ASSIGNMENTS

* Sheet 2 deadline extended until 16th Dec
* Sheet 3 now up
* For emailed submissions, please use a decent scanner!

Correction to last lecture:

mapping cones.

$$A^\cdot \xrightarrow{f} B^\cdot$$

$C_f^\cdot$ mapping cone

Better sign convention:

$A[1]$ complex with $A[1]^i = A^{i+1}$

$$d^i_{A[1]} = -d^{i+1}_A$$

Then let $C_f^i = A^{i+1} \oplus B^i$

$$d^i_{C_f}((a,b)) = (-d^{i+1}_A a,\ d^i_B b + f(a)).$$

Last time: homotopy cats K^+, K^- etc.

- make more things isomorphisms.

Localization of Categories

Def^n Let $\mathcal{C}$ cat., S collection of morphisms in $\mathcal{C}$. The localization $S^{-1}\mathcal{C}$ is a cat. with a functor $\mathcal{C} \to S^{-1}\mathcal{C}$ st

- any morphism in S goes to an isomorphism in new cat. $S^{-1}\mathcal{C}$
- $S^{-1}\mathcal{C}$ is universal among cats with this property.

(Might not exist, but unique up to equiv. of cats if it exists)

If $\mathcal{C}$ is <u>small</u>, this exists: define morphisms in $S^{-1}\mathcal{C}$ to be "zigzags"

$$\begin{tikzcd}[column sep=small]
 & U_1 \arrow[dl] \arrow[dr] & & U_3 \arrow[dl] \arrow[dr] & & U_n \arrow[dl] \arrow[dr] & \\
X & & U_2 & & \cdots & & Y
\end{tikzcd}$$

in S.

problem: not obvious if this is a <u>set</u> if $\mathcal{C}$ not small.

<u>Example</u> $K^?(\mathcal{C})$ = localiz^n of $Ch^?(\mathcal{C})$ at homotopy equivs.

(This is <u>not</u> automatic!)

<u>Thm</u> If $\mathcal{C}$ = Ab or R-Mod, then the localiz^n of $K(\mathcal{C})$ at quasi-isos exists, & we call it $D(\mathcal{C})$, the <u>unbounded derived category</u>.

(Key lemma: any zigzag as above in $K(\mathcal{C})$ with S = (quasi-isos) collapses to a "roof" $\swarrow\searrow$.)

$D(\mathcal{C})$ is a triangulated cat., & $K(\mathcal{C}) \longrightarrow D(\mathcal{C})$ sends dist. tris to dist. tris. By construction, cohomology functors factor thru $D(\mathcal{C})$.

Inj. objects and $D^+(\mathcal{C})$

$\mathcal{C}$ any ab cat with enough injs.

Prop $D^+(\mathcal{C})$, localiz^n of $K^+(\mathcal{C})$ at quasi-isos, exists, and is given by $K^+(I)$ = full subcat of $K^+(\mathcal{C})$ consisting of complexes of inj. objects.

Sketch of pf Need to show

- any cplx in Ch^+ is quasi-iso. to one with injective terms. — send $X^\cdot$ to $\mathrm{Tot}(I^{\cdot\cdot})$, $I^{\cdot\cdot}$ a CE res^n of $X^\cdot$
- any quasi-iso. between inj cplxes is a homotopy equiv — refinement of a question from Sheet 2!

$\square$

Def^n For any additive $F: \mathcal{C} \to \mathcal{D}$, can define $R_+(F): D^+(\mathcal{C}) \to D^+(\mathcal{D})$ as the functor given by F on $K^+(I)$ "total right derived functor".

Upshot: if F is left exact, then the $R^i(F)$ are the cohomology functors applied to $R_+(F)$.

Advantage of this: total derived functors compose as expected

$$R_+(G \circ F) = R_+(G) \circ R_+(F)$$

- encodes Grothendieck sp. seq.!

"Lots of formulae that ought to work at complex level really do work in derived category."

E.g. for $G \rhd H$ groups, M a G-module,

have $R\Gamma(G, M) \in D^+(\underline{Ab})$,

and $$R\Gamma(G/H, R\Gamma(H, M)) = R\Gamma(G, M).$$

- implies HS spectral seq.

Similarly $D^-(\mathcal{C})$ exists when $\mathcal{C}$ has enough projs, + get total left derived functors.

Some constructions in $D^{+/-}(\underline{R\text{-Mod}})$

- for R commutative,
  derived tensor product

$$D^-(R\text{-Mod}) \times D^-(R\text{-Mod}) \to D^-(R\text{-Mod})$$
$$A, \quad B \longmapsto A \otimes^{\mathbb{L}} B$$

If A, B conc. in degree 0 this has i^{th} cohomology $\mathrm{Tor}^R_{-i}(A^0, B^0)$.

e.g. $A \otimes^{\mathbb{L}} (B \otimes^{\mathbb{L}} C)$
$= (A \otimes^{\mathbb{L}} B) \otimes^{\mathbb{L}} C$ — looks very messy written as sp. seq.

- $\underline{\text{derived Hom}}$ (RHom)

$$D_-(R\text{-Mod})^{opp} \times D^+(R\text{-Mod}) \to D^+(Ab)$$
$$A, \quad B \longrightarrow \mathrm{RHom}(A, B)$$

recovers Ext^i functors.

E.g. universal coeff thms for (singular or simplicial) coh:

$$C^{\cdot}(X, A) = C^{\cdot}(X, \mathbb{Z}) \otimes^{\mathbb{L}}_{\mathbb{Z}} A$$
$$C^{\cdot}(X, A) = \mathrm{RHom}_{\mathbb{Z}}(C_{\cdot}(X), A)$$

(cohomology is "derived dual" of homology.)

Final example:

Serre/Grothendieck duality.

k field, X/k projective variety.

$\mathcal{F}$ sheaf of $\mathcal{O}_X$-modules,

<u>coherent</u> ($\Leftrightarrow$ loc. fin. gen $/\mathcal{O}_X$).

Serre duality: if X smooth dim d

$\mathcal{F}$ locally free

$H^{d-i}(X, \mathcal{F})$ is k-dual of

$H^i(X, \mathcal{H}om(\mathcal{F}, \omega_X))$

$\omega_X = \Lambda^d(\Omega^1_{X/k})$ "dualizing sheaf"

Grothendieck: allow any coherent $\mathcal{F}$

and any X (maybe singular).

$\exists$ obj of $D^+(\mathrm{Coh}_X)$, $\underline{\omega}_X$

st $R\mathrm{Hom}(R\Gamma(X, \mathcal{F}), k)$

$= R\Gamma(X, R\mathcal{H}om(\mathcal{F}, \underline{\omega}_X))$

for all $\mathcal{F} \in \mathrm{Coh}_X$.

If X is smooth, $\underline{\omega}_X = \omega_X[-d]$

but thm applies to any proj. var. X