Last time:

- spectral seqs
- hyper-derived functors
- 2 sp. seqs of hypercoh:

$$\left.\begin{array}{l} {}^{I}E_2^{pq} = H^p\left((R^qF)(X^\cdot)\right) \\ {}^{II}E_2^{pq} = (R^pF)(H^qX^\cdot) \end{array}\right\} \Rightarrow \mathbb{R}^{p+q}F(X^\cdot)$$

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§5.5 Grothendieck's Sp. Seq.

$\mathcal{C}, \mathcal{D}, \mathcal{E}$ ab. cats

$F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{E}$ left exact

$\mathcal{C}, \mathcal{D}$ enough injs

Def^n Say F, G satisfy Grothendieck cond^n if F sends inj. objects of $\mathcal{C}$ to G-acyclic objs of $\mathcal{D}$.

E.g. if F sends injs to injs, this holds for any G.

Thm If F, G satisfy Gr. cond^n, then $\forall X \in \mathrm{Obj}\,\mathcal{C}$ have a sp. seq.

$$E_2^{pq} = R^p(G)\, R^q(F)(X) \Rightarrow R^{p+q}(G \circ F)(X)$$

Proof Let $I^\cdot$ inj res^n of X.
Consider $R^n(G)(F(I^\cdot))$.

2 sp. seqs. converging to this:

$${}^{I}E_2^{pq} = H^p((R^qG)(F(I^\cdot)))$$ vanishes unless $q=0$.

so $R^p(G)(FI^\cdot) = H^p(G(F(I^\cdot)))$
$= R^p(G\circ F)(X)$.

$${}^{II}E_2^{pq} = R^p(G)(H^q(F(I^\cdot)))$$
$$(R^pG)(R^qF)(X).$$ $\square$

Useful criterion for verifying Grothendieck cond^n:

Lemma If $R: \mathcal{C} \to \mathcal{D}$ is additive + has a left adj^t $L: \mathcal{D} \to \mathcal{C}$, and L is exact, then R sends injectives to injectives.

Pf I inj in $\mathcal{C} \iff \mathrm{Hom}_{\mathcal{C}}(-, I)$ exact

$$\mathrm{Hom}_{\mathcal{D}}(-, R(I)) = \mathrm{Hom}_{\mathcal{C}}(L(-), I)$$
composite of 2 exact functors $\Rightarrow$ exact.
So $R(I)$ is inj. $\square$

Thm (Hochschild–Serre):
G group, $H \trianglelefteq G$ normal subgp
Then for any G-module M, $\exists$ sp. seq

$$E_2^{pq} = H^p(G/H, H^q(H,M)) \Rightarrow H^{p+q}(G,M).$$

Pf Apply Groth. sp. seq. to functors of H-inv'ts & G/H-inv'ts.
Need to show "inv_H" sends injs to G/H-acyclics
but inv_H has a left adjt
$G/H\text{-Mod} \longrightarrow G\text{-Mod}$, "inflation"
(regarding a G/H mod as a G-mod on which H happens to act trivially)
– identity on underlying ab. gps, hence exact. $\square$

Another cute application: recall
Cohomology of sheaves = derived functors of global sections.

$X \xrightarrow{f} Y$ cts map of top. spaces.

$\mathcal{F}$ sheaf on X
define $f_*(\mathcal{F})$ = sheaf on Y
$f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$ V open in Y.
(check: this is a sheaf.)
left-exact $\underline{\mathrm{AbSh}_X} \to \underline{\mathrm{AbSh}_Y}$

Def^n "higher direct images" $= R^i(f_*)(\mathcal{F})$.

$\Gamma(X,-)$ global sections. Easy to see that $\Gamma(Y, f_*\mathcal{F}) = \Gamma(X,\mathcal{F})$

Question: Is Grothendieck cond^n satisfied?

Answer: Yes, because f_* has a left adjt

$$f^* : \underline{AbSh_Y} \to \underline{AbSh_X}$$

$$f^*(G)(U) = \varinjlim_{\substack{V \supset f(U) \\ \text{open}}} G(V)$$ [maybe need to sheafify this]

Can check f^* is exact (consider stalks)

So f_* preserves injectives.

Hence:

Thm (Leray): for any sheaf $\mathcal{F}$ on X $\exists$ sp. seq.

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X,\mathcal{F}).$$

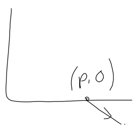
This example motivated the whole theory of sp. seqs & hom. algebra. Worked out by Leray while in a POW camp during World War 2.

§ 5.6 Cookbook: how to get information from sp. seqs.

We've seen lots of examples now — how to get info out of them?

Let E_r^{pq} sp. seq. in some ab cat $\mathcal{C}$ converging to $(X^n)_{n \geq 0}$.

Edge maps consider $(p,0)$ spot



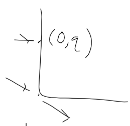
from E_2 onwards outgoing diff. is 0, so $E_{r+1}^{(p,0)}$ is a quot. of $E_r^{(p,0)}$

$\Rightarrow$ have maps $E_2^{p0} \to E_3^{p0} \to \dots \to E_\infty^{p0}$

& $E_\infty^{p0} = \dfrac{Fil^p X^p}{\underbrace{\cancel{Fil^{p+1} X^p}}_{=0}}$ so $E_\infty^{p0} \hookrightarrow X^p$.

Hence have "edge maps" $E_r^{p0} \to X^p$ any $r \geq 2$.

Similarly for any $r \geq 1$ have maps



$$X^q \to E_r^{0q}$$

In particular, E_2^{00} has maps to + from X^0 giving isomorphism

$$X^0 \cong E_2^{00}.$$

Prop "5-term exact seq":

$$0 \to E_2^{10} \xrightarrow[(edge)]{} X^1 \xrightarrow[(edge)]{} E_2^{01} \xrightarrow[(d_2^{01})]{} E_2^{20} \to X^2.$$

This is about as much as one can say in this generality.

Degeneration of sp. seqs

Say E degenerates at the r^{th} page (or "degenerates at E_r")
if all differentials are 0 from r^{th} page onwards.

Clearly we then have $E_r^{pq} = E_\infty^{pq}$ $\forall p,q$.

E.g. recall Hodge-de Rham sp. seq. for a compact cpx mfld X:

$$E_1^{pq}: H^q(X, \Omega^p_{hol}) \Rightarrow H^{p+q}_{dR}(X)$$

Thm (Hodge): for $X = \mathcal{X}(\mathbb{C})$, $\mathcal{X}$ smooth proj. alg variety, this sp. seq. degenerates at E_1, so have

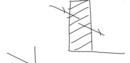
$$\mathrm{Fil}^p H^{p+q}_{dR} / \mathrm{Fil}^{p+1} \cong H^q(X, \Omega^p_{hol}).$$

More straightforward examples: if E_r^{pq} is zero outside some region, often get degeneration automatically.

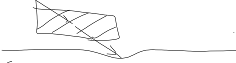
Prop. • If $E_2^{pq} = 0$ for $p > N$, then E degenerates at $r = N+1$.

• If $E_2^{pq} = 0$ for $q > N$, E degenerates at $r = N+2$.

Pf Draw a picture:



all differentials have 0 source or target for r large enough.



□

Special cases:

- If E_2 has only one nonzero row or col, the abutment X^n is just the E_2 term corresponding

- If E_2 has only 2 nonzero columns wlog E_2^{0q} and E_2^{1q}, then E degenerates at $r=2$, + get short exact seqs

$$0 \to E_2^{1,n-1} \to X^n \to E_2^{0,n} \to 0$$

$\forall n$.

If only 2 nonzero rows E_2^{p0}, E_2^{p1} then degeneration is at E_3, not E_2 but

$$E_3^{n0} = \frac{E_2^{n0}}{\operatorname{im}(E_2^{n-2,1})} \text{ etc,}$$

so get long exact seq

$$\cdots \to E_2^{n-2,1} \to E_2^{n0} \to X^n \to E_2^{n-1,1} \to \cdots$$

Example: duality of homology & cohomology

X nice top. space (e.g. a manifold)

$C.(X)$ chain complex (formal sums of simplices in X)

– cplx of free ab. gps

$H_i(X) =$ homology of $C.(X)$

$H^i(X) =$ (co)homology of dual cplx

$$C^{\cdot}(X) = \mathrm{Hom}(C.(X), \mathbb{Z})$$

Prop $\exists$ sp. seq. $E_2^{pq} = \mathrm{Ext}^p_{\mathbb{Z}}(H_q(X), \mathbb{Z})$

$$\Rightarrow H^{p+q}(X, \mathbb{Z}).$$

Pf compute $\mathbb{R}^i(F)(Y^{\cdot})$ via 2 sp seqs,

$F = \mathrm{Hom}(-, \mathbb{Z}) : Ab^{opp} \to Ab$

$Y^{\cdot} =$ image of $C.$ in $Ch(Ab^{opp})$

one sp. seq. collapses because $Y^{\cdot}$ cplx of injs

so 2^{nd} seq gives result. $\square$

We saw $\mathrm{Ext}^p_{\mathbb{Z}}(-, \mathbb{Z}) = 0$ for $p \geq 2$,

so only 2 nonzero columns at E_2

$\Rightarrow$ SES,

$$0 \to \mathrm{Ext}^1(H_{p-1}(X), \mathbb{Z}) \to H^p(X)$$
$$\to \mathrm{Hom}(H_p(X), \mathbb{Z}) \to 0.$$

Chapter 6: Derived Categories

§6.1 The homotopy category

<u>Def</u>

Def^n for $\mathcal{C}$ ab. cat. let

$K(\mathcal{C})$ = cat. with same objs as $Ch(\mathcal{C})$, but morphisms = homotopy classes of cochain maps.

Similarly $K^+(\mathcal{C})$ complexes with $A^n = 0$, $n \ll 0$

K^- bounded-above complexes

K^b bounded cplexes

$K^?$ are additive cats, + have canonical functors $Ch^?(\mathcal{C}) \to K^?(\mathcal{C})$.

Note cohomology functors $H^i : Ch^?(\mathcal{C}) \to \mathcal{C}$ factor thru $K^?(\mathcal{C})$

& inj. resolutions unique up to iso in $K^+(\mathcal{C})$

proj " " " in $K^-(\mathcal{C})$.

Problem The K's are not ab. cats (although this is far from obvious, see eg Holm–Jørgensen–Rouquier §2.6)

So kernels + cokernels don't "work well".

Def^n If $A^\cdot \xrightarrow{f} B^\cdot$ cochain map, define mapping cone $C_f^\cdot$ = cplx with i^{th} term $B^i \oplus A^{i+1}$, differentials

$$d(b, a) = (f(a) - db, da).$$

Check: homotopic morphisms give homotopy-eq^{ct} mapping cones.

So any map $X \xrightarrow{f} Y$ in $K^?(\mathcal{C})$ extends to a triangle

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[dl] \\ C_f \arrow[u, "{[+1]}"] & \end{tikzcd}$$

(i.e. have a map $C_f \to X[1]$, cplx with i^{th} term X^{i+1})

Def^n A distinguished triangle in $K^?(\mathcal{C})$ is a triangle of obj's + morphisms

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u, "{[+1]}"] & \end{tikzcd}$$

isomorphic to one of the above form.

Def^n A <u>triangulated category</u> is an additive cat $\mathcal{T}$ with

- shift functors $[n] : \mathcal{T} \to \mathcal{T}$, $n \in \mathbb{Z}$
- a class of distinguished triangles

satisfying a bunch of axioms, chosen st the cats $K^?(\mathcal{C})$ for $\mathcal{C}$ abelian are automatically triangulated.

Def^n If $\mathcal{T}$ triangulated & $\mathcal{A}$ abelian, a <u>cohomological functor</u> is a functor $F : \mathcal{T} \to \mathcal{A}$ sending distinguished tris to long exact seqs:

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u, "{[+1]}"] & \end{tikzcd} \rightsquigarrow$$

$$F(Z[-1]) \to FX \to FY \to FZ \to F(X[1]) \to \cdots$$

E.g. • H^0 is a coh. functor on $K^?(\mathcal{C})$, & clearly $H^0(X[n]) = H^n(X)$

• for any tri. cat $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(X, -)$ is a coh. functor $\forall X \in \mathrm{Ob}(\mathcal{T})$.

(NB: if $\mathcal{T} = K(Ab)$, $X = \mathbb{Z}$ in degree 0, can check $\mathrm{Hom}_{\mathcal{T}}(\mathbb{Z}, Y) = H^0(Y)$.)