TCC Homological Algebra: Assignment #3

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6/12/19

This is the last of 3 problem sheets. Solutions should be submitted to me (by email, or via my pigeonhole for Warwick students) by **noon on 6/1/20**. This problem sheet will be marked out of a total of 20; the number of marks available for each question is indicated. Questions marked [*] are optional and not assessed.

Note that rings are not necessarily commutative, but are always assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1. The notation Ab denotes the category of abelian groups, and R-Mod the category of left modules over the ring R. If $\mathcal{C}$ is an abelian category, then $\mathrm{Ch}(\mathcal{C})$ denotes the category of cochain complexes over $\mathcal{C}$, and $\mathrm{Ch}^+(\mathcal{C})$ the full subcategory of bounded-below complexes.

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1. *(Borrowed from Pete Clark)* Let R be a commutative ring and M, N be R-modules.
   (a) [1 point] Show that the groups $\mathrm{Ext}^i_R(M,N)$ are also naturally R-modules.
   (b) [2 points] Let $r \in R$ and let $\mu : N \to N$ be the map $x \mapsto rx$. Show that for any i, the map $\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,N)$ induced by μ via the functoriality of $\mathrm{Ext}^i(M,-)$ is also multiplication by r. Show a similar result for the multiplication-by-r map $M \to M$.

2. Let G be a group and $H \trianglelefteq G$ a subgroup isomorphic to $(\mathbf{Z}, +)$.
   (a) [1 point] Show that for any G-module M, we have $H^i(H,M) = 0$ for $i \notin \{0,1\}$.
   (b) [1 point] Show that there is a long exact sequence

$$\dots \to H^n(G/H, H^0(H,M)) \to H^n(G,M) \to H^{n-1}(G/H, H^1(H,M)) \to H^{n+1}(G/H, H^0(H,M)) \to \dots$$

3. [2 points] Let E be a (first-quadrant, cohomological) spectral sequence in Ab converging to $(X^n)_{n \ge 0}$, and suppose there is some r such that $E_r^{p,q}$ is finitely-generated for all (p,q) and zero for almost all (p,q). Show that X^n is finitely-generated for all n and zero for almost all n, and we have

$$\sum_{p,q} (-1)^{p+q} \operatorname{rank}\left(E_r^{p,q}\right) = \sum_n (-1)^n \operatorname{rank}\left(X^n\right).$$

   [*] Formulate and prove an analogous statement with "finitely-generated" replaced by "finite".

4. [2 points] Let $G = \mathrm{SL}_2(k)$, where k is a finite field of characteristic $\ne 2$. Let M be k^2, with G acting via the standard left-multiplication action on column vectors. Show that $H^i(G,M) = 0$ for all i. *[Hint: Apply the Hochschild–Serre spectral sequence to $Z(G) \trianglelefteq G$.]*

5. Let R be a ring and let $f : A^\bullet \to B^\bullet$ be a morphism in $\mathrm{Ch}(R\text{-Mod})$. Recall the definition of the *mapping cone* $C^\bullet_f$ of f (with the corrected sign conventions given in Lecture 8).
   (a) [1 point] Show that $C^\bullet_f$ is a cochain complex, and the obvious projection and inclusion maps $g : C^\bullet_f \to A^\bullet[1]$ and $h : B^\bullet \to C^\bullet_f$ are cochain maps.
   (b) [2 points] Show that all three compositions $f \circ g$, $g \circ h$, and $h \circ f$ are null-homotopic.
   (c) [2 points] Show that if $g : A^\bullet \to B^\bullet$ is another morphism homotopic to f, then the complex $C^\bullet_g$ is homotopy-equivalent to $C^\bullet_f$, compatibly with the morphisms from $B^\bullet$ and to $A^\bullet[1]$.

6. Let $F : \mathcal{C} \to \mathcal{D}$ be a left-exact functor between abelian categories, with $\mathcal{C}$ having enough injectives. We defined the hyperderived functors $\mathbf{R}^i(F)$ as functors $\mathrm{Ch}^{\geqslant 0}(\mathcal{C}) \to \mathcal{D}$, where $\mathrm{Ch}^{\geqslant 0}(\mathcal{C})$ is the full subcategory of $\mathrm{Ch}^+(\mathcal{C})$ consisting of complexes that are zero in degrees < 0.

(a) [1 point] Show that there is a unique extension of the functors $\mathbf{R}^i(F)$ to $\mathrm{Ch}^+(\mathcal{C})$ satsfying $\mathbf{R}^i(F)(X) = \mathbf{R}^0(F)(X[i])$.

(b) [1 point] Show that if $f : X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $\mathrm{Ch}^{\geqslant 0}(\mathcal{C})$, then it induces isomorphisms $\mathbf{R}^i(F)(X) \to \mathbf{R}^i(F)(Y)$ for all Y.

7. [4 points] *(Suggested by Sarah Zerbes)* Let $\mathcal{C}, \mathcal{D}$ be abelian categories with $\mathcal{C}$ having enough injectives, $F : \mathcal{C} \to \mathcal{D}$ a left-exact functor, and $f : A^\bullet \to B^\bullet$ a morphism of complexes in $\mathrm{Ch}^{\geqslant 0}(\mathcal{C})$. Let $C^\bullet = C_f^\bullet[-1]$, so we also have $C \in \mathrm{Ch}^{\geqslant 0}(\mathcal{C})$; this shifted mapping cone is sometimes known as the *mapping fibre*.

   Show that there is a spectral sequence in $\mathcal{D}$ converging to $\mathbf{R}^{p+q}(F)(C^\bullet)$, such that for each $q \geq 0$, the q-th row on the E_1 page, $E_1^{\bullet q}$, is the mapping fibre of the morphism $R^q(F)(f) : R^q(F)(A^\bullet) \to R^q(F)(B^\bullet)$ in $\mathrm{Ch}^{\geqslant 0}(\mathcal{D})$.