hom_{\mathcal{C}}(A^j, B^{j+i})$.

Can equip $Hom(-)$ with differentials
st $\ker(d^0) =$ cochain maps $A^\cdot \to B^\cdot$
then $im(d^{-1}) =$ null-homotopic maps.

Notation Say $f^\cdot, g^\cdot$, cochain maps $A^\cdot \to B^\cdot$,
are homotopic if $f - g$ is null-homotopic.

Prop (i) If $f^\cdot, f'^\cdot : A^\cdot \to B^\cdot$
$g^\cdot, g'^\cdot : B^\cdot \to C^\cdot$ are homotopic,
then $g \circ f$ homotopic to $g' \circ f'$.

(ii) if $f : A^\cdot \to B^\cdot$ null-homotopic,
$F : \mathcal{C} \to \mathcal{D}$ any additive functor, then
$F(f^\cdot)$ is null-homotopic.

(iii) Null-homotopic maps induce 0 on
$H^i(-)$.

Proof (i), (ii) straightforward.

for (iii):

$$\begin{tikzcd} & A^i \arrow[r, "d^i"] \arrow[dl, "s^i"'] \arrow[d, "f"] & A^{i+1} \arrow[dl, "s^{i+1}"] \\ B^{i-1} \arrow[r, "d^{i-1}"'] & B^i & \end{tikzcd}$$

composite $\ker(d_A^i) \to A^i \xrightarrow{f^i} B^i$
factors thru d_B^{i-1} via s^i, hence
$\ker(d_A^i) \to H^i(A) \to H^i(B)$
factors thru the map $\mathrm{im}(d_B^{i-1}) \to \ker(d_B^i)$
+ is thus 0. $\square$

Prop Let $X, Y \in \mathrm{Obj}(\mathcal{C})$, $f: X \to Y$,
$I^\cdot, J^\cdot$ inj res^{n}s of X, Y.
Then $\exists\ \tilde{f} \in \mathrm{Hom}_{Ch(\mathcal{C})}(I^\cdot, J^\cdot)$
inducing f on H^0, & $\tilde{f}$ is unique up to homotopy.

Corollary If $I^\cdot, J^\cdot$ inj res^{n}s of same obj X, then $\exists$ maps of complexes

$$\left.\begin{array}{l} \alpha: I^\cdot \to J^\cdot \\ \beta: J^\cdot \to I^\cdot \end{array}\right\} \text{inducing } id_X \text{ on } H^0,$$

and $\alpha \cdot \beta$, $\beta \cdot \alpha$ are homotopic to the identity.
("Inj res^{n}s are unique up to homotopy equivalence")

Pf of corollary
Identity id_X must lift to some
$\alpha: I^\cdot \to J^\cdot$
& symmetrically $\beta: J^\cdot \to I^\cdot$
Composite $\beta \cdot \alpha : I^\cdot \to I^\cdot$ is a
lifting of id_X, but so is $id_{I^\cdot}$
$\Rightarrow \beta \cdot \alpha$ is homotopic to $id_{I^\cdot}$ by uniqueness statement of prop.
& similarly $\alpha \cdot \beta$.

Proof of Prop^n

Existence: induct on i.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, dotted] & I^0 \arrow[d, "\tilde{f}^0"] \\ Y \arrow[r, hook] & J^0 \end{tikzcd}$$

diag^l arrow $X \to J^0$ must lift to some $\tilde{f}^0$ because J^0 is inj.

Now suppose $\tilde{f}^j$ constructed for $0 \leqslant j \leqslant i$

$$\begin{tikzcd} \cdots & I^{i-1} \arrow[r] \arrow[d] & I^i \arrow[r] \arrow[d] \arrow[dr] & I^{i+1} \arrow[d, "\tilde{f}^{i+1}"] \\ \cdots & J^{i-1} \arrow[r] & J^i \arrow[r] & J^{i+1} \end{tikzcd}$$

Diag^l map $I^i \to J^{i+1}$ triv^l on image of I^{i-1}

so
$$\begin{tikzcd} 0 \arrow[r] & I^i/\mathrm{im}(I^{i-1}) \arrow[r] \arrow[dr] & I^{i+1} \arrow[d, dotted] \\ & & J^{i+1} \end{tikzcd}$$

from injectivity of J^{i+1}

<u>Remark</u> Don't actually need I's to be injective, or that J's be exact!

Uniqueness of $\tilde{f}$ up to homotopy (sketch): again use induction on i to build homotopies $\Box$

Prop Let $0 \to X \to Y \to Z \to 0$ SES in $\mathcal{C}$, $I_X^\cdot$, $I_Z^\cdot$ inj res^{n}s of X, Z, then $\exists$ inj res^n $I_Y^\cdot$ st $I_Y^i = I_X^i \oplus I_Z^i$ $\forall i$, and inclusion & proj^n maps

$$0 \to I^{\cdot}_X \to I^{\cdot}_Y \to I^{\cdot}_Z \to 0$$

are liftings of maps $X \to Y \to Z$.

("Horseshoe Lemma")

<u>Proof</u>

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & I^0_X \arrow[r] \arrow[d] & I^1_X \arrow[r] & \cdots \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & I^0_X \oplus I^0_Z \arrow[d] & & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & I^0_Z \arrow[r] \arrow[d] & I^1_Z \arrow[r] & \cdots \\
 & 0 & 0 & &
\end{tikzcd}$$

Can extend $X \to I^0_X$ to $Y \to I^0_X$ & compose with $I^0_X \hookrightarrow I^0_X \oplus I^0_Z$

By Snake Lemma, $Y \to I^0_X \oplus I^0_Z$ is injective, and

$$0 \to I^0_X/X \to I^0_Y/Y \to I^0_Z/Z \to 0$$

is an SES: now repeat with these in place of X, Y, Z to define I^1_Y

+ continue inductively $\square$