TCC Homological Algebra: Assignment #1 (Solutions)

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This is the first of 3 problem sheets. Solutions should be submitted to me (by email, or via my pigeonhole for Warwick students) by **noon on 22nd November**. This problem sheet will be marked out of a total of 20; the number of marks available for each question is indicated. Questions marked [*] are optional and not assessed.

Note that rings are not necessarily commutative, but are always assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1.

Categories, functors, and natural transformations

1. [3 points] Let $\mathcal{C}$ be a category admitting a faithful functor $F : \mathcal{C} \to \underline{\mathrm{Set}}$, and $\phi : X \to Y$ a morphism in $\mathcal{C}$.

(a) Show that:

i. If $F(\phi)$ is injective, then ϕ is a monomorphism.

Solution: Let $\alpha, \beta : Z \to X$ be morphisms such that $\phi \circ \alpha = \phi \circ \beta$. Then $F(\phi) \circ F(\alpha) = F(\phi) \circ F(\beta)$; since $F(\phi)$ is injective, it follows that $F(\alpha) = F(\beta)$. As F is an injection $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\underline{\mathrm{Set}}}(FX, FY)$, it follows that $\alpha = \beta$. Thus ϕ is a monomorphism.

ii. If $F(\phi)$ is surjective, then ϕ is an epimorphism.

Solution: Let $\alpha, \beta : Y \to Z$ be morphisms such that $\alpha \circ \phi = \beta \circ \phi$. Thus $F(\alpha) \circ F(\phi) = F(\beta) \circ F(\phi)$. Since $F(\phi)$ is surjective, this implies $F(\alpha) = F(\beta)$, and since F is faithful, hence $\alpha = \beta$. Thus ϕ is an epi.

(b) Show that if $\mathcal{C} = \underline{\mathrm{Ring}}$ and F is the forgetful functor, then the converse of (i) is true: if ϕ is a monomorphism, then ϕ is set-theoretically injective. *(Hint:* $\mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], R) = R$.*)*

Solution: For any ring r and $r \in R$, there is a unique homomorphism $\mathbf{Z}[X] \to R$ sending X to r, namely $\sum a_i X^i \mapsto \sum a_i r^i$. Hence evaluation at X gives a bijection of sets

$$\mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], R) \to R.$$

Moreover, this is functorial in R, i.e. for $\phi : R \to S$ a homomorphism the map $R \cong \mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], R) \to \mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], S) \cong S$ is just ϕ regarded as a map of sets $R \to S$.

With this in hand, we see that if ϕ is a monomorphism, it gives an injection

$$\mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], R) \to \mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], S),$$

and by the naturality assertion this shows that ϕ is injective as a map of sets $R \to S$.

(c) By considering the inclusion map $\mathbf{Z} \to \mathbf{Q}$, or otherwise, show that the converse of (ii) is false in this case.

> **Solution:** We shall show that $\iota : \mathbf{Z} \to \mathbf{Q}$ is an epimorphism. Since it is clearly not surjective, this shows that the converse to (ii) is false.
>
> Let ϕ_1, ϕ_2 be two homomorphisms of rings $\mathbf{Q} \to R$ whose restrictions to $\mathbf{Z}$ agree, and let this common restriction be ϕ. Then, for any $n \geq 1$, the product $\phi_1(1/n) \times \phi(n) \times \phi_2(1/n)$ is equal to both $\phi_1(1/n)$ and $\phi_2(1/n)$ (by associativity of multiplication) and hence $\phi_1(1/n) = \phi_2(1/n)$. Thus $\phi_1(m/n) = \phi_2(m/n)$ for all $m/n \in \mathbf{Q}$.

2. Let Ring be the category of rings and ring homomorphisms, and $u : \underline{\text{Ring}} \to \underline{\text{Grp}}$ the functor sending a ring R to the group $R^\times$ of invertible elements of R (and acting on morphisms in the obvious way).
   (a) [1 point] Is u full?

   > **Solution:** No, it is not full: there is a unique homomorphism of rings $\mathbf{Z} \to \mathbf{Z}$, but $\mathrm{Hom}_{\underline{\text{Grp}}}(u(\mathbf{Z}), u(\mathbf{Z}))$ has order 2.

   (b) [*] Is u faithful?

   > **Solution:** No. The simplest counterexample I could find was $\mathbf{Z}[X]$: this ring has no units except ± 1, but $\mathrm{Hom}_{\underline{\text{Ring}}}(\mathbf{Z}[X], \mathbf{Z}[X])$ is huge (you can send X to anything).

3. Let k be a field. Let $\mathcal{C}$ denote the category with objects $\{0, 1, \dots\}$ and $\mathrm{Hom}_{\mathcal{C}}(m, n)$ defined to be the space of $n \times m$ matrices over k (with composition defined as matrix multiplication); and let $\mathcal{D}$ denote the category of all finite-dimensional k-vector spaces.
   (a) [1 point] Verify that there is a functor $F : \mathcal{C} \to \mathcal{D}$ sending n to $\mathbf{R}^n$ (you should explain what it does to morphisms).

   > **Solution:** We send a matrix $M \in \mathrm{Hom}_{\mathcal{C}}(m, n)$ to the homomorphism $\mathbf{R}^m \to \mathbf{R}^n$ given by the left-multiplication action of M on column vectors.

   (b) [*] Show (directly, without quoting the general criterion from §1.4) that F has a quasi-inverse.

   > **Solution:** For every finite-dimensional vector space V, choose a specific basis $\mathcal{B}_V$ of V. For simplicity we can and do suppose that $\mathcal{B}_{\mathbf{R}^n}$ is the standard basis. Then we define a functor $G : \mathcal{D} \to \mathcal{C}$ by sending V to $\mathbf{R}^{\dim V}$ for each V, and each morphism $\alpha : V \to W$ to the matrix of α with respect to the bases $\mathcal{B}_V$ and $\mathcal{B}_W$.
   >
   > We now need to show that G is a quasi-inverse of F. By construction, $G \circ F$ is the identity functor; so it suffices to construct an isomorphism of functors $\nu : F \circ G \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$. Clearly $F \circ G$ acts on objects by sending V to $\mathbf{R}^{\dim(V)}$, and on morphisms by sending $\alpha : V \to W$ to the morphism $\mathbf{R}^{\dim V} \to \mathbf{R}^{\dim W}$ given by the matrix of α in the chosen basis. So, for each V, let $\nu(V)$ be the map $\mathbf{R}^{\dim(V)} \to V$ sending the standard basis vectors to $\mathcal{B}_V$; this intertwines $(F \circ G)(\alpha)$ with α for every morphism α.

Limits and adjunctions

4. [2 points] Prove the following result (part B of a lemma stated in §1.6): if $f_1, f_2 : X \to Y$ are two morphisms in a category $\mathcal{C}$, and $g : Y \to Z$ is a monomorphism in $\mathcal{C}$, then the pair $(g \circ f_1, g \circ f_2)$ has an equaliser if and only if (f_1, f_2) has an equaliser, and the two equalisers coincide.

> **Solution:** Since g is a monomorphism, we see that for any homomorphism $h : Z \to X$, we have
>
> $$(f_1 \circ h = f_2 \circ h) \iff (g \circ f_1 \circ h = g \circ f_2 \circ h).$$

Hence any equaliser of f_1 and f_2 also satisfies the universal property required to be the equaliser of $g \circ f_1$ and $g \circ f_2$, and vice versa.

5. Let $(\mathcal{P}, \succ)$ be a partially ordered set, and $\mathcal{J}$ the category with objects $\mathcal{P}$ and a single homomorphism $x \to y$ if $x \succ y$.

   (a) [2 points] Show that if there exists a greatest element in $\mathcal{P}$, then $\mathcal{J}$-diagrams have limits in any category. Formulate and prove a similar statement for colimits.

   Solution: Suppose $\mathcal{P}$ has a greatest element p_0; then p_0 is an initial object of $\mathcal{J}$.
   Let $D : \mathcal{J} \to \mathcal{C}$ is a $\mathcal{J}$-diagram. A cone of D is the data of an object X and morphisms $\phi_p : X \to D(p)$ for every $p \in \mathcal{P}$ compatible with composition; but this implies that for every $p \in \mathcal{P}$ we have $\phi_p = D(\iota_p) \circ \phi_{p_0}$, where $\iota_x : p_0 \to p$ is the unique homomorphism $p_0 \to p$ in $\mathcal{J}$. This shows that the cone $(X, (\phi_p)_{p \in \mathcal{P}})$ is entirely determined by the single morphism $X \to D(x_0)$: it is the composition of the single homomorphism $\phi_{x_0} : X \to D(x_0)$ with the "stupid cone" $(D(p_0), D(\iota_p)_{p \in \mathcal{P}})$. Hence this latter cone is universal, i.e. the diagram D has a limit.
   The dual statement is that if $(\mathcal{P}, \succ)$ has a least element, then $\mathcal{J}$-diagrams have colimits in any category $\mathcal{C}$. We prove this as follows: if $\succ^{\mathrm{op}}$ is the opposite relation of $\succ$, then the poset category associated to $(\mathcal{P}, \succ^{\mathrm{op}})$ is $\mathcal{J}^{\mathrm{op}}$, so:
   $(\mathcal{P}, \succ)$ has a least element
   $\Leftrightarrow (\mathcal{P}, \succ^{\mathrm{op}})$ has a greatest element
   $\Rightarrow \mathcal{J}^{\mathrm{op}}$-diagrams have limits in $\mathcal{C}^{\mathrm{op}}$ for any $\mathcal{C}$
   $\Leftrightarrow \mathcal{J}$-diagrams have colimits in $\mathcal{C}$ for any $\mathcal{C}$.
   [Optional follow-up question: if $\mathcal{J}$-diagrams have limits in any category, does $(\mathcal{P}, \succ)$ have a greatest element?]

   (b) [1 point] Suppose $(\mathcal{P}, \succ) = (\mathbb{N}, \geq)$. Show that a $\mathcal{J}$-diagram in Ab is equivalent to the data of a collection of abelian groups A_i and morphisms $\phi_i : A_{i+1} \to A_i$ for all $i \in \mathbf{N}$, and that the inverse limit

   $$\varprojlim_i A_i = \left\{ (x_i) \in \prod_{i \in \mathbf{N}} A_i : \phi_i(x_{i+1}) = x_i \ \forall i \right\}$$

   satisfies the universal property of the category-theoretic limit.

   Solution: If we are given abelian groups A_i and morphisms ϕ_i as above, then we can extend this to a $\mathcal{J}$-diagram in a unique way by sending the unique morphism $m \to n$ in $\mathcal{J}$, for integers $m \geq n$, to the composite $\phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{m-1}$ (understood as the identity if $m = n$).
   We need to show that if B is any other abelian group with homomorphisms $\beta_i : B \to A_i$ for all i satisfying $\beta_i = \phi_i \circ \beta_{i+1}$, then there is a unique homomorphism $\beta_\infty : B \to \varprojlim A_i$ whose projection to A_i is β_i for every i. To show existence, we let β_∞ be the map sending $b \in B$ to $(\beta_i(b))_{i \in \mathbf{N}}$. To show uniqueness, let β' be another homomorphism with the same property; then for any $b \in B$, the element $\beta'(b)$ must have i-th term $\beta_i(b)$ for all i, so we must have $\beta'(b) = \beta_\infty(b)$.

6. [2 points] Show that the forgetful functor Top $\to$ Set has both a left and a right adjoint, and describe both of these explicitly.

Solution: If X is a set, let $L(X)$ denote the discrete topology on X (every subset is open), and let $R(X)$ denote the indiscrete topology on X (only X and $\varnothing$ are open). These are clearly functors Set $\to$ Top.

If T is any topological space, any map of sets $X \to T$ is continuous with the topology $L(X)$, and any map of sets $T \to X$ is continuous with the topology $R(X)$. So if F denotes the forgetful functor, we have

$$\mathrm{Hom}_{\underline{\mathrm{Set}}}(X, F(T)) = \mathrm{Hom}_{\underline{\mathrm{Top}}}(L(X), T) \quad \text{and} \quad \mathrm{Hom}_{\underline{\mathrm{Set}}}(F(T), X) = \mathrm{Hom}_{\underline{\mathrm{Top}}}(T, R(X)).$$

Thus L is left adjoint to F, and R right adjoint.

7. [3 points] Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors (L the left adjoint, R the right adjoint).
   (a) Show that if $\mathcal{J}$ is a small category, and D is a $\mathcal{J}$-diagram in $\mathcal{D}$ which has a limit, then $R(D)$ also has a limit and $R(\lim D) = \lim R(D)$ (i.e. R is *continuous*).
   [Hint: Use L to construct a map from cones of $R(D)$ to cones of D.]

Solution: *[This was perhaps a little harder than intended, sorry!]*
Step 1: Follow the hint. Let $\Delta = (C, (\phi_x)_{x \in \mathcal{J}})$ be a cone of $R(D)$, so $C \in \mathrm{Obj}\,\mathcal{C}$ (the "peak" of the cone) and $\phi_x \in \mathrm{Hom}(C, R(D(x))$ are homomorphisms for each $x \in \mathrm{Obj}\,\mathcal{J}$ compatible with the morphisms in $R(D)$. We shall construct from Δ a cone $L(\Delta)$ of D in $\mathcal{D}$, whose "peak" is $L(C)$.
For each $x \in \mathrm{Obj}\,\mathcal{J}$, the morphism $\phi_x \in \mathrm{Hom}_{\mathcal{C}}(C, RD(x))$ corresponds via the adjunction to a morphism $\phi'_x \in \mathrm{Hom}(L(C), D(x))$. I claim that $L(\Delta) = (L(C), \phi'_x)$ is a cone of D in $\mathcal{D}$: what we need to check is that for each morphism $\alpha : x \to y$ in $\mathcal{J}$, the commutativity of the diagram

$$\begin{array}{ccc} C & & \\ {\scriptstyle \phi_x}\downarrow & \searrow^{\phi_y} & \\ RD(x) & \xrightarrow[RD(\alpha)]{} & RD(y) \end{array}$$

implies that of the diagram

$$\begin{array}{ccc} L(C) & & \\ {\scriptstyle \phi'_x}\downarrow & \searrow^{\phi'_y} & \\ RD(x) & \xrightarrow[D(\alpha)]{} & RD(y). \end{array}$$

This is an instance of the naturality of the adjunction: we assumed that $\mathrm{Hom}_{\mathcal{C}}(C, R(-)) = \mathrm{Hom}_{\mathcal{D}}(L(C), -)$ as an isomorphism of functors $\mathcal{D} \to \underline{\mathrm{Set}}$, so we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(C, RD(x)) & \longrightarrow & \mathrm{Hom}(C, RD(y)) \\ \wr\!\!\downarrow & & \downarrow\!\wr \\ \mathrm{Hom}(LC, D(x)) & \longrightarrow & \mathrm{Hom}(LC, D(y)) \end{array}$$

where the vertical arrows are the adjunction maps, the top horizontal arrow is composition with $RD(\alpha)$, and the bottom vertical arrow is composition with $D(\alpha)$. Since the top horizontal arrow sends ϕ_x to ϕ_y, it follows that the bottom one sends ϕ'_x to ϕ'_y. Thus $L(\Delta)$ is a cone of D.
Step 2: Use it to check the claim. Suppose D has a limit, i.e. there is a universal cone $\Delta^{\mathrm{univ}} = (\mathcal{L}, (\Phi_x)_{x \in \mathcal{J}})$ of D such that any cone $(Z, (\psi_x)_{x \in \mathcal{J}})$ is obtained from Δ^{univ} by composition with a unique homomorphism $Z \to \mathcal{L}$. Applying R, we obtain a cone $R(\Delta^{\mathrm{univ}}) = (R(\mathcal{L}), (R(\Phi_x))_{x \in \mathcal{J}})$ of $R(D)$ in $\mathcal{C}$. We shall show that $R(\Delta^{\mathrm{univ}})$ is universal among cones of $R(D)$.
Suppose $\Delta = (C, (\phi_x)_{x \in \mathcal{J}})$ is a cone of $R(D)$ in $\mathcal{C}$, and let $L(\Delta)$ be its image under Step 1. The adjunction gives a bijection between morphisms $u : C \to R(\mathcal{L})$ and morphisms

$u' : L(C) \to \mathcal{L}$; and the naturality of the adjunction shows that for each $x \in \mathcal{J}$ this fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(C, R\mathcal{L}) & \xrightarrow{R(\Phi_x)\circ(-)} & \mathrm{Hom}(C, RD(x)) \\ \downarrow \wr\!| & & \downarrow \wr\!| \\ \mathrm{Hom}(LC, \mathcal{L}) & \xrightarrow{\Phi_x\circ(-)} & \mathrm{Hom}(LC, D(x)) \end{array}$$

By construction, for each x, the left vertical arrow sends the morphism ϕ_x giving the cone structure of Δ to the morphism ϕ'_x giving the cone structure of $L(\Delta)$. A morphism $u \in \mathrm{Hom}(C, R\mathcal{L})$ is compatible with the cone structures if and only if it gets sent to $\phi_x \in \mathrm{Hom}(C, RD(x))$ for every x; and similarly $u' \in \mathrm{Hom}(LC, \mathcal{L})$ is compatible with the cone structures if it gets sent to ϕ'_x for all x. Thus the commutativity of the diagram shows that u is compatible with the cone structures if and only if u' is.

The universal property of Δ^{univ} tells us that there is one and only one morphism $u' : LC \to \mathcal{L}$ compatible with the cone structures. So we deduce that there is one and only one morphism $u : C \to R(\mathcal{L})$ compatible with the cone structures, which is exactly the statement that $R(\Delta^{\mathrm{univ}})$ is the limit of RD in $\mathcal{D}$.

[A more systematic way to describe the same argument is to define a category $\mathrm{Co}_{\mathcal{D}}(D)$ *of cones of* D*, with a suitable notion of morphism, so that a limit of* D *is the same thing as an initial object in this category. Clearly* R *gives a functor* $\tilde{R} : \mathrm{Co}_{\mathcal{D}}(D) \to \mathrm{Co}_{\mathcal{C}}(RD)$*. Step 1 gives a map of* objects $\tilde{L} : \mathrm{Obj}\,\mathrm{Co}_{\mathcal{C}}(RD) \to \mathrm{Obj}\,\mathrm{Co}_{\mathcal{D}}(D)$*, and an easy check shows that one can make this into a functor. The last commutative diagram shows that* $\tilde{L}$ *is left adjoint to* $\tilde{R}$*. Since right adjoints have to send initial objects to initial objects, we are now done.]*

(b) Show that the opposite functor $L^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is **right** adjoint to R^{op}. By applying part (a) to the opposite functors, or otherwise, show that L is *cocontinuous*, i.e. preserves all colimits.

Solution: We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(X, L^{\mathrm{op}}(Y)) &= \mathrm{Hom}_{\mathcal{D}}(L(Y), X) \\ &= \mathrm{Hom}_{\mathcal{C}}(Y, R(X)) \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(R^{\mathrm{op}}(X), Y). \end{aligned}$$

This shows that L^{op} is right adjoint to R^{op}. *[Strictly speaking one should check naturality of these bijections but I'm not mean enough to insist upon it.]*

If $\mathcal{J}$ is a small category and $D : \mathcal{J} \to \mathcal{C}$ is a diagram, then a colimit of D is the same thing as a limit of $D^{\mathrm{op}} : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. Since L^{op} is a right adjoint functor, it preserves limits, so L^{op} sends a limit of D^{op} to a limit of $L^{\mathrm{op}}(D^{\mathrm{op}})$, i.e. L sends a colimit of D to a colimit of $L(D)$.

Additive categories, kernels, cokernels

8. [2 points] Let $\mathcal{C}$ be an additive category. Show that there is an isomorphism $X_1 \oplus (X_2 \oplus X_3) \cong (X_1 \oplus X_2) \oplus X_3$ for any three objects X_1, X_2, X_3 of $\mathcal{C}$.

Solution: Let $U = X_1 \oplus (X_2 \oplus X_3)$.

We claim there are maps $p_j : U \to X_j$ and $i_j : X_j \to U$, for $j = 1, 2, 3$, such that $p_j \circ i_k$ is 0 if $j \neq k$, and id_{X_j} if $j' = j$, and $\sum_j (i_j \circ p_j) = \mathrm{id}_U$. The maps $p_1 : U \to X_1$ and $i_1 : X_1 \to U$ are given to us, and we define p_2 and p_3 to be the composites $U \to (X_2 \oplus X_3) \to X_2$ and $U \to (X_2 \oplus X_3) \to X_3$ of the structure maps in the two 2-term direct sums, and similiarly for i_2 and i_3. Then the composition relations are immediate from the corresponding relations for the 2-term direct sums.

Similarly, there are maps $i'_j : X_j \to U'$ and $p'_j : U' \to X_j$, where $U' = (X_1 \oplus X_2) \oplus X_3$; this time it is i_3 that is given, and i_1 and i_2 are obtained as composites.

Using these, we define a map $\alpha : U \to U'$ as the sum $i'_1 \circ p_1 + i'_2 \circ p_2 + i'_3 \circ p_3$, and $\alpha' = \sum i_j \circ p'_j : U' \to U$.

Using the formulae for $p_j \circ i_k$, and the linearity of composition, we find that $\alpha \circ \alpha' = \sum i'_j \circ p'_j = \mathrm{id}_{U'}$. Similarly, using the formulae for $p'_j \circ i'_k$, we conclude that $\alpha' \circ \alpha = \mathrm{id}_U$.

[Note that this is the unique isomorphism $U \cong U'$ compatible with the maps to and from the X_i.]

9. [3 points] Let $\mathcal{C}$ be the full subcategory of Ab consisting of torsion-free abelian groups. Show that all morphisms in $\mathcal{C}$ have kernels and cokernels. Give an example to show that the cokernel of a morphism in $\mathcal{C}$ may not coincide with the cokernel of the same morphism in Ab. Is $\mathcal{C}$ an abelian category?

Solution: If $X \to Y$ is a morphism in $\mathcal{C}$, then it has a kernel K in Ab, and the kernel is a subgroup of X and is therefore itself torsion-free; since K satisfies a universal property in a bigger category, it certainly satisfies the same property in $\mathcal{C}$.

This doesn't work for cokernels; the "multiplication-by-2" map $\mathbf{Z} \to \mathbf{Z}$ has a cokernel in Ab which is obviously not an object of $\mathcal{C}$. However, if we define $\mathrm{coker}_{\mathcal{C}}(\phi)$ to be the quotient of $\mathrm{coker}_{\underline{\mathrm{Ab}}}(\phi)$ by its torsion subgroup, then any homomorphism from $\mathrm{coker}_{\underline{\mathrm{Ab}}}(\phi)$ to an object of $\mathcal{C}$ will be trivial on the torsion subgroup and thus factors (uniquely) through $\mathrm{coker}_{\mathcal{C}}(\phi)$ as required.

The multiplication-by-2 example shows that this is not always the same as the cokernel in Ab. It also shows that $\mathcal{C}$ is not abelian: this morphism has zero kernel and cokernel, but it is not an isomorphism (since if it were an isomorphism in $\mathcal{C}$, it would have to remain an isomorphism in the larger category Ab and thus have zero cokernel in Ab as well).