Modular Forms + Rep^s of GL_2

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§0. Motivation

Mod forms = fcns $\mathcal{H} \to \mathbb{C}$ transforming nicely under $\Gamma < SL_2\mathbb{Z}$

Hecke operators: send MFs to MFs.

Nice collection of operators T_n – (almost) simultaneously diag^{ble}.

· Why these operators?

· How does this generalize to other kinds of automorphic forms?

Chapter 1 Reps of locally profinite gps

Reference: Bushnell – Henniart chap. 1

§3.1 Defs

· A topological group is a group that is also a top. space, such that

gp operation $*$: $G \times G \to G$
inversion $G \to G$
$\Big\}$ cts

(Jargon: "group obj in category of top spaces")

Eg: Any group w. discrete top

- $(\mathbb{R}, +)$, $(\mathbb{R}^{\times}, \times)$, $(\mathbb{Q}_p, +)$ etc with usual top

- $GL_2(\mathbb{Q}_p)$

Fact In any top gp, open subgps are also closed.

Def G is **locally profinite** if every open nhd of 1_G contains an open compact subgp.

Exercise If G is loc. prof., $H \leq G$ closed, then H is loc. prof., + if H normal, so is G/H.

Prop If G loc prof, any open compact $K \subset G$ is profinite, i.e. $K \longrightarrow \varprojlim_{U} K/U$ is an iso of top gps,

U ranges over open cpct normal subgps of K.

Exercise: prove this.

Fact G loc. prof $\iff$ G loc. compact + totally disconnected

Eg: · any discrete group

· any profinite gp, even very silly ones like $\prod_{\mathbb{R}} \mathbb{Z}/2$

§1.2 Local fields

F nonarchimedean local field

$\mathcal{O}$ ring of integers (complete DVR)

v valuation on F normalized so

a generator ϖ of max^l ideal of $\mathcal{O}_F$

has $v(\varpi) = 1$

$\mathcal{O}_F/\mathfrak{p}_F$ finite, size q (prime power)

$|\ |$ absolute value, $|x| = q^{-v(x)}$

Prop $GL_n(F)$ is a loc prof. ~~top~~ gp.

Proof $K_m = \left\{ g \in GL_n(\mathcal{O}) : g = \mathrm{Id} \bmod \mathfrak{p}_F^m \right\}$

form a basis of nhds of 1, + they're open + cpct subgps.

$\Rightarrow$ Lots of loc prof gps as closed subgps.

1.3 Smooth + admissible reps

G loc prof gp

<u>Defn</u> A repn of G is a $\mathbb{C}$-vs V with a left action of G by linear maps.

V is <u>smooth</u> if every $v \in V$ has open stabilizer in G.

V is <u>admissible</u> if $\forall$ K open cpct in G, V^K is finite-dim!

Note top. on $\mathbb{C}$ plays <u>no role</u> – could also take $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}}_\ell$

<u>Rep_G</u> category of G-reps

<u>Smo_G</u> smooth reps

<u>Adm_G</u> admissible smooth

<u>Duals</u> If $V \in \mathrm{Rep}_G$, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is in Rep_G too, but doesn't preserve smooth reps.

Let $V^\vee = \left\{ \lambda \in V^* : \exists \text{ open } K \text{ st } \lambda \in (V^*)^K \right\}$

$\in \underline{\mathrm{Smo}}_G$.

<u>Fact</u> If $V \in \underline{\mathrm{Smo}}_G$, nat^l map

$V \to (V^\vee)^\vee$ exists + is injective.

It's a bij^n if + only if $V \in \underline{\mathrm{Adm}}_G$

Exercise Let $G = \mathbb{Z}_p^\times$, $V = \{$locally const. fns $G \to \mathbb{C}\}$

(i) Show V is smooth + adm.

(ii) Show V^* not smooth

(iii) Show $V^\vee$ isomorphic (non-canonically) to V.

Thm (Jacquet) Suppose G/K countable for some (hence any) open cpct K. If $V \in \mathrm{Sm}_G$ is irreducible, then $\mathrm{End}_G(V) = \mathbb{C}$. ("Schur's Lemma")

Pf Let $\phi \neq 0 \in \mathrm{End}_G(V)$. Then $\mathrm{Ker}\,\phi = 0$ and $\mathrm{Im}(\phi) = V$. So $\mathrm{End}_G V$ is a division algebra. If $\phi \notin \mathbb{C}$, ϕ must be transcendental$/\mathbb{C}$.

Claim The set $\{(\phi - a)^{-1} : a \in \mathbb{C}\}$ is linearly indep.

Pf Any linear relation gives a polynomial killing ϕ.

But if $v \neq 0 \in V$, $v \in V^K$ some K and $\{gv : g \in G/K\}$ spans V. So $\dim V$ countable + eval^n at v is an inj^n $\mathrm{End}\,V \hookrightarrow V$. $\lightning$

Corollary If G/K countable, V irred, $Z(G)$ acts on V by a character $G \to \mathbb{C}^\times$

1.4 Induced Rep's

G loc prof, $H \leq G$ closed

Have a restriction $\mathrm{Smo}_G \to \mathrm{Smo}_H$

Can we go back?

Def^n For $W \in \mathrm{Smo}_H$, set

$$\mathrm{Ind}_H^G(W) = \left\{ f : G \to W \;\middle|\; \begin{array}{l} f(hg) = h \cdot f(g) \\ \quad \forall h \in H, \\ \text{and } \exists K \text{ open cpct in } G \\ \text{st } f(gk) = f(g) \ \forall g \in G, \\ \qquad k \in K \end{array} \right\}$$

with G acting by right translation.

Thus Ind_H^G is a functor $\mathrm{Smo}_H \to \mathrm{Smo}_G$.

Thm (Frobenius reciprocity):
for $W \in \mathrm{Smo}_H$, $V \in \mathrm{Smo}_G$,

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) = \mathrm{Hom}_H(V, W)$$

Slogan: Induction is right adjoint to restriction.

Can also consider a variant,
$c\text{-}\mathrm{Ind}_H^G(W)$ = fcns in $\mathrm{Ind}_H^G(W)$ with compact support modulo H.

Will see later that c-Ind is left adjt to restriction if H open in G.

Prop If G/H is compact, the functor $\mathrm{Ind}_H^G = \mathrm{cInd}_H^G$ sends admissible reps to admissible reps.

Proof Let $K \subset G$ open cpct.
WTS V^K fin dim'l $\quad (V = \mathrm{Ind}_H^G W,$
W adm H-rep$)$

G/H cpct $\Rightarrow$ $H \backslash G / K$ is finite.

If $f \in V^K$, f uniquely determined by $f(x_1) \dots f(x_n)$, x_i representatives for $H \backslash G / K$

but $f(x_i) \in W^{\overbrace{H \cap x_i K x_i^{-1}}^{\text{open cpct}}}$

thus $V^K \hookrightarrow \bigoplus_{i=1}^{n} W^{H \cap x_i K x_i^{-1}}$ finite-dim'l $\square$

1.5 Haar Measure + the Modulus Character

Thm (Haar): If G loc prof, $\exists$ finitely additive fcn $\mu : \left\{ \begin{matrix} \text{open subsets} \\ \text{of } G \end{matrix} \right\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ finite on open compact sets, nonzero on nonempty ones, st $\mu(gS) = \mu(S) \ \forall g \in G$ (left Haar measure) + μ is unique up to scaling by $\mathbb{R}_{>0}^{\times}$.

(Easy proof for loc prof gps: choose an open cpct K, give it measure 1, then $\mu(L)$ determined for every $L \leqslant K$ open cpct subgp)

Any left Haar measure gives a map

$$\int_G d\mu : \underset{\substack{| \\ \text{loc const, cptly supported} \\ \text{fns on } G}}{C_c^\infty(G)} \longrightarrow \mathbb{C}$$

$$\int_G f(g)\, d\mu(g)$$

<u>Prop</u> If G loc. prof., $\exists$ character $\delta_G : G \to \mathbb{C}^\times$, st for any left HM μ,

$\mu(Sg) = \delta(g)\,\mu(S) \quad \forall g \in G$, S open.

<u>Pf</u> $S \mapsto \mu(Sg)$ is also a left HM.

Note δ_G takes <u>+ve real</u> values, so δ_G is trivial on any compact subgp.

If δ_G trivial, say G is <u>unimodular</u> (eg compact gps, abelian gps)