Afterthoughts from last week

- <u>Notation</u> A map of complexes
  $f : A^\bullet \to B^\bullet$ is a <u>homotopy equivalence</u>
  if $\exists\, g : B^\bullet \to A^\bullet$ st $g \circ f$ homotopic to $\mathrm{id}_{A^\bullet}$
  $f \circ g$ h'topic to $\mathrm{id}_{B^\bullet}$

So injective resolutions are unique up to homotopy equiv.

If f is a homotopy equiv, it is also a quasi-iso,
<u>but</u> if F is any additive functor,
f h.equiv $\Rightarrow$ $F(f)$ h.equiv
(<u>not</u> true for quasi-isos!)

- In proof of Horseshoe Lemma:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & I^0_X \arrow[r] \arrow[d] & \cdots \\
 & Y \arrow[r, dotted] \arrow[d] & I^0_X \oplus I^0_Z \arrow[d] & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & I^0_Z \arrow[r] & \cdots \\
 & 0 & &
\end{tikzcd}$$

Dotted arrow:
choose $Y \to I^0_X$ extending $X \to I^0_X$
also have a map $Y \to I^0_Z$ by composing
$Y \to Z \to I^0_Z$
dotted arrow: <u>sum</u> of these.

Chapter 4 : Derived Functors

§4.1 Setup

$\mathcal{C}, \mathcal{D}$ ab cats, $F: \mathcal{C} \to \mathcal{D}$ left-exact

SES $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$

$\rightsquigarrow \quad 0 \to FX \to FY \to FZ \dashrightarrow ??$

Can we extend this to a LES?
Is there a "best way" to extend to the right?
(universal δ-functor)

Assume $\mathcal{C}$ has enough injectives.

<u>Def^n</u> For $i \geq 0$, $X \in \mathrm{Obj}(\mathcal{C})$,

let $R^i(F)(X) = H^i(F(I^\cdot_X)) \in \mathrm{Obj}\,\mathcal{D}$

where $I^\cdot_X$ is an inj res^n of X. (well-def up to isomorphism)

For $f: X \to Y$, $R^i(F)(f) =$ map on H^i
induced by a lifting $\tilde{f}: I^\cdot_X \to I^\cdot_Y$
of f. ($\tilde{f}$ well-def up to homotopy,
so well-def. on H^i.)

This makes $R^i(F)$ into functors. (strictly speaking, need to <u>choose</u> an inj res^n for each X, but changing these changes the functor by a nat^l isomorphism).

Prop

1. R^iF are additive functors
2. $R^0F = F$
3. if F exact, $R^iF = 0 \quad \forall i \geq 1$
4. Nat' transfmns $F \Rightarrow G$ induce $R^iF \Rightarrow R^iG$ all i
5. if X injective, $R^iF(X) = 0 \quad \forall i \geq 1$.

Pf: Straightforward. For #2 note

$0 \to X \to I_X^0 \to I_X^1$ exact

So $0 \to FX \to FI_X^0 \to FI_X^1$ exact

i.e. $FX = \ker(FI_X^0 \to FI_X^1) = R^0(F)(X)$. $\square$

Prop If $0 \to X \to Y \to Z \to 0$ SES, have LES

$$\begin{tikzcd} 0 \arrow[r] & FX \arrow[r] & FY \arrow[r] & FZ \arrow[dll, out=0, in=180] \\ & R^1FX \arrow[r] & R^1FY \arrow[r] & R^1FZ \arrow[dll, out=0, in=180] \\ & R^2FX \arrow[r] & \cdots \end{tikzcd}$$

Pf By Horseshoe Lemma, may assume $I_Y = I_X \oplus I_Z$

Thus $0 \to FI_X^{\cdot} \to FI_Y^{\cdot} \to FI_Z^{\cdot} \to 0$

exact in $Ch(\mathcal{D})$

+ now take LES of cohomology. $\square$

Similarly if $F: \mathcal{C} \to \mathcal{D}$ right-exact, $\mathcal{C}$ having enough projectives, define

$L_i(F)(X) = H_i(F(P_\cdot))$ $\quad P_\cdot \to X$ proj res^n.

& can continue exact seqs to left

$\cdots \to L_1FY \to L_1FZ \to FX \to FY \to FZ \to 0$.

(& can also extend this to contravariant functors.)

§4.2 The Ext functor

= derived functors of Hom.

$\mathcal{C}$ ab cat, $A, B \in \mathrm{Obj}(\mathcal{C})$.

Def^{n} If $\mathcal{C}$ has enough injs, let

$$\mathrm{Ext}^i_{\mathcal{C}}(A,B) = R^i(\mathrm{Hom}(A,-))(B) \in \mathrm{Obj}(\mathrm{Ab})$$

Clearly functorial in B, but also in A, because homs $A \to A'$ give nat^{l} transf^{n}'s of Hom functors.

So Ext^i is a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$.

E.g. $\mathrm{Ext}^i_{\mathrm{Ab}}(\mathbb{Z}/n, \mathbb{Z})$ $n \geq 1$:

$[\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}]$ is our resolution of $\mathbb{Z}$

$\mathrm{Hom}(\mathbb{Z}/n, \text{this}) = [0 \to \mathbb{Z}/n]$

so $\mathrm{Ext}^i = \begin{cases} 0 & i \neq 1 \\ \mathbb{Z}/n & i = 1. \end{cases}$

If $\mathcal{C}$ has enough projectives can also consider

$R^i\mathrm{Hom}_{\mathcal{C}}(-,B) : \mathcal{C}^{op} \to \mathrm{Ab}$

(compute using inj res^{n}'s in $\mathcal{C}^{op}$, ie proj res^{n}'s in $\mathcal{C}$.)

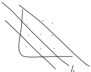
Prop ("Balancing Ext") If $\mathcal{C}$ has enough injs & projs, these constructions agree.

Sketch Pf $B \to I^{\cdot}$, $P_{\cdot} \to A$ inj/proj res^{n}'s.

Want isos $H^i(\mathrm{Hom}(P_{\cdot}, B)) \simeq H^i(\mathrm{Hom}(A, I^{\cdot}))$

Let $X^{pq} = \mathrm{Hom}(P_p, I^q)$,

$$T^n = \bigoplus_{p+q=n} X^{pq}$$



Can make $T^\cdot$ into a cplx (sum over diag'l stripes)
& there are quasi-isos

$$\mathrm{Hom}(A, I^\cdot) \to T^\cdot \leftarrow \mathrm{Hom}(P_\cdot, B).$$

<u>Upshot</u> Can compute Ext using proj res^{n}s when they exist — usually <u>easier</u>.

Eg $[\mathbb{Z} \xrightarrow{n} \mathbb{Z}]$ proj res^n of $\mathbb{Z}/n$
so recovers previous computation of $\mathrm{Ext}^\cdot(\mathbb{Z}/n, \mathbb{Z})$

Why the name "Ext"?

<u>Def^n</u> An extension of A by B is a SES

$$0 \to B \to E \to A \to 0 \qquad \text{some } E.$$

Say 2 ext^{n}s are equiv^t if $\exists$ diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal, "id"'] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal, "id"] & 0 \\
0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

<u>Fact</u> $\exists$ bijection $\mathrm{Ext}^1_{\mathcal{C}}(A,B) \cong \left\{ \begin{matrix} \text{equiv classes} \\ \text{of ext}^n\text{s} \end{matrix} \right\}$.

Given as follows: assume enough injs
apply $\mathrm{Hom}(A, -)$ to ext^n seq:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(A,E) \arrow[r] & \mathrm{Hom}(A,A) \ni id_A \arrow[dll, bend left, "\delta"] \\
& \mathrm{Ext}^1(A,B) & &
\end{tikzcd}$$

send our ext^n to $\delta(id_A)$.

§4.3 Group Cohomology

G group, $\underline{G\text{-Mod}}$ cat. of ab. gps. with $\mathbb{Z}$-linear left G-action

$= \underline{\mathbb{Z}[G]\text{-Mod}}$ $\quad \mathbb{Z}[G]$ group ring.

$F: \underline{G\text{-Mod}} \to \underline{Ab}$ $\quad M \mapsto M^G$: invariants functor.

<u>Def</u>ⁿ $H^i(G,M) = R^i(F)(M)$.

Have $F = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$

so $H^i(G,M) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M)$.

Can compute using either inj res^{n}s of M or proj res^{n}s of $\mathbb{Z}$

E.g. $G = \langle g \rangle$ infinite cyclic

$$\left[\mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G]\right] \text{ proj res}^n \text{ of } \mathbb{Z}$$

so $H^i(G,M) =$ coh. of cplx

$$\left[M \xrightarrow{g-1} M\right]$$

ie $H^0(G,M) = M^G$

$$H^1(G,M) = M/(g-1)M$$

$H^1(G,M) = M/_{(g-1)M}$.

$H^i = 0 \quad i \notin \{0,1\}$

In fact $\exists$ canonical <u>free</u> res^n of $\mathbb{Z}$ ("bar resolution")

<u>Def^n</u> let X_i = free $\mathbb{Z}$-mod on symbols $(g_0, \dots, g_i)$ $g_i \in G$.

G-action: $g \cdot (g_0, \dots, g_i) = (gg_0, \dots, gg_i)$

$d : X_i \to X_{i-1}$ sends $(g_0 \cdots g_i)$ to

$$\sum_{j=0}^{i} (-1)^j (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_i) \in X_{i-1}.$$

Check: this is a chain cplx,

X_i is $\mathbb{Z}G$-free (symbols with $g_0 = id$ are basis)

and $X_i \to [\mathbb{Z}]$ is a free, hence proj, res^n.

Hence $H^i(G,M) = H^i(\mathrm{Hom}_{\mathbb{Z}G}(X_\bullet, M))$

$$\| $$

$C^\bullet(G,M)$ cochain complex.

($\simeq$ fcns on $G \times \dots \times G$ with funny differential)

§4.4 Tor and Group Homology

R ring, $\underline{R\text{-Mod}}$ left modules
$\underline{\text{Mod-}R}$ right modules

$\otimes : \underline{\text{Mod-}R} \times \underline{R\text{-Mod}} \longrightarrow \underline{Ab}$

right exact in both factors.

__Def__ $\mathrm{Tor}_i^R(A,B) = L_i(A\otimes -)(B)$

__Prop__ This is isomorphic to $L_i(-\otimes B)(A)$
(balancing Tor).

Eg $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, A)$
$[\mathbb{Z} \xrightarrow{n} \mathbb{Z}]$ proj res^n of $\mathbb{Z}/n$

so $\mathrm{Tor}_0(\mathbb{Z}/n, A) = A/nA$
$\mathrm{Tor}_1(\mathbb{Z}/n, A) = A[n]$ n-torsion.

Special case: $R = \mathbb{Z}[G]$

$$H_i(G,M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

left derived functors of $\mathbb{Z} \otimes_{\mathbb{Z}G} M$

$$= M \big/ \langle (g-1)M : g\in G \rangle$$

(largest quotient on which G acts trivially, called __coinvariants__)

§4.5 Cohomology of Sheaves

X top. space

<u>Def^n</u> A <u>presheaf</u> on X with values in an ab. cat. $\mathcal{C}$ is the data of:

- for each open U an obj $\mathcal{F}(U)$
- for each inclusion $V \subseteq U$ a map $\mathcal{F}(U) \to \mathcal{F}(V)$ (compat. w composition)

Say $\mathcal{F}$ is a sheaf if for all coverings

$$U = \bigcup_{i \in I} U_i,$$

$$0 \to \mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{\substack{i,j \in I \\ i \neq j}} \mathcal{F}(U_i \cap U_j)$$

is <u>exact</u>. (need to assume $\mathcal{C}$ has products!)

Key case $\mathcal{C} = \underline{Ab}$ (abelian sheaves)

<u>Fact</u> inclusion $\underline{AbSh}(X) \hookrightarrow \underline{AbPSh}(X)$ has a <u>left adjoint</u>, "sheafification".

Lemma $\underline{\text{AbSh}(X)}$ is an abelian category.

(note: cokernel in $\underline{\text{AbSh}}$ = sheafification of presheaf cokernel)

and has enough injectives.

Have a functor $\Gamma : \underline{\text{AbSh}(X)} \to \underline{\text{Ab}}$

$\mathcal{F} \longmapsto \mathcal{F}(X)$, "global sections"

This turns out to be left exact but not right exact in gen'l.

Defn $H^i(X, \mathcal{F}) = R^i(\Gamma)(\mathcal{F})$

Useful to be aware of: if X is a manifold, $H^i(X, \text{constant sheaf } \underline{A}) \cong H^i(X, A)$

for A abelian gp.

However sheaf coh. works well for nasty spaces, e.g. Zariski top. of an alg variety.