TCC Modular Forms and Representations of GL_2: Assignment #1 (Solutions)

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This is the first of 3 problem sheets for this course. Questions *not* marked $*$ are assessed, and students taking this course for credit should submit their solutions to me (by email, or via my pigeonhole for Warwick students) by **noon on Monday 5th November**. Solutions will be distributed that afternoon, so late submissions will not be accepted. The total mark available for the assessed questions is 20.

Questions marked with one or more $*$'s are included for your own interest, and will not be given a numerical mark, but if you would like some (brief) feedback on your answers you are welcome to submit them to me anyway. The number of stars is intended as a rough indication of difficulty.

Throughout the sheet, F denotes a nonarchimedean local field with ring of integers $\mathcal{O}$ and residue field of cardinality q; ϖ denotes an arbitrary uniformiser of $\mathcal{O}$; and $|x|$, for $x \in F$, denotes the absolute value normalised by $|\varpi| = 1/q$.

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1. [1 point] Show that in a topological group, every open subgroup is also closed.

> **Solution:** Let G be a topological group and $H \leqslant G$ an open subgroup. Then any coset gH is also open, since it is the image of H under the homeomorphism $G \to G$ given by translation by g. Since an arbitrary union of open sets is open, $G - H = \bigcup_{\substack{g \in G/H \\ g \neq 1}} gH$ is open, so H is closed.

2. [1 point] Show that if G is locally profinite and compact, then every open neighbourhood of 1_G contains a normal open compact subgroup.

> **Solution:** Let U be an open neighbourhood of 1_G. Since G is locally profinite, there exists an open compact subgroup H such that $H \subseteq U$. For any $g \in G$, the conjugate gHg^{-1} is open (being the image of H under a homemorphism), and it depends only on the image of g in G/H. Since $[G : H]$ is finite, and finite intersections of open sets are open, the intersection $K = \bigcup_{g \in G/H} gHg^{-1}$ is open; it is clearly normal in G, and $K \subseteq H \subseteq U$.

3. [3 points] Let $G = \mathbf{Z}_p^\times$, for p prime, and let $V = \{\text{locally constant functions } G \to \mathbf{C}\}$.

   (a) Show that V is smooth and admissible.

> **Solution:** Smoothness: let $f \in V$. For all $x \in G$ there is an open set $U(x) \ni x$ such that $f|_{U(x)}$ is constant. The opens $U(x)$ cover G, and since G is compact, we can find a finite subcovering $G = U_1 \cup \cdots \cup U_n$ such that $f|_{U_i}$ is constant for all i. We may assume each U_i is of the form $a_i + p^{k_i}\mathbf{Z}_p$ for some $k_i \geqslant 1$ and $a_i \in G$, since these form a basis for the

topology. Let $N \geqslant \max(\{k_i\})$, so that $(1+p^N\mathbf{Z}_p)U_i = U_i$ for all i. Then $1+p^N\mathbf{Z}_p$ is an open subgroup contained in $\operatorname{Stab} f$.

Admissibility: let K be any open compact. Then any $f \in V^K$ is uniquely determined by its values on any set of representatives for the finite quotient G/K, so V^K is finite-dimensional.

(b) Show that V has infinitely many (distinct) irreducible subrepresentations.

Solution: Let χ be a Dirichlet character of conductor p^n for some n. Then χ generates a G-invariant subspace of V, which is 1-dimensional and hence irreducible.

(c) Show that the abstract dual V^* is not smooth.

Solution: Consider the linear functional $\lambda(f) = f(1)$. If there were open compact K such that $\lambda \in (V^*)^K$, then we would have $f(k) = f(1)$ for all $f \in V$ and $k \in K$, which is clearly absurd.

(d) [*] Show that the smooth dual $V^\vee$ is (non-canonically) isomorphic to V.

Solution: *(based on solution by Lambert A'Campo, Imperial)* Let μ be a left Haar measure for G and consider the map $\phi : V \to V^\vee$ sending f to the linear functional

$$f' \mapsto \int_G f(x)f'(x)\,\mathrm{d}\mu(x).$$

The image of ϕ is in $V^\vee$ because $\operatorname{Stab}(f) \leqslant \operatorname{Stab}\phi(f)$. Now we show that ϕ is injective. Let $f \in V$ and suppose $\phi(f) = 0$. Then in particular $\phi(f)(\overline{f}) = \int_G |f|^2\,\mathrm{d}\mu = 0$. Since $|f|^2 \geqslant 0$, this implies $f = 0$ by the properties of the integral. It remains to prove that ϕ is a surjection. Let $T \in V^\vee$ and $H = \operatorname{Stab} T$, and define $f(x) = \sum_{g\in H/G} \frac{T(1_{Hg})}{\mu(H)} 1_{Hg}$. Then one checks that $\phi(f) = T$.

4. [2 points] Let G be a locally profinite group. Show that the natural inclusion of $\underline{\mathrm{Smo}}_G$ in $\underline{\mathrm{Rep}}_G$ has a right adjoint (and describe the adjoint functor).

Solution: For $V \in \underline{\mathrm{Rep}}_G$, let $V^{\mathrm{sm}} = \bigcup_{K \text{ open cpct}} V^K$. Clearly $V^{\mathrm{sm}} \in \underline{\mathrm{Smo}}_G$, $(-)^{\mathrm{sm}}$ is a functor, and the obvious inclusion $V^{\mathrm{sm}} \hookrightarrow V$ gives a natural transformation from $(-)^{\mathrm{sm}}$ to the identity functor.

We want to show that for any $W \in \underline{\mathrm{Smo}}_G$ and $V \in \underline{\mathrm{Rep}}_G$, we have

$$\mathrm{Hom}_G(W, V^{\mathrm{sm}}) = \mathrm{Hom}_G(W, V).$$

Composing with the inclusion $V^{\mathrm{sm}} \to V$ gives injective maps $\mathrm{Hom}_G(W, V^{\mathrm{sm}}) \to \mathrm{Hom}_G(W, V)$, which (by construction) are natural in both W and V; so it suffices to show that these maps are surjective. However, if $\phi \in \mathrm{Hom}_G(W, V)$ and $w \in W$, then $w \in W^K$ for some K, so $\phi(w) \in V^K$; hence ϕ factors through V^{sm} as required.

5. (a) [1 point] Show that every compact open subgroup of $\mathrm{GL}_2(F)$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathcal{O})$.

Solution: Consider the action of $\mathrm{GL}_2(F)$ on the set of $\mathcal{O}$-lattices in F^2. It is clear that the action is transitive, so it suffices to show that every open compact stabilises some lattice.

Let Λ be the standard lattice $\mathcal{O}^2$, whose stabiliser is $\mathrm{GL}_2(\mathcal{O})$. If K is any open compact, then $K \cap \mathrm{GL}_2(\mathcal{O})$ has finite index in K, so the K-orbit of Λ is finite. The sum of any finite collection of lattices is a lattice, so summing over the K-orbit of Λ gives a lattice Λ_K which is K-invariant.

(b) [**] Show that this is not true for SL_2 in place of GL_2.

6. [2 points] Show that $\mathrm{GL}_n(F)$ is unimodular for all $n \geqslant 1$.

Solution: Using the Cartan decomposition, it suffices to show that $\delta_G(\gamma) = 1$ when $\gamma = \mathrm{diag}(1, \dots, 1, \varpi, \dots, \varpi)$, where the number of ϖ terms is m for some $1 \leqslant m \leqslant n-1$. Consider the subgroup U of $\mathrm{GL}_n(\mathcal{O})$ consisting of terms in which the bottom left $m \times n$ block is in $\varpi\mathcal{O}$. Then γ conjugates U onto its opposite $\bar{U}$; but U is the image of $\bar{U}$ under the inverse-transpose automorphism of $\mathrm{GL}_n(\mathcal{O})$, hence U and $\bar{U}$ have the same index in $\mathrm{GL}_n(\mathcal{O})$, and we are done.

7. Let G be locally profinite, and let μ be a left Haar measure on G. For $f \in C_c^\infty(G)$ and $g \in G$, write $g \cdot f$ for the function $x \mapsto f(xg)$.

(a) [1 point] Show that the linear functional $\lambda : C_c^\infty(G) \to \mathbf{C}$ defined by

$$\lambda(f) = \int_{x \in G} \delta_G(x)^{-1} f(x)\, \mathrm{d}\mu(x)$$

satisfies $\lambda(g \cdot f) = \lambda(f)$ for all $f \in C_c^\infty(G)$ and $g \in G$.

Solution: It suffices to check this for $f = \mathbf{1}_{Ua}$ the indicator function of a set of the form Ua, where U is an open compact subgroup and $a \in G$, since these span $C_c^\infty(G)$. We have

$$\lambda(f) = \delta(a)^{-1}\mu(Ua) = \delta_G(a)^{-1}\delta_g(a)\mu(U) = \mu(U)$$

which is independent of a. Since $g \cdot f = \mathbf{1}_{Uag^{-1}}$ [*sic; not Uag*], we are done.

(b) [2 points] Show that if λ' is a linear map $C_c^\infty(G) \to \mathbf{C}$ such that $\lambda'(g \cdot f) = \lambda'(f)$ for all $f \in C_c^\infty(G)$ and $g \in G$, then there is a constant $\alpha \in \mathbf{C}$ such that $\lambda' = \alpha\lambda$.

Solution: Fix one open compact K_0 and let $\alpha = \mu(K_0)^{-1}\lambda'(\mathbf{1}_{K_0})$. Then $\lambda'(f) = \lambda(f)$ for $f = \mathbf{1}_{K_0}$.
If $K \leqslant K_0$ is any smaller open compact subgroup, then

$$\lambda'(1_{K_0}) = \sum_{\gamma \in K_0/K} \lambda'(\gamma \cdot 1_K) = [K : K_0]\lambda'(1_K),$$

and similarly for λ. Thus $\lambda'(1_K) = \alpha\lambda(1_K)$ for all open compacts $K \leqslant K_0$. Since the functions $g \cdot 1_K$, for $K \leqslant K_0$ open compact and $g \in G$, span $C_c^\infty(G)$ we have $\lambda'(f) = \alpha\lambda(f)$ for all f.

8. (a) [*] Show that any open normal subgroup of $\mathrm{GL}_2(F)$ must contain $\mathrm{SL}_2(F)$.

(b) [*] Hence show that a finite-dimensional irreducible smooth representation of $\mathrm{GL}_2(F)$ must be of the form $\chi \circ \det$, for some smooth $\chi : F^\times \to \mathbf{C}^\times$.

9. A smooth representation of a locally profinite group G is said to be *unitarizable* if there is a pairing $\langle -, - \rangle : V \to \mathbf{C}$ such that the following conditions hold:

- V is linear in the second variable and conjugate-linear in the first;

- we have $\langle gx, gy\rangle = \langle x, y\rangle$ for all $x, y \in V$ and $g \in G$;
- $\langle y, x\rangle = \overline{\langle x, y\rangle}$;
- $\langle x, x\rangle > 0$ if $x \neq 0$.

(a) [*] Show that if V is smooth, admissible, and unitarizable, any G-invariant subspace $W \subseteq V$ has a G-invariant complement.

(b) [1 point] Show that if $H \leqslant G$ is closed and $W \in \underline{\mathrm{Smo}}_G$ is unitarizable, then the normalised induction $V = \text{c-Ind}_H^G \left(W \otimes (\delta_H^{-1}\delta_G)^{1/2}\right)$ is unitarizable.

Solution: By hypothesis there is some pairing $\langle -, -\rangle_W$ on W satisfying the hypotheses. Let $f_1, f_2 \in \text{c-Ind}_H^G \left(W \otimes (\delta_H^{-1}\delta_G)^{1/2}\right)$. Then the function $g \mapsto \langle f_1(g), f_2(g)\rangle_W$ is transforms by $(\delta_H^{-1}\delta_G)$ under left translation by H, and is compactly supported modulo H; so the integral

$$\langle f_1, f_2\rangle = \int_{H\backslash G} \langle f_1(g), f_2(g)\rangle_W \,\mathrm{d}\mu(g)$$

is defined, up to a positive real scaling factor depending on our choice of measures. It manifestly satisfies the first three conditions, and if $\langle f, f\rangle = 0$, then $\langle f(g), f(g)\rangle$ is a smooth function taking non-negative real values whose total integral is 0, so it must be identically 0 and we conclude $f = 0$.

(c) [2 points] Show that if the representation $I(\chi, \psi)$ of $\mathrm{GL}_2(F)$ is unitarizable, then the character χ/ψ must be either unitary, or real-valued. (Hint: Consider the representation $I(\overline{\chi}, \overline{\psi})$.)

Solution: If V is a smooth unitarizable representation then $v \mapsto \langle -, v\rangle$ is a non-zero G-equivariant map from V into the dual space $(\overline{V})^\vee$, where $\overline{V}$ is the representation which is V as an abelian group but with the $\mathbf{C}$-linear structure flipped by complex conjugation. Clearly $\overline{I(\chi, \psi)} = I(\overline{\chi}, \overline{\psi})$, so if $I(\chi, \psi)$ is unitarizable then

$$\mathrm{Hom}_G\left(I(\chi, \psi), I(\overline{\chi}^{-1}, \overline{\psi}^{-1})\right) \neq 0.$$

By a result from lectures this implies that we must have either $\{\chi = \overline{\chi}^{-1}, \psi = \overline{\psi}^{-1}\}$ or $\{\chi = \overline{\psi}^{-1}, \psi = \overline{\chi}^{-1}\}$. In the first case, both χ and ψ are unitary, hence so is χ/ψ. In the second case, we find $\chi/\psi = \overline{\psi}^{-1}/\overline{\chi}^{-1} = \overline{\chi/\psi}$, so χ/ψ is real-valued.

(d) [**] Show that the Steinberg representation of $\mathrm{GL}_2(F)$ is unitarizable.

10. Prove the two parts of the first lemma from §2.3, concerning representations of the additive group $N \cong (F, +)$:

(a) [2 points] If $V \in \underline{\mathrm{Smo}}_N$, then the kernel of $V \mapsto V_N$ coincides with the space

$$\left\{v \in V : \int_{N_0} n \cdot v \,\mathrm{d}\mu_N(n) = 0 \quad \text{for some open compact } N_0 \subset N \text{ (depending on } v)\right\}$$

Solution: The kernel of $V \to V_N$ is the subspace $V(N)$ spanned as a $\mathbf{C}$-vector space by all vectors of the form $nv - v$, for $n \in N$. Let $V'(N)$ be the other space in question.
To show $V(N) \subseteq V'(N)$ it suffices to show that $nv - v \in V'(N)$, for all $n \in N$ and $v \in V$. Choose $N_0 \leqslant N$ open compact containing n; then

$$\int_{N_0} n' \cdot (nv - v)\,\mathrm{d}n' = \int_{N_0} n'nv\,\mathrm{d}n' - \int_{N_0} n'v\,\mathrm{d}n' = 0$$

by the translation-invariance of the measure. *(Note that N is commutative, so there is no need to distinguish between left and right translation invariance.)*

Conversely, if $v \in V'(N)$, then choose some N_0 such that $\int_{N_0} nv \, \mathrm{d}n = 0$. Then we have

$$v = v - \frac{1}{\mu(N_0)} \int_{N_0} nv \, \mathrm{d}n = -\int_{N_0} (nv - v) \, \mathrm{d}n$$

which is a finite linear combination of the generators of $V(N)$. Thus $v \in V(N)$.

(b) [2 points] The functor $V \mapsto V_N$ is exact on $\underline{\mathrm{Smo}}_N$.

Solution: It is standard that coinvariants are right exact; so we need only show that if $f : A \hookrightarrow B$ is an injective map of N-representations, then $A_N \to B_N$ is also injective. However, if $a \in A$ maps to 0 in B_N, by part (a) there is N_0 such that $\int_{N_0} nf(a) = 0$ as an element of B. Thus $f(\int_{N_0} na) = 0$, but f is injective, so $\int_{N_0} na = 0$ in A and thus $a = 0$ in A_N.

11. [*] Let χ, ψ be smooth characters of $F^\times$ with $\chi \neq \psi$. We saw in lectures that $\mathrm{Hom}_G(I(\chi, \psi), I(\psi, \chi))$ is 1-dimensional. In this exercise we'll construct an explicit homomorphism between the two (in a special case).

   (a) Show that if the inequality $|\chi(\varpi)/\psi(\varpi)| < 1$ holds for one uniformiser ϖ, this holds for every uniformiser ϖ.

   (b) Suppose the inequality of (a) holds, and let $f \in I(\chi, \psi)$. Show that the integral

   $$\tilde{f}(g) = \int_N f\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) ng \right) \, \mathrm{d}\mu_N(n)$$

   converges absolutely. You may find it helpful to note the matrix identity

   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

   (c) Show that in the situation of (a), the map $f \mapsto \tilde{f}$ is a non-zero element of $\mathrm{Hom}_G(I(\chi, \psi), I(\psi, \chi))$.

   (d) [**] What is happening if $\chi/\psi = |\cdot|$?

12. [**] Let G be locally profinite, V an irreducible smooth representation of G, and $H \leqslant G$ an open normal subgroup such that $G/H \cong (\mathbf{Z}/2)^n$ for some n.

   (a) Show that $V|_H$ is a direct sum of finitely many irreducible H-subrepresentations, and the number of these factors divides 2^n. (Hint: Use induction to reduce to the case $n = 1$, and consider the subspace $g \cdot W$ where W is an irreducible H-subrepresentation and $g \in G - H$.)

   (b) Show that $V|_H$ is irreducible if and only if there is no nontrivial character $\chi : G/H \to \mathbf{C}^\times$ such that $V \cong V \otimes \chi$. (Hint: Show that $\mathrm{Ind}_H^G(V|_H) \cong \bigoplus_\chi V \otimes \chi$.)

   (c) Describe the decomposition into irreducible $\mathrm{SL}_2(F)$-representations of every irreducible, non-supercuspidal representation of $\mathrm{GL}_2(F)$. (Hint: $\mathrm{GL}_2(F)/(F^\times \cdot \mathrm{SL}_2(F))$ is finite.)

   (d) [***] Can you find an irreducible representation of $\mathrm{GL}_2(F)$ which decomposes into more than two $\mathrm{SL}_2(F)$ subrepresentations?