TCC Modular Forms and Representations of GL_2: Assignment #3 (Solutions)

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If $\Gamma \leqslant \mathrm{SL}_2(\mathbf{Z})$ and L is a subfield of $\mathbf{C}$, we define $M_k(\Gamma, L)$ to be the L-subspace of $M_k(\Gamma)$ consisting of forms with q-expansion coefficients in L, and similarly $S_k(\Gamma, L)$. For $f \in M_k(\Gamma)$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$, we let f^σ be the formal q-expansion $\sum \sigma(a_n) q^n$, where $f = \sum a_n q^n$.

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1. [1 point] Prove the formula relating the global Kirillov function to q-expansions, $a_n(f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -)) = n^t \phi_f(nx)$.

Solution: We compute that for $f \in M_{k,t}$ we have

$$\begin{aligned}\phi_f(nx) &= a_1(f(\left(\begin{smallmatrix} nx & 0 \\ 0 & 1\end{smallmatrix}\right), -)) \\ &= a_1(f(\left(\begin{smallmatrix} n & 0 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -)) \\ &= a_1(f(\left(\begin{smallmatrix} n & 0 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -)) \\ &= a_1\left(f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -) \mid_{k,t} \left(\begin{smallmatrix} n & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1}\right)\end{aligned}$$

From the definition of the weight k, t action, we have

$$\left(f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -) \mid_{k,t} \left(\begin{smallmatrix} n & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1}\right)(\tau) = n^{-t} f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), \tfrac{\tau}{n}),$$

so $a_1\left(f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -) \mid_{k,t} \left(\begin{smallmatrix} n & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1}\right) = n^{-t} a_n(f(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right), -))$.

2. [1 point] Let Π be a cuspidal automorphic representation of weight (k, t), and $f \in S_k(\Gamma_1(N))$ its normalised new vector. Show that if f transforms under the diamond operators $\langle d \rangle$ via the character $\varepsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, then the central character of the automorphic representation Π is the character $\|\cdot\|^{2t-k}\underline{\varepsilon}$, where $\underline{\varepsilon}$ is the adelic character attached to ε (as in Q7 of Sheet 1).

Solution: We saw in lectures that the action of $\left(\begin{smallmatrix} \varpi_\ell & 0 \\ 0 & \varpi_\ell\end{smallmatrix}\right)$ corresponds to $\ell^{k-2t}\langle \ell \rangle$ under the identification $(S_{k,t})^{U_1(N)} \cong S_k(\Gamma_1(N))$. Thus the central character of Π has to map ϖ_ℓ to $\ell^{k-2t}\varepsilon(\ell)$ for all but finitely many primes ℓ, which is also true of the character $|\cdot|^{2t-k}\underline{\varepsilon}$. By the uniqueness assertion of Sheet 2 Q7a these two characters of $\mathbf{A}_f^\times / \mathbf{Q}_{>0}^\times$ must be the equal.

3. [*] Let χ be a quadratic Dirichlet character, and Π a cuspidal automorphic representation such that $\Pi = \Pi \otimes \chi$ *[NB: such examples do exist]*. Let $\chi' \neq \chi$ be another quadratic Dirichlet character. Show that the representation $\Pi' = \Pi \otimes \chi'$ satisfies $\Pi_\ell \cong \Pi'_\ell$ for a set of primes ℓ of density $\geqslant \frac{3}{4}$.

4. [2 points] Show (without using Shimura's rationality theorems) that if $f \in M_{k,t}$ then the function $f^*(g,\tau) = \overline{f(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) g, -\bar{\tau})}$ is also in $M_{k,t}$, and that $(g \cdot f)^* = g \cdot f^*$.

Solution: Clearly $f^*(g,-)$ has moderate growth, and since $\overline{\exp(2\pi i(-\bar{\tau}))} = \exp(2\pi i\tau)$, the locally-uniform convergence of the q-expansion of f shows that f^* is holomorphic in τ. *[Alternatively, one can check directly from the Cauchy-Riemann equations that if h is holomorphic on $D \subset \mathbf{C}$ then $\tau \mapsto \overline{h(\bar{\tau})}$ is holomorphic on $\bar{D}$.]*

For $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ set $\gamma^* = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) \gamma \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Then

$$\begin{aligned} f^*(\gamma g, \tau) &= \overline{f(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)\gamma g, -\bar{\tau})} \\ &= \overline{f(\gamma^* \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) g, -\bar{\tau})} \\ &= \overline{\left(f(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) g, -) \mid_{k,t} (\gamma^*)^{-1}\right)(-\bar{\tau})} \end{aligned}$$

If $\gamma^{-1} = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then $(\gamma^*)^{-1} = \begin{pmatrix}a & -b\\ -c & d\end{pmatrix}$, so this is

$$\begin{aligned} \overline{(ad-bc)^t(c\bar{\tau}+d)^{-k} f(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) g, -\bar{\tau})} &= (ad-bc)^t(c\tau+d)^{-k}\overline{f(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) g, -\bar{\tau})} \\ &= (f^*(g,-) \mid_{k,t} \gamma^{-1})(\tau). \end{aligned}$$

Thus f transforms like a modular form under the left action of $\mathrm{GL}_2^+(\mathbf{Q})$. Finally $(g\cdot f)^*(g',\tau) = \overline{(g\cdot f)(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) g', -\bar{\tau})} = f^*(g'g,\tau)$, which implies in particular that f^* is fixed by right-translation under any subgroup fixing f, so $f^* \in M_{k,t}$ and the map $f \to f^*$ is $\mathrm{GL}_2(\mathbf{A}_f)$-equivariant.

5. [2 points] Let $f \in S_{k,t}(\mathbf{Q})$, for some $k, t \in \mathbf{Z}$, so that the Kirillov function ϕ_f of f takes values in $\mathbf{Q}_\infty$ and satisfies $\sigma(\phi_f(x)) = \phi_f(\chi(\sigma)x)$ for all $\sigma \in \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Show that the same is true of the Kirillov function of $\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right) f$, for any $a \in \mathbf{A}_f^\times, b \in \mathbf{A}_f$. *[You may **not** use Shimura's theorem that $S_{k,t}(\mathbf{Q})$ is $\mathrm{GL}_2(\mathbf{A}_f)$-stable.]*

Solution: We have $\phi_{\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right)f}(\chi(\sigma)x) = \theta(\chi(\sigma)bx)\phi_f(\chi(\sigma)ax)$, and $\phi_{\left(\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right)f}(x)^\sigma = \theta(bx)^\sigma\phi_f(ax)^\sigma$. By hypothesis $\phi_f(\chi(\sigma)ax) = \phi_f(ax)^\sigma$ so it suffices to prove that $\theta(bx)^\sigma = \theta(\chi(\sigma)bx)$ for all b, x, σ. We may take $b = 1$ without loss of generality.

For any given $x \in \mathbf{A}_f$, $\theta(x)$ is a root of unity of order R, where R is the order of x in the torsion group $\mathbf{A}_f/\hat{\mathbf{Z}} \cong \mathbf{Q}/\mathbf{Z}$.

By the definition of the cyclotomic character χ, we have $\theta(x)^\sigma = \theta(x)^m$ where m is any integer congruent to $\chi(\sigma)$ mod R. However, we also have $\theta(x\chi(\sigma)) = \theta(mx)$ (since $(m - \chi(\sigma))x \in \hat{\mathbf{Z}}$); so $\theta(x)^m = \theta(\chi(\sigma)x)$.

6. [3 points] Let $N \geqslant 1$. Define

$$S_k'(\Gamma_1(N), \mathbf{Q}) = \left\{ f \in S_k(\Gamma_1(N), \mathbf{Q}(\zeta_N)) : f^\sigma = \langle \chi(\sigma) \rangle f \ \forall \sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \right\}.$$

(a) Show that $S_k'(\Gamma_1(N), \mathbf{Q})$ spans $S_k(\Gamma_1(N))$ over $\mathbf{C}$.

(b) Show that for any integer t the Atkin–Lehner operator W_N, defined by $W_N(f) = f|_{k,t} \left(\begin{smallmatrix}0 & -1\\ N & 0\end{smallmatrix}\right)$, is a bijection
$$S_k(\Gamma_1(N), \mathbf{Q}) \cong S_k'(\Gamma_1(N), \mathbf{Q}).$$

[Hint: Consider the group $\{\gamma \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : \gamma = \left(\begin{smallmatrix}1 & *\\ 0 & *\end{smallmatrix}\right) \bmod N\}$.]

Solution: Let U' the group in the Hint. By a theorem of Shimura stated in the lectures, we have

$$S_{k,t}^{U'} = S_{k,t}(\mathbf{Q})^{U'} \otimes_{\mathbf{Q}} \mathbf{C}.$$

Since we have $\det(U') = \widehat{\mathbf{Z}}^\times$ and $U' \cap \mathrm{GL}_2^+(\mathbf{Q}) = \Gamma_1(N)$, a theorem from earlier in the course tells us that the map $f \mapsto f(1,-)$ is a bijection $S_{k,t}^{U'} \cong S_k(\Gamma_1(N))$. Hence if I is the space of functions $\{f(1,-) : f \in S_{k,t}(\mathbf{Q})^{U'}\}$, we have $S_k(\Gamma_1(N)) = I \otimes_{\mathbf{Q}} \mathbf{C}$.

I claim that I is precisely the space $S_k'(\Gamma_1(N), \mathbf{Q})$. This clearly proves (a). To prove the claim, we check that the action of $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for $a \in \hat{\mathbf{Z}}^\times$, on $S_{k,t}(\mathbf{Q})^{U'}$ coincides with the classical diamond operator $\langle a \rangle$ on $S_k(\Gamma_1(N))$.

Finally we note that $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{Q})$ conjugates U into U', so the action of this matrix gives a bijection between $S_{k,t}(\mathbf{Q})^U$ and $S_{k,t}(\mathbf{Q})^{U'}$ and thus between $S_k(\Gamma_1(N), \mathbf{Q})$ and $S_k'(\Gamma_1(N), \mathbf{Q})$. This is (b).

7. [4 points] Show that $X_0(16)$ has 6 cusps, of which 4 are defined over $\mathbf{Q}$. What is the field of definition of the remaining two?

Solution: By standard complex-analytic theory (see e.g. §3 of Diamond + Shurman), we find that a set of representatives for the cusps is given by $\{1, \frac{1}{2}, \frac{1}{4}, \frac{-1}{4}, \frac{1}{8}, \infty\}$. In particular, every cusp has the form $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \cdot \infty$ for some $a \in \mathbf{Z}$. Conversely, if $a \in \mathbf{Z}$ then the equivalence class of $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \cdot \infty$ depends only on a mod 16, and is given by

cusp	values of a mod 16
1	(units)
$\frac{1}{2}$	$\{2, 6, 10, 14\}$
$\frac{1}{4}$	$\{4\}$
$-\frac{1}{4}$	$\{12\}$
$\frac{1}{8}$	$\{8\}$
∞	$\{0\}$

A theorem from the lectures tells us that the image of $\gamma \cdot \infty$ under the action of $\sigma \in \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ is given by $\gamma' \cdot \infty$ where γ' is any element of $\mathrm{SL}_2(\mathbf{Z})$ whose image in $\mathrm{SL}_2(\mathbf{Z}/16)$ coincides with that of $\begin{pmatrix} \chi(\sigma) & \\ & 1 \end{pmatrix} \gamma \begin{pmatrix} \chi(\sigma)^{-1} & \\ & 1 \end{pmatrix}$.

[Note that $\begin{pmatrix} \chi(\sigma) & \\ & 1 \end{pmatrix} \gamma \begin{pmatrix} \chi(\sigma)^{-1} & \\ & 1 \end{pmatrix}$ is in $\mathrm{SL}_2(\hat{\mathbf{Z}})$ but is not in $\mathrm{SL}_2(\mathbf{Z})$ in general, so it doesn't make sense to let it act on $\mathbf{P}^1(\mathbf{Q})$; we have to choose some γ' close enough and act by that.]

Hence if γ has the form $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$, we can take $\gamma' = \left(\begin{smallmatrix} 1 & 0 \\ a' & 1 \end{smallmatrix}\right)$ where a' is anything congruent to $\chi(\sigma)^{-1} a$ mod 16.

Inspecting the tables of values of a mod 16, we see that the cusps $\{1, \frac{1}{2}, \frac{1}{8}, \infty\}$ correspond to subsets of $\mathbf{Z}/16\mathbf{Z}$ which are preserved by multiplication by units, so these cusps are defined over $\mathbf{Q}$. The remaining two cusps $\pm\frac{1}{4}$ are fixed by multiplication by units that are 1 mod 4, and they are interchanged by the action of units that are 3 mod 4, so they correspond to a conjugate pair of points defined over the quadratic field $\mathbf{Q}(\zeta_4) = \mathbf{Q}(i)$.

8. [*] Let F be a nonarchimedean local field, and π_1, π_2 irreducible infinite-dimensional representations of $\mathrm{GL}_2(F)$. Let χ, ψ be any two characters of $F^\times$ such that $\chi\psi$ is the product of the central characters of the π_i. Show that there is a non-zero homomorphism of $\mathrm{GL}_2(F)$-representations $\pi_1 \otimes \pi_2 \to I(\chi, \psi)$. *[Hint: Consider first the case where at least one of the π_i is supercuspidal.]*

9. Recall the functions $f_\Phi(g, s)$ and $\tilde{f}_\Phi(g, s)$ defined in Jacquet's local Rankin–Selberg theory. *[The parameter s was omitted from the notation in the lecture, but we include it here.]*

(a) [1 point] Show that if $\mathrm{Re}(s)$ is sufficiently large that $|q^{-2s}\omega(\varpi)| < 1$, then the integral defining $f_\Phi(g,s)$ converges for all g and Φ.

Solution: We defined

$$f_\Phi(g,s) = |\det g|^s \cdot \int_{t\in F^\times} \Phi((0,t)g)\omega(t)|t|^{2s}\,\mathrm{d}^\times t.$$

We claim that the integral is absolutely convergent under the given hypotheses. It suffices to assume $g = 1$. Since $|\omega(t)| = 1$ for $t \in \mathcal{O}^\times$ the absolute-value integral is

$$(*) \cdot \sum_{m\in\mathbf{Z}} \int_{t\in\mathcal{O}^\times} (|\Phi(0,\varpi^m t)|)\cdot(|q^{-2s}\omega(\varpi)|)^m.$$

For $m \ll 0$ the integrand is zero, and for $m \gg 0$ the term $\Phi(0,\varpi^m t)$ is just $\Phi(0,0)$; hence the integral is bounded by

$$(\text{finite sum}) + (\text{const})\cdot\sum_{m\geqslant 0}(q^{-2s}|\omega(\varpi)|)^m$$

which is finite under the stated hypothesis on s.

(b) [1 point] Show that whenever $f_\Phi(g,s)$ is defined, we have $f_\Phi(-,s) \in I\left(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1}\right)$.

Solution: We need to check the following:

(i) there is some open U such that $f_\Phi(gu,s) = f_\Phi(g,s)$, for all $g \in G$ and $u \in U$;

(ii) $f_\Phi(\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) g, s) = |a/d|^s\omega^{-1}(d)$ for all $\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \in B$.

For (i), we note that if $\det(u) \in \mathcal{O}^\times$ we have $f_\Phi(gu,s) = f_{u\cdot\Phi}(g,s)$, so it suffices to show that $C_c^\infty(F^2)$ is a smooth representation of $\mathrm{GL}_2(F)$. The space $C_c^\infty(F^2)$ is spanned by indicator functions of sets of the form $(a+\mathfrak{P}^n, b+\mathfrak{P}^n)$. If a,b are in $\mathfrak{P}^{-m}$ then this set is preserved by the open subgroup $\{u : u = 1 \bmod p^{m+n}\}$.
For (ii), we have

$$\begin{aligned} f_\Phi(\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) g, s) &= |ad|^s|\det g|^s \int_{F^\times} \Phi((0,t)\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) g)\omega(t)|t|^{2s}\,\mathrm{d}^\times t \\ &= \frac{|ad|^s|\det g|^s}{\omega(d)|d|^{2s}} \int_{F^\times} \Phi((0,dt)g)\omega(dt)|dt|^{2s}\,\mathrm{d}^\times t \\ &= \frac{|ad|^s}{\omega(d)|d|^{2s}} f_\Phi(g,s) \end{aligned}$$

as required.

(c) [*] Let $s_0 \in \mathbf{C}$. Show that the following are equivalent:

- there exists some $\Phi \in C_c^\infty(F^2)$ and $g \in \mathrm{GL}_2(F)$ such that $f_\Phi(g,s)$ has a pole at $s = s_0$;
- the representation $I\left(|\cdot|^{s_0-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s_0}\omega^{-1}\right)$ is reducible with a 1-dimensional subrepresentation.

Show that if these conditions are satisfied, then the limit

$$\lim_{s\to s_0}(s-s_0)\cdot f_\Phi(g,s)$$

exists for all g and Φ, and as a function of g it lies in the 1-dimensional subrepresentation of $I(\dots)$.

(d) [*] Use (c) to show that if at least one of π_1 and π_2 is supercuspidal, then $L(\pi_1 \times \pi_2, s)$ is identically 1 unless π_1 is isomorphic to a twist of π_2.

10. [2 points] Let F be a nonarchimedean local field. Let θ be a character $F \to \mathbf{C}^\times$ trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$, and let μ denote the Haar measure on F such that $\mu(\mathcal{O}) = 1$.

(a) For $\phi \in C_c^\infty(F)$, define $\hat{\phi}$ by

$$\hat{\phi}(x) = \int_F \phi(u)\theta(xu)\,\mathrm{d}\mu(u).$$

Show that $\hat{\phi} \in C_c^\infty(F)$, and $\hat{\hat{\phi}}(x) = \phi(-x)$.

Solution: We first make a preliminary reduction. Let $\phi \in C_c^\infty(F)$ and define Φ by $\Phi(x) = \phi(ax+b)\theta(cx)$ for some $a, b, c \in F$ (with $a \neq 0$). A change of variable shows that

$$\hat{\Phi}(x) = \left(|a|^{-1}\theta(-\tfrac{bc}{a})\right)\theta(-\tfrac{b}{a}x)\hat{\phi}(\tfrac{1}{a}x + \tfrac{c}{a}).$$

Applying this again with ϕ replaced by $\hat{\phi}$ and a, b, c by $a' = 1/a$, $b' = c/a$ and $c' = -b/a$, we end up with

$$\hat{\hat{\Phi}}(x) = \left(|a'|^{-1}\theta(-b'c'/a')\right)\left(|a|^{-1}\theta(-bc/a)\right)\theta(-\tfrac{b'}{a'}x)\hat{\hat{\phi}}(\tfrac{1}{a'}x + \tfrac{c'}{a'}) = \hat{\hat{\phi}}(ax-b)\theta(-cx).$$

The first formula shows that if $\hat{\phi} \in C_c^\infty(F)$ then we also have $\hat{\Phi} \in C_c^\infty(F)$. The second shows that if $\hat{\hat{\phi}}(x) = \phi(-x)$, then we also have $\hat{\hat{\Phi}}(x) = \Phi(-x)$.
Since $C_c^\infty(F)$ is spanned by functions of the form $\mathbf{1}_{\mathcal{O}}(a+bx)$, it follows that these two relations hold for all $\phi \in C_c^\infty$ if and only if they hold for the single function $\phi = \mathbf{1}_{\mathcal{O}}$.
For this ϕ, we have $\hat{\phi}(x) = \int_{\mathcal{O}} \theta(xu)\,\mathrm{d}\mu(u)$. If $x \in \mathcal{O}$, then the integrand is identically 1, so the integral is just $\mu(\mathcal{O}) = 1$. On the other hand, if $x \notin \mathcal{O}$ then $u \mapsto \theta(xu)$ is a non-trivial smooth character of $\mathcal{O}$, so $\int_{\mathcal{O}} \theta(xu)\mathrm{d}\mu(u)$ is zero. Thus $\hat{\phi} = \phi$ in this case; in particular, we have $\hat{\phi} \in C_c^\infty(F)$, and $\hat{\hat{\phi}}(x) = \phi(x) = \phi(-x)$. So we are done.

(b) For $\Phi \in C_c^\infty(F^2)$, define $\hat{\Phi}$ by

$$\hat{\Phi}(x,y) = \iint_{F\times F} \Phi(u,v)\theta(xv - yu)\,\mathrm{d}\mu(u)\mathrm{d}\mu(v).$$

Show that $\hat{\hat{\Phi}} = \Phi$. *[Hint: $C_c^\infty(F^2)$ is spanned by functions of the form $\Phi(x,y) = \phi_1(x)\phi_2(y)$.]*

Solution: Letting Φ have the form given in the Hint, we compute that

$$\hat{\Phi}(x,y) = \hat{\phi}_1(-y)\hat{\phi}_2(x) = \rho_1(x)\rho_2(y),$$

where $\rho_1(x) := \hat{\phi}_2(x)$ and $\rho_2(y) := \hat{\phi}_1(-y)$. Iterating the argument, we have $\hat{\hat{\Phi}}(x,y) = \hat{\rho}_1(-y)\hat{\rho}_2(x)$; but $\hat{\rho}_1(-y) = \hat{\hat{\phi}}_2(-y) = \phi_2(y)$, and $\hat{\rho}_2(x) = \hat{\hat{\phi}}_1(-x) = \phi_1(x)$.

11. [3 points] Let $k \geqslant 0$ be an integer, $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$, and $\Phi \in C_c^\infty(\mathbf{A}_f^2)$. Show that the Eisenstein series $E_\Phi^k(g,\tau,s)$ and $\tilde{E}_\Phi^k(g,\tau,s)$ transform like elements of $M_{k,k/2}$ under left translation by $\mathrm{GL}_2^+(\mathbf{Q})$. (You may assume that the sums concerned are absolutely convergent.)

Solution: *[Everybody who attempted this question got it right, and typesetting it is a pain, so I'm not going to provide a model solution.]*