MOD FORMS & GL_2 REPS
LECTURE 2

- No lecture 8th Nov
  - 6 Dec instead
- Credit - email me to confirm
- Problem sheet 1 on web page (revised) - deadline 5th Nov

Last time Haar measure

Duality Theorem

Want to understand dual of (c)-$\mathrm{Ind}_H^G(V)$

Idea "integrate over $H\backslash G$"

Would like G-int measure on $H\backslash G$
= linear func^l on $C_c^\infty(H\backslash G)$

In gen^l need a funny twist

Prop Let $\theta: H \to \mathbb{C}^\times$ smooth char. Then $\exists$ nonzero G-equiv^t map

$$c\text{-}\mathrm{Ind}_H^G(\theta) \longrightarrow \mathbb{C}$$

iff $\theta = \delta_H^{-1}(\delta_G|_H)$, + it's unique up to scalars if so

Pf See Bushnell-Henniart §3.4

Write this as

$$f \mapsto \int_{H\backslash G} f(g)\, d\mu_{H\backslash G}(g)$$

where f transforms via $\delta_H^{-1}\delta_G$ under H on left.

Thm (Duality Thm)

For $V \in \mathrm{Smo}_H$, we have

$$\left(c\text{-}\mathrm{Ind}_H^G V\right)^\vee = \mathrm{Ind}_H^G\left(V^\vee \otimes \delta_H^{-1}\delta_G\right)$$

Sketch of pf can pair

$$c\text{-}\mathrm{Ind}(V) \times \mathrm{Ind}(V^\vee \otimes \delta_H^{-1}\delta_G)$$
$$\to c\text{-}\mathrm{Ind}(\delta_H^{-1}\delta_G) \xrightarrow{\int_{H\backslash G}} \mathbb{C}$$

This turns out to be perfect. $\square$

Define normalized induction

$$I_H^G(V) = \mathrm{Ind}\left((\delta_H^{-1}\delta_G)^{\frac{1}{2}} \otimes V\right)$$

If $H\backslash G$ cpct we have

$$I_H^G(V)^\vee = I_H^G(V^\vee)$$

(Normalization is annoying if base field is not alg closed)

(Deligne "Langlands is very sure what $\sqrt{p}$ is. I have never been so sure")

Chap 2 The principal series of $GL_2(F)$

F nonarch local field
q, $|x|$ etc as before

§2.1 Some subgroups + decomp^n's

$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ Borel subgp

$B = T \ltimes N$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

$K_0 = GL_2(\mathcal{O})$

Prop (i) $G = B \cup BwB$, $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
(Bruhat decomp^n)

(ii) $G = BK_0$ (Iwasawa decomp^n)

(iii) $G = \bigsqcup_{\substack{a \leq b \\ \in \mathbb{Z}}} K_0 \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K_0$
(Cartan decomp^n)

For (ii) note $G/B = \mathbb{P}^1(F)$
$= \mathbb{P}^1(\mathcal{O})$
$= K_0/(K_0 \cap B)$

Prop (a) B is not unimodular
(b) G is unimodular.

Pf (a) $B = T \ltimes N$ δ_B has to be triv^l on N as every elt of N is contained in an open cpct subgp
also triv^l on centre
take $g = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$ Let $B_0 = B \cap K_0$
If $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_0$, $g^{-1}hg = \begin{pmatrix} a & \varpi b \\ 0 & d \end{pmatrix}$

compute $\delta_B(g) = \dfrac{\mu(B_0 g)}{\mu(B_0)} = \dfrac{\mu(g^{-1}B_0 g)}{\mu(B_0)}$

$[B_0 : g^{-1}B_0 g] = q = \# \mathcal{O}/\varpi\mathcal{O}$

So $\delta_B(g) = \frac{1}{q}$

More gen^ly $\delta_B \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \left| t/r \right|$

(b) using Cartan decomp, enough to show $\delta_G \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} = 1$

Let $\mathcal{U} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 \quad c \in \mathfrak{p} \right\}$

g conjugates $\mathcal{U}$ to $\overline{\mathcal{U}}$ (subgp $b \in \mathfrak{p}$)

$\mathcal{U}$ image of $\overline{\mathcal{U}}$ under inverse transpose
$\Rightarrow$ same index in K_0
$\Rightarrow$ same Haar measure $\Rightarrow \delta_G(g) = 1$

(NB reductive gps / F always unimodular)

§22 The rep^s $I(\chi, \psi)$

Let $\chi : F^\times \to \mathbb{C}^\times$ smooth

(ie $\chi|_{\mathcal{O}^\times}$ factors thru $(\mathcal{O}/\varpi^n)^\times$ some n)

$\chi(\varpi)$ arbitrary

ψ another such char

$\chi \boxtimes \psi$ character $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \chi(a)\psi(d)$ of B

$$I(\chi, \psi) = I_B^G(\chi \boxtimes \psi)$$

$$= \Big\{ f : G \to \mathbb{C} \;\; f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \left|\tfrac{a}{d}\right|^{\frac{1}{2}} \chi(a)\psi(d) f(g)$$

and $\exists\, K$ open cpct st $f(gk) = f(g)\ \forall k \in K \Big\}$

Exercise show 2nd cond^n is equi^t to "f locally const," if 1st holds

Prop (a) $I(\chi, \psi)$ is smooth + admissible as a G-rep^n

(b) $I(\chi^{-1}, \psi^{-1}) = I(\chi, \psi)^\vee$

(c) $I(\chi, \psi)$ has central character $\chi\psi$ □

Prop If $\chi/\psi = |\ |^{\pm 1}$, $I(\chi, \psi)$ reducible

Pf If $\chi/\psi = |\ |^{-1}$, then $\exists$ 1-dim^l subspace spanned by

$$f(g) = |\det g|^{\frac{1}{2}} \psi(\det g)$$

By duality $\exists$ 1-dim^l quotient if $\chi/\psi = |\ |$ □

Def^n The Steinberg rep^n St is the kernel of the map $I(|\,|^{\frac12}, |\,|^{-\frac12}) \longrightarrow \mathbb{C}$

Thm (i) If $\chi/\psi \neq |\,|^{\pm 1}$, $I(\chi,\psi)$ is irred, and $I(\chi,\psi) \cong I(\psi,\chi)$

(ii) If $\chi/\psi = |\,|$, then $I(\chi,\psi)$ has codim 1 subrep $St \otimes \psi|\,|^{\frac12}$ and this is irreducible

(iii) If $\chi/\psi = |\,|^{-1}$, $I(\chi,\psi)$ has 1-dim^l subrep + the quotient is $St \otimes \psi|\,|^{\frac12}$

(In particular $St^\vee = St$)

(iv) There are no further isomorphisms between these rep^{n}s

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Corollary Let V be an irred rep^n of G. Then TFAE

(i) V is a subquot of some $I(\chi,\psi)$

(ii) V is a sub of some $I(\chi,\psi)$

(iii) N-coinvariants $V_N \neq 0$

Pf (i)$\Leftrightarrow$(ii) comes from thm

(ii)$\Leftrightarrow$(iii) is Frobenius reciprocity □

If V does not satisfy these say V is **supercuspidal**

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§2.3 A hint at the proof

Lemma (i) Let $V \in \underline{Sm}_N$. Then kernel of $V \to V_N$ (N-coinv^{t}s) is given by

$$\left\{ v \in V \;\middle|\; \int_{N_0} nv \, d\mu_N(n) = 0 \text{ for some open cpct } N_0 \leq N \right\}$$

(ii) $V \mapsto V_N$ is an exact functor on $\underline{Sm}_N$ (obviously right exact)

Def^n For $V \in Sm_B$ let $J_B(V) = V_N \otimes \delta_B^{\frac12} \in \underline{Sm}_T$

Then $Hom_G(V, I_B^G(\chi \boxtimes \psi)) = Hom_T(J_B(V), \chi \boxtimes \psi)$

Prop Let $\eta = \chi \boxtimes \psi$ char of T

$\eta^w = \psi \boxtimes \chi$

Then $\exists$ SES of B-rep^{n}s

$$0 \to V \to I(\eta)|_B \to \eta \otimes \delta_B^{-\frac12} \to 0$$

Moreover $V \cong c\text{-}Ind_T^B(\eta^w \otimes \delta_B^{\frac12})$

Proof Uses Mackey theory: understand $Ind_H^G(-)$ restricted to subgp J in terms of $H \backslash G / J$

Use $G = B \sqcup BwB$
(B ↑ closed, BwB ↑ open)

Evaluation at 1_G gives B-hom $I(\eta) \to \eta \otimes \delta_B^{-\frac12}$

If $f \in Ker(\text{this})$, support$(f) \subset \mathbb{P}^1(F)$ $(= G/B)$ has to be disjt from some open nbd of 1

Hence contained in compact subset of $\mathbb{A}^1(F)$

So evaluation $f(w)$, which is a T-hom to $\eta^w \otimes \delta_B^{\frac12} \longleftrightarrow$ B-hom to $Ind_T^B(\eta^w \otimes \delta_B^{\frac12})$

Lands in $c\text{-}Ind_T^B(-)$. Can show this map is an iso

Prop $J_B(V)$ is 1-dim^l and $\cong \eta^w$

Pf This turns out to be a question about N-invariant integration on $C_c^\infty(N)$ – 1-dim^l space by question from problem sheet

Thus

$$0 \to V_N \to I(\eta)_N \to (\eta \otimes \delta_B^{\frac12})_N \to 0$$

(Lemma (ii))

+ we conclude $J_B(I(\eta))$ is 2-dim^l + chars of T appearing are η and η^w

Corollary $Hom_G(I(\eta), I(\eta'))$ is 0 unless $\eta' = \eta$ or $\eta' = \eta^w$ and if $\eta \neq \eta^w$, in these cases it is 1-dim^l

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Remarks

(i) Slightly delicate to extract an explicit hom $I(\eta) \to I(\eta^w)$

(ii) Same pf shows that if θ is a nontriv char of N, then $\dim Hom_N(I(\eta), \theta) = 1$ ("uniqueness of Whittaker functionals")

(iii) Let V as above and $W = ker(V \to V_N)$

W codim 2 sub of $I(\eta)$

Fact W is irred as a B-rep^n – leads easily to proof of full thm

(iv) G/B is a G-variety with an open B-orbit (**spherical variety**)