$U \subset Gl_2(\mathbb{A}_f)$ open cpct

$$Y(U) = {}_{Gl_2^+(\mathbb{Q})}\backslash^{Gl_2(\mathbb{A}_f)\times\mathcal{H}}/_U$$

$$= {}_{Gl_2(\mathbb{Q})}\backslash^{Gl_2(\mathbb{A}_f)\times(\mathbb{C}-\mathbb{R})}/_U$$

If $g_1 \cdots g_r \in Gl_2(\mathbb{A}_f)$ st $\det(g_1) \cdots \det(g_r)$ are reps for ${}_{\mathbb{Q}_+^\times}\backslash^{\mathbb{A}_f^\times}/_{\det(U)}$, then $Y(U) =$

$\coprod_{i=1}^{r} \Gamma_{g_i}\backslash\mathcal{H}$ where $\Gamma_{g_i} = Gl_2^+(\mathbb{Q}) \cap g_i U g_i^{-1}$

Delicate step: prove that (this union) $\to Y(U)$ is surj, i.e. that

$$Gl_2(\mathbb{A}_f) = \dot{\bigcup_{i=1}^{r}} Gl_2^+(\mathbb{Q})\, g_i U.$$

This comes from strong approx for SL_2.

<u>In particular</u>: if $\det(U) = \hat{\mathbb{Z}}^\times$ then $Y(U) = \Gamma\backslash\mathcal{H}$, $\Gamma = U \cap Gl_2^+(\mathbb{Q})$

but $\exists$ many U with the same Γ in gen^l.

E.g. $U = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N$

$U' = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod N$

$Y(U)$ and $Y(U')$ are both $\Gamma_1(N)\backslash\mathcal{H}$,

but actions of $\left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \hat{\mathbb{Z}}^\times\right\}$ are <u>different</u>.

Number field case

F totally real # fld

$\sigma_1 \cdots \sigma_d$ embeddings $F \hookrightarrow \mathbb{R}$

$GL_2^+(F) := \{ g \in GL_2(F) : \sigma_i(\det g) > 0 \ \forall i \}$

acts on $\prod_{i=1}^{d} \mathcal{H}$, acting via σ_i on ith term

for $U \subset GL_2(\mathbb{A}_{F,f})$ open cpct, set

$$Y(U) = GL_2^+(F) \backslash GL_2(\mathbb{A}_{F,f}) \times \mathcal{H}^d / U$$

(Hilbert modular variety)

components indexed by $\boxed{F_+^\times \backslash \mathbb{A}_{F,f}^\times / \det(U)}$

If F has nontriv^l class gp, this quotient is <u>never</u> triv^l – disconnected quotients are a fact of life!

Chap 6 Modular Forms via Adeles

6.1 Recap of mod forms

For $f : \mathcal{H} \to \mathbb{C}$, $g \in GL_2^+(\mathbb{R})$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$, define

$$(f|_{(k,t)} g)(\tau) = (\det g)^t (c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

<u>Def^n</u> For $\Gamma \leqslant GL_2^+(\mathbb{Q})$ commensurable with $SL_2(\mathbb{Z})$

(i.e. $\Gamma \cap SL_2(\mathbb{Z})$ has finite index in Γ and in $SL_2\mathbb{Z}$)

a modular form of level Γ & wt (k,t) is a fcn $f : \mathcal{H} \to \mathbb{C}$ st

- f is holomorphic
- $f|_{k,t}\,\gamma = f \ \forall \gamma \in \Gamma$
- $(f|_{k,t}\,\gamma)(\tau)$ bounded as $\operatorname{Im}\tau \to \infty$ $\forall \gamma \in GL_2^+(\mathbb{Q})$

If $(f|_{k,t}\,\gamma)(\tau) \to 0 \ \forall \gamma$, say f is a <u>cusp form</u>

Standard fact: $S_{k,t}(\Gamma)$, $M_{k,t}(\Gamma)$ are finite-dim^l + $\exists$ nat^l inner product on $S_{k,t}(\Gamma)$

---

§6.2 Adelic picture

Choose $(k,t) \in \mathbb{Z} \times \mathbb{R}$ as before

<u>Def^n</u> An adelic mod form of wt (k,t) is a fcn

$$F : GL_2(\mathbb{A}_f) \times \mathcal{H} \longrightarrow \mathbb{C}$$

st

(i) $F(g,\tau)$ holo in τ $\forall g$

(ii) $F(gu,\tau) = F(g,\tau)$ $\forall g \in GL_2(\mathbb{A}_f)$, $\tau \in \mathcal{H}$, $u \in U$ for some open cpct U (depending on F)

(iii) $F(\gamma g, -) = F(g,-)|_{k,t}\,\gamma^{-1}$ $\forall \gamma \in GL_2^+(\mathbb{Q})$

(iv) $\forall g \in GL_2(\mathbb{A}_f)$, $F(g,\tau)$ bounded as $\operatorname{Im}(\tau) \to \infty$

If $F(g,\tau) \to 0 \ \forall g$ say F is a cusp form.

This gives spaces $M_{k,t} \supset S_{k,t}$ which are $GL_2(\mathbb{A}_f)$-rep^{n}s.

<u>Fact</u> These are admissible smooth & if $t = k/2$, $S_{k,t}$ is <u>unitarizable</u> (messy computation)

<u>Prop</u> If $U \subset GL_2(\mathbb{A}_f)$ open cpct, $g_1 \cdots g_r$ as before, then

$$(S_{k,t})^U = \bigoplus_{i=1}^{r} S_{k,t}(\Gamma_{g_i})$$

via evaluation at $g_1, \cdots, g_r$

(In particular it's finite-dim^l $\Rightarrow$ admissibility)

Similarly $M_{k,t}$.

In particular, if $U = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$ we <u>recover</u> $S_{k,t}(\Gamma_1(N))$ as inv^{t}s in an adm. smooth rep^n of $GL_2(\mathbb{A}_f)$.

<u>Prop</u> If $U_1(N)$ is above subgp, and ϖ_p = idele which is 1 at all places except p and a unif^{zr} at p, then $\left[U_1(N) \begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix} U_1(N) \right]$ acts on $(M_{k,t})^{U_1(N)}$ as the classical Hecke op $p^{1-t} U_p$ resp. $p^{1-t} T_p$

$$\left[U\begin{pmatrix}\varpi_p & 0\\ 0 & 1\end{pmatrix}U\right] = p^{1-t}\begin{cases}U_p & p \mid N,\\ T_p & p \nmid N\end{cases}$$

Assume $p \mid N$ first.

Then

$$\left[U\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}U\right]f = \sum_{a\in \mathbb{Z}/p}\begin{pmatrix}\varpi & a\\ 0 & 1\end{pmatrix}f$$

evaluate at $(1,\tau)$:

$$\sum_a f\left(\begin{pmatrix}\varpi & a\\ 0 & 1\end{pmatrix},\tau\right)$$

$$= \sum_a f\left(\begin{pmatrix}p & a\\ 0 & 1\end{pmatrix},\tau\right) \quad (\text{by } U\text{-invariance})$$

$$= \sum_a \left[f(1,-)\Big|_{k,t}\begin{pmatrix}p & a\\ 0 & 1\end{pmatrix}^{-1}\right](\tau)$$

$$= \sum_a p^{-t} f\left(1,\frac{\tau-a}{p}\right)$$

$$= p^{1-t}\, U_p(f(1,-))(\tau).$$

Similar if $p \nmid N$ with one extra coset $\begin{pmatrix}1 & 0\\ 0 & \varpi_p\end{pmatrix}$ – massage by multiplying on right by U to get sth. in $GL_2^+(\mathbb{Q})$.

Similarly $\left[U_1(N)\begin{pmatrix}\varpi_p & 0\\ 0 & \varpi_p\end{pmatrix}U_1(N)\right]$, $p \nmid N$, is $p^{k-2t}\langle p\rangle$ (Exercise: what if $p \mid N$?)

– so nebentype character of a classical mod form encodes action of <u>centre</u>.

<u>§6.3</u> <u>Hilbert mod forms</u> (<u>sketch</u>)

F tot real, $GL_2^+(F) \curvearrowright \mathcal{H}^d$ as before.

<u>Defn</u> Hilbert mod forms of wt $(\underline{k},\underline{t}) \in \mathbb{Z}^d \times \mathbb{R}^d$ are fcns

$$f : GL_2(\mathbb{A}_{F,f}) \times \mathcal{H}^d \longrightarrow \mathbb{C}$$

st
- $f(g,\tau)$ holo in each coord of τ $\forall g$
- $f(gu,\tau) = f(g,\tau)$ some U dep. on f
- $f(\gamma g,-) = f(g,-)|_{(\underline{k},\underline{t})}\gamma^{-1}$

[Here $(f|_{(\underline{k},\underline{t})}\gamma)(\tau) = \left[\prod_i \dfrac{\sigma_i(\det\gamma)^{t_i}}{(\sigma_i(c)\tau_i+\sigma_i(d))^{k_i}}\right] \times f(\gamma\tau)$

- (some boundedness condn at cusps)

Again get spaces $S_{(\underline{k},\underline{t})} \subset M_{(\underline{k},\underline{t})}$ which are adm smooth $GL_2(\mathbb{A}_{F,f})$ repns, + U-invts are finite direct sums of spaces of fcns int under subgps $\Gamma_g \subset GL_2^+(F)$

In this world the only sensible defn of $T_{\mathfrak{p}}$, $U_{\mathfrak{p}}$, $\langle\mathfrak{p}\rangle$ etc is the adelic one.

(Again similar picture for non-tot-real F using prod. of copies of $\mathcal{H}$ and $\mathcal{H}_3$)

<u>Prop</u> $S_{(\underline{k},\underline{t})} = M_{(\underline{k},\underline{t})} = 0$ unless $\exists\, r \in \mathbb{R}$ st $k_i - 2t_i = r$ $\forall i$.

<u>Pf</u> Suppose U open cpct. Then

$$\left\{\varepsilon \in \mathcal{O}_F^\times \;\middle|\; \begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix} \in U\right\}$$

has finite index in $\mathcal{O}_F^\times$ & any $f \in (M_{(\underline{k},\underline{t})})^U$ must satisfy

$$f(g,\tau) = f\left(g\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix},\tau\right)$$

$$= f\left(\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}g,\tau\right)$$

$$= \left[f(g,-)\Big|_{(\underline{k},\underline{t})}\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}^{-1}\right](\tau)$$

$$= \left(\prod_i \sigma_i(\varepsilon)^{k_i-2t_i}\right) f(g,\tau).$$

So must have this prod $= 1$ $\forall$ such ε.

Know from Dirichlet unit thm that $(\log|\sigma_1(\varepsilon)|, \dots, \log|\sigma_d(\varepsilon)|)$ for $\varepsilon \in \mathcal{O}_F^\times$ spans a codim 1 subspace of $\mathbb{R}^d$ $\Rightarrow$ $(k_i - 2t_i)_{i=1\dots d}$ lies in a 1-diml space spanned by $(1,\dots,1)$. $\square$

This makes it difficult to choose a "natural" normalizn for t_i's given the k_i's.

Chapter 7 Multiplicity One

§7.1 Restricted tensor products

Recall "most" = "all but finitely many"

F # field

<u>Def^n</u> Suppose we have a collection of vector spaces X_v, v prime of F & vectors $x_v^0 \in X_v$, [nontriv' for most v]

Define $\bigotimes_v{}'(X_v, x_v^0)$ as subspace of $\bigotimes_v X_v$ spanned by tensors $\otimes x_v$ st $x_v = x_v^0$ for most v.

Often drop x_v^0 from notation + write

$$\bigotimes_v{}' X_v$$

<u>Two key examples</u>

- <u>Hecke algs</u>: $X_v = \mathcal{H}(GL_2 F_v)$

  $x_v^0 = e_{K_v}$, $K_v = GL_2(\mathcal{O}_v)$

Then $\bigotimes_v{}'(X_v, x_v^0)$ is $\mathcal{H}(GL_2 \mathbb{A}_{F,f})$.

- <u>Irred rep^{n}s</u>: let Π irred smooth $GL_2(\mathbb{A}_{F,f})$ rep^n.

<u>Flath's Tensor Product Thm</u>: $\exists$ uniquely determined irred rep^{n}s $\Pi_v \circlearrowleft GL_2(F_v)$, $\phi_v^0 \in (\Pi_v)^{K_v}$ (mostly nonzero), st

$$\Pi \cong \bigotimes_v{}'(\Pi_v, \phi_v^0)$$

(This is <u>purely formal</u> – no actual idea.)

In particular, if $\Pi \subset S_{(k,t)}$ irred subrep^n, get smooth irred reps Π_v for every prime v and most Π_v are spherical.

If U is a subgp of form $\underset{GL_2(F_v)}{\underset{\cap}{U_v}} \times \underset{\prod'_{w \neq v} GL_2(F_w)}{\underset{\cap}{U^v}}$

then $\Pi^U \cong$ sum of fin. many copies of $\Pi_v^{U_v}$

as $\mathcal{H}(GL_2 F_v, U_v)$ rep.

$$\mathcal{H}(GL_2 F_v, U_v) \subseteq \mathcal{H}(GL_2(\mathbb{A}_{F,f}), U).$$

---

§7.2 Global Kirillov models

$S_{k,t}$ as before ($F = \mathbb{Q}$ now)

<u>Def^n</u> For $f \in S_{k,t}$, let ϕ_f = fcn on $\mathbb{A}_f^\times$ def by

$$\phi_f(x) = \left(\begin{array}{c}\text{coeff of } e^{2\pi i \tau} \\ \text{in Fourier exp}^n \text{ of} \\ f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \tau\right)\end{array}\right)$$

(note Fourier exp^n can involve $e^{2\pi i a \tau}$ for $a \notin \mathbb{Z}$).

<u>Prop</u>

(i) ϕ_f is a smooth fcn on $\mathbb{A}_f^\times$ supported in a cpct subset of $\mathbb{A}_f$.

(ii) If $n \in \mathbb{Q}_+^\times$, then

$$\phi_f(nx) = a_n\left(f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \tau\right)\right)$$

(iii)

$$\phi_{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f}(x) = \Theta(bx)\, \phi_f(ax)$$

Θ some nontriv char $\mathbb{A}_f \to \mathbb{C}^\times$

(iv) $f \longmapsto \phi_f$ is an injection.