Modular forms + reps of GL(2)

Lecture 3

Note on last lecture

$H \leqslant G$, integration map

$$C\text{-}\mathrm{Ind}_H^G(\delta_H^{-1}\delta_G|_H) \xrightarrow{\int} \mathbb{C}$$

<u>NB</u> Can normalize st if $f \in C\text{-}\mathrm{Ind}$ st $f(g) \in \mathbb{R}_{\geqslant 0}$ $\forall g$, then $\int f \geqslant 0$ and $\int f = 0$ only if $f = 0$

(Needed for problem sheet)

Supercuspidal rep^{n}s = irred rep^{n}s of $GL_2(F)$ not subquots of any $I(\chi, \psi)$

<u>Fact</u> If $p \neq 2$, $\exists$ explicit description of supercuspidals

<u>Def^n</u> An <u>admissible pair</u> is a pair (E, χ), where

- E/F is a quadratic ext^n
- $\chi : E^\times \to \mathbb{C}^\times$ smooth char

st (a) χ doesn't factor thru $\mathrm{Norm}_{E/F}$

(b) if E is ramified, $\chi|_{1+\mathfrak{p}_E}$ doesn't factor thru norm

Then $\exists$ map

$$\begin{pmatrix}\text{admissible}\\ \text{pairs } (E,\chi)\end{pmatrix} \longrightarrow \begin{pmatrix}\text{supercuspidal}\\ GL_2F \text{ reps}\end{pmatrix}$$

& all supercuspidals arise in this way (for $p \neq 2$)

Chapter 3 Hecke Algebras

§3.1 Def^ns

G loc prof gp, unimodular μ Haar measure

$C_c^\infty(G)$ = loc const, cpctly supp fcns on G

<u>Def^n</u> For $V \in \underline{Smo}_G$, $v \in V$, $F \in C_c^\infty(G)$, let

$$F * v = \int_G F(g)\, g\, v \, d\mu(g) \in V$$

In particular can take $V = C_c^\infty(G)$ with $(gF)(h) = F(g^{-1}h)$, so

$$(F * G)(h) = \int_G F(g)\, G(g^{-1}h)\, d\mu(g)$$

<u>Exercise</u> This is <u>associative</u>, so $C_c^\infty(G)$ becomes a ring $\mathcal{H}(G)$, and any $V \in \underline{Smo}_G$ an $\mathcal{H}(G)$-module

<u>Problem</u> $\mathcal{H}(G)$ has no identity elt unless G is discrete

<u>Def^n</u> $K \leqslant G$ open cpct

$$e_K = \frac{1}{\mu(K)} \mathbb{1}_K$$

Then $e_K e_K = e_K$, so the subring

$$\mathcal{H}(G, K) = e_K \mathcal{H}(G) e_K$$

is unital, with e_K as identity

For $V \in \underline{Smo}_G$, $e_K V = V^K$, + in particular, $\mathcal{H}(G,K)$ = fcns inv^t under left + right K-translation

<u>Prop</u> (i) As $\mathbb{C}$-vector space, $\mathcal{H}(G,K)$ has basis $[KgK]$

$$= \frac{1}{\mu(K)} \mathbb{1}_{KgK}, \quad g \in K\backslash G/K$$

(ii) $[KgK][KhK]$

$= \sum_\gamma c_\gamma [K\gamma K]$, where c_γ is the integer $\mu(KgK \cap \gamma K h^{-1} K)/\mu(K)$ ($=0$ for all but finitely many γ)

<u>Pf</u> (i) is clear (ii) is an easy exercise from def^n of multiplication

If $V \in \underline{\mathrm{Smo}_G}$ is <u>irred</u>, then V^K is a simple $H(G,K)$-module (or 0)

<u>Thm</u> (B-H §4.3) This gives a bij^n

$$\left(\begin{matrix}\text{irred } V\\ \text{st } V^K \neq 0\end{matrix}\right)\Big/_{\text{iso}} \xrightarrow{\sim} \left(\begin{matrix}\text{simple}\\ \mathcal{H}(G,K)\text{-mods}\end{matrix}\right)\Big/_{\text{iso}}$$

<u>§3.2</u> <u>Spherical Hecke algebras</u>

We take $G = GL_2(F)$
$K = GL_2(\mathcal{O})$

Cartan decomp^n $\mathcal{H}(G,K)$ spanned by $\left[K\begin{pmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{pmatrix}K\right]$, $a \geq b \in \mathbb{Z}$

Let $S = \begin{bmatrix}\varpi & 0\\ 0 & \varpi\end{bmatrix}$, $T = \begin{bmatrix}\varpi & 0\\ 0 & 1\end{bmatrix}$

<u>Will show</u> $\mathcal{H}(G,K) \cong \mathbb{C}[S, S^{-1}, T]$

Let $A_0 = \mathbb{C}[S, S^{-1}]$ central subring of $\mathcal{H}$

$A_n = \mathbb{C}$-span of $\begin{bmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{bmatrix}$ st $0 \leq a-b \leq n$

$\quad = A_0$-span of $\begin{bmatrix}\varpi^a & 0\\ 0 & 1\end{bmatrix}$, $0 \leq a \leq n$

<u>Lemma</u> $\forall n \geq 1$, $T * \begin{bmatrix}\varpi^n & 0\\ 0 & 1\end{bmatrix}$

$= c\begin{bmatrix}\varpi^{n+1} & 0\\ 0 & 1\end{bmatrix}$ modulo A_{n-1}, some $c \geq 1$

<u>Pf</u> Any double coset in support of $T * \begin{bmatrix}\varpi^n & 0\\ 0 & 1\end{bmatrix}$ has det ϖ^{n+1} up to units, + is in $M_{2\times 2}(\mathcal{O})$, so it is either

$$\begin{bmatrix}\varpi^{n+1} & 0\\ 0 & 1\end{bmatrix} \text{ or } \underbrace{\begin{bmatrix}\varpi^a & 0\\ 0 & \varpi^b\end{bmatrix},\ a+b=n+1,\ a,b \geq 1}_{\text{all in } A_{n-1}}$$

definitely is in support, so $c \neq 0$ $\square$

<u>Proof of thm</u>

We claim that A_n is spanned as an A_0-mod by $(1, T, \dots, T^n)$ + these are A_0-LI

Clear for $n=0$, $n=1$

+ via lemma, follows $\forall n$ by induction $\square$

§33 Unramified principal series

χ, ψ to be smooth unramified chars of $F^\times$ (trivial on $\mathcal{O}^\times$, thus of form $\chi(x) = \alpha^{v(x)}$, $\alpha = \chi(\varpi)$
$\psi(x) = \beta^{v(x)}$, $\beta = \psi(\varpi)$)

Prop $I(\chi, \psi)^K$ is 1-dim^l, and S acts as $\alpha\beta$, T as $q^{\frac{1}{2}}(\alpha+\beta)$

Pf Since $G = BK$, $I(-)^K$ has dim ≤ 1 for any χ, ψ, but pf of lemma earlier in course shows $I(-)^K = (\chi \otimes \psi)^{K \cap B}$, 1-dim^l

S-action is clear (since S central)

If ϕ basis of $I(-)^K$,

$$(T * \phi)(1) = \frac{1}{\mu(K)} \int_G T(g)\,(g\,\phi)(1)\, dg$$

$$= \frac{1}{\mu(K)} \int_{K\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)K} \phi(g)\, dg$$

$$= \sum_{g \in K\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)K/K} \phi(g)$$

$$= \sum_{a \in \mathcal{O}/\mathfrak{p}} \phi\begin{pmatrix}\varpi & a\\ 0 & 1\end{pmatrix} + \phi\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}$$

all $= \phi\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right) = q^{-\frac{1}{2}}\alpha\,\phi(1)$ $\qquad$ $= q^{+\frac{1}{2}}\beta\,\phi(1)$

so $(T * \phi)(1) = \left[q\,(q^{-\frac{1}{2}}\alpha) + q^{\frac{1}{2}}\beta\right]\phi(1)$ $\square$

Equ^{t}ly consider "Satake polynomial"

$X^2 - q^{\frac{1}{2}}TX + S \in \mathcal{H}(G,K)[X]$

α, β = roots of Satake poly acting on $I(\chi, \psi)^K$

Corollary Every irred rep^n V of $GL_2(F)$ st $V^K \neq 0$ is one of the following

- $I(\chi, \psi)$ for some unram χ, ψ st $\chi/\psi \neq |\ |^{\pm 1}$
- one-dim^l reps $\nu \cdot \det$, ν unram

Pf These exhaust the possible simple $\mathcal{H}(G,K)$-modules (since $\mathcal{H}(G,K)$ is commutative $\Rightarrow$ all simple mods 1-dim^l)

§34 The Iwahori-Hecke algebra

Let $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \quad c \in \mathfrak{p} \right\}$
(Iwahori subgp)

Prop (a) If $V = I(\chi, \psi)$, χ, ψ unram,
$\chi/\psi \neq |\ |^{\pm 1}$, then V^I is 2-dim^l

(b) $\left[I \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} I \right]$ acts w eigenvalues $\{ q^{\frac{1}{2}}\alpha, q^{\frac{1}{2}}\beta \}$
$\quad \underset{U}{\shortparallel}$

Pf Assume $\alpha \neq \beta$

I has 2 orbits on $\mathbb{P}^1(\mathcal{O})$, by Bruhat decomp^n for $GL_2(k)$, represented by 1 and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$V^I = \left(\chi \boxtimes \psi \; \delta_B^{-\frac{1}{2}} \right)^{B \cap I} \oplus \left(\text{——} \right)^{B \cap wIw^{-1}}$

2-dim^l

We compute that if $f \in V^I$,
$\left[I \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} I \right] * f = \sum_{a \in \mathcal{O}/\mathfrak{p}} f\begin{pmatrix} \varpi & a \\ 0 & 1 \end{pmatrix}$

$= q \quad q^{-\frac{1}{2}} \alpha \, f(1)$

So eval^n at 1 is a nonzero linear fun^l on V^I factoring thru proj^n to $U = q^{\frac{1}{2}}\alpha$ eigenspace. By symmetry, $q^{\frac{1}{2}}\beta$ e'space also nonzero $\square$

(If $\alpha = \beta$, U acts with matrix $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} q^{\frac{1}{2}}$)

§4 Local New Vectors

§4.1 Statement

Let V irred rep^n of $GL_2(F)$
Assume V not 1-dim^l
Let $U_n = \left\{ g \in GL_2(\mathcal{O}) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p}^n \right\}$

<u>Thm</u> (Casselman)
(a) $\exists n$ st $V^{U_n} \neq 0$
(b) If c = least n st $V^{U_n} \neq 0$, then V^{U_c} is 1-dim^l v any basis vect
(c) $\forall n \geq c$, V^{U_n} has dim^n $n-c+1$ + $\left\{\begin{pmatrix} 1 & 0 \\ 0 & \varpi^a \end{pmatrix} v, \; 0 \leqslant a \leqslant n-c\right\}$, is a basis of V^{U_n}

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§4.2 The Kirillov Model

<u>Lemma</u> (i) Let $W \in \underline{\mathrm{Smo}}_N^{W \neq 0}$, $(N \cong (F,+))$
Then $\exists$ char θ $N \to \mathbb{C}^\times$, and $\psi \in \mathrm{Hom}_N(W, \theta)$, st $\psi \neq 0$
Moreover, for any nonzero $w \in W$, $\exists$ ψ and θ as above st $\psi(w) \neq 0$
(ii) For V as in theorem, $V^N = 0$

<u>Pf</u> (i) suffices to show that $\forall N_0$ open cpct, $\exists \theta$ st
$$\int_{N_0} \theta(n)^{-1} n w \, dn \neq 0$$
But this is easy from character theory of N_0/N_1, for any N_1 fixing w
(ii) see Exercise 4.4.2 in Bump

<u>Thm</u> (Kirillov) for any non-triv char θ of N, $\boxed{\dim \mathrm{Hom}_N(V, \theta) = 1}$
(local multiplicity one thm)
– pfs in Bump or Jacquet–Langlands