PROBLEM SHEET 2

* Deadline next Friday
* note corrected version now online

<u>Notation</u> For $x \in \mathbb{A}_f$ let $\|x\| = \prod_{\ell \text{ prime}} |x_\ell|_\ell$

<u>Easy check</u>: if $x \in \mathbb{Q}^{\times +}$, $\|x\| = \frac{1}{x}$

$\mathcal{E}_{k,t}$ = orth comp. of $S_{k,t}$ in $M_{k,t}$

$$\alpha : \mathcal{E}_{k,t} \to \mathbb{C}$$
$$f \mapsto f(1,\infty)$$

<u>Prop</u> if $a, d \in \mathbb{Q}^\times$, $ad > 0$, $x \in \mathbb{A}_f$

then $\alpha\left(\begin{pmatrix} a & x \\ 0 & d \end{pmatrix} f\right) = d^k (ad)^{-t} \alpha(f)$

<u>Pf</u>: Clear if $x \in \mathbb{Q}$ also; follows $\forall x$ by Strong approx. $\square$

For χ_1, χ_2 finite-order characters of $\mathbb{A}_f^\times / \mathbb{Q}^{\times +}$, define

$$\alpha_{\chi_1,\chi_2}(f) = \int_{(\mathbb{A}_f^\times/\mathbb{Q}^{\times +})^2} \chi_1(a)^{-1} \chi_2(d)^{-1} \left\|\tfrac{d}{a}\right\|^{k/2} \alpha\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} f\right)$$

(here we take $t = k/2$)

Then
- α_{χ_1,χ_2} is identically 0 unless $\chi_1(-1) = \chi_2(-1) \cdot (-1)^k$
- α_{χ_1,χ_2} is a hom of $B(\mathbb{A}_f)$-rep^{n}s

$$\mathcal{E}_{k,k/2} \longrightarrow \left(\chi_1 \|\cdot\|^{k/2}\right) \boxtimes \left(\chi_2 \|\cdot\|^{-k/2}\right)$$

or eqv^{t}ly a hom of $GL_2(\mathbb{A}_f)$ reps

$$\mathcal{E}_{k,k/2} \longrightarrow \mathrm{Ind}_{B(\mathbb{A}_f)}^{GL_2(\mathbb{A}_f)}(\text{this char})$$
$$= \bigotimes_\ell{}' \underbrace{I\left(|\cdot|_\ell^{(k-1)/2} \chi_{1,\ell},\ |\cdot|_\ell^{(1-k)/2} \chi_{2,\ell}\right)}_{\text{irreducible } \forall \ell \text{ if } k \neq 2}$$

<u>Prop</u>

(i) If $k \geq 3$, then the map

$$\mathcal{E}_{k, k/2} \longrightarrow \bigoplus_{\substack{(\chi_1, \chi_2) \\ \chi_1(-1)\chi_2(-1) = (-1)^k}} \left(\bigotimes_v{}' I(\ldots) \right)$$

is an isomorphism

(ii) If $k=2$, the map is injective, + its image is the kernel of the maps

$$I(\chi \| \|^{\frac{1}{2}}, \chi \| \|^{-\frac{1}{2}}) \longrightarrow (\chi \circ \det)$$

for pairs of form (χ, χ)

<u>Pf</u> Injectivity: if $f \in \mathcal{E}_{k, k/2}$ maps to 0

$\Leftrightarrow \quad \alpha(g f) = 0$

But this says $f(g, \infty) = 0 \ \forall g$

replace g with γg, $\gamma \in GL_2^+(\mathbb{Q})$

$\Rightarrow f(g, \gamma^{-1}\infty) = 0 \ \forall (g, \gamma)$

$\Rightarrow f \in S_{k,t} \cap \mathcal{E}_{k,t} = 0$

Surjectivity: unravels to statement in classical theory about existence of Eis series w. specified values @ cusps.

$\square$

<u>Upshot</u> If $k \geq 3$, $\mathcal{E}_{k,t}$ is a $\bigoplus$ of distinct, irreducible, $\underbrace{\text{generic}}_{\text{(no 1-dim}^{\text{l}}\text{ local factors)}}$ rep$^{\text{n}}$s of $GL_2(\mathbb{A}_f)$

$\Rightarrow$ oldform/newform theory works as it should

If $k=2$, weird stuff happens!

— old + new subspaces not disjoint, etc

<u>Remarks</u>

* Weight 1 works, but need to use <u>unordered</u> pairs of characters
* Can "put back" missing factors in wt 2 by using <u>nearly holomorphic</u> mod forms — "E_2"

Chapter 9: The Galois Action

Notation $\mathbb{Q}_\infty = \mathbb{Q}($all roots of unity$)$

$Gal(\mathbb{Q}_\infty/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times$

$\chi : Gal(\cdot) \longrightarrow \hat{\mathbb{Z}}^\times$

$\chi(\sigma) = a \bmod m \iff \sigma(\zeta) = \zeta^a \ \forall \zeta \in \mu_m$

9.1 Rat'l structure on $M_{k,t}$ & $S_{k,t}$

Let $k, t \in \mathbb{Z}$.

Define $S_{k,t}(\mathbb{Q}_\infty) = \{ f : \phi_f$ is $\mathbb{Q}_\infty$ valued$\}$

$M_{k,t}(\mathbb{Q}_\infty) = \{ f : \phi_f\ \mathbb{Q}_\infty$-valued & $\alpha\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. $\}$

Thm (Shimura): $M_{k,t}(\mathbb{Q}_\infty)$ and $S_{k,t}(\mathbb{Q}_\infty)$ are $GL_2(\mathbb{A}_f)$-stable, + span $M_{k,t}$, $S_{k,t}$ / $\mathbb{C}$

Can't expect ϕ_f to be valued in anything smaller (need values of Θ)

Def^n $S_{k,t}(\mathbb{Q}) = \{ f \in S_{k,t}(\mathbb{Q}_\infty)$ st $\sigma(\phi_f(x)) = \phi_f(\chi(\sigma)x)$ $\forall x \in \mathbb{A}_f^\times, \sigma \in Gal(\mathbb{Q}_\infty/\mathbb{Q}) \}$

$M_{k,t}(\mathbb{Q})$ sim (using α and φ_f)

Thm (also Shimura): $M_{k,t}(\mathbb{Q}), S_{k,t}(\mathbb{Q})$ are $GL_2(\mathbb{A}_f)$-stable and

$S_{k,t} = \mathbb{C} \otimes_{\mathbb{Q}} S_{k,t}(\mathbb{Q})$ (& same for $M_{k,t}$)

Pf 2 approaches:

* Explicit: find gens of $\mathrm{Frac}\left(\bigoplus_{k \geq 0} M_{k,0}\right)$ which have explicit q-exp's

  Cf. Rohrlich's article in Cornell-Silverman-Stevens

* Geometric magic: interpret $Y(U)$ as a moduli space □

Corollary If $U = U_1(N)$, then any $f \in M_{k,t}(\mathbb{Q})^{U}$ has ϕ_f $\mathbb{Q}$-valued

$\left[\begin{pmatrix} \hat{\mathbb{Z}}^\times & 0 \\ 0 & 1 \end{pmatrix} \subset U_1(N) \right]$

NB $\exists$ a second rat'l structure on M_k where Galois acts thru $\begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathbb{Z}}^\times \end{pmatrix}$

- used in Euler system theory (want Siegel units def / $\mathbb{Q}$)

9.2 Canonical Models

$U \subset GL_2(\mathbb{A}_f)$ open cpct

$$Y(U) = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H} / U$$

$$\cap$$

$$X(U) = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times (\mathcal{H} \cup \mathbb{P}^1_{\mathbb{Q}}) / U$$

$X(U)$ is a compact cplx mfld

<u>Fact</u> $X(U) = \mathrm{Proj}\left(\bigoplus_{k \geq 0} M_{k,0}(U) \right)$

$\Rightarrow X(U)$ is ($\mathbb{C}$-pts of) an alg variety $/\mathbb{C}$

<u>Def^n</u> $X_{\mathbb{Q}}(U) = \mathrm{Proj}\left(\bigoplus_{k \geq 0} \underbrace{M_{k,0}(U, \mathbb{Q})}_{U\text{-inv}^t\text{s in } M_{k,0}(\mathbb{Q})} \right)$

<u>canonical model</u> of $X(U)$

<u>Galois action on cusps</u>

$C(U) = X(U) - Y(U)$ is a finite subset of $X(U)$

$$= GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathbb{P}^1_{\mathbb{Q}} / U$$

$$\underbrace{B^+(\mathbb{Q})}_{\mathrm{Stab}(\infty)} \backslash GL_2 \mathbb{A}_f / U$$

<u>Prop</u> $C(U) \subset X_{\mathbb{Q}}(U)(\overline{\mathbb{Q}})$, and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by

$$\sigma \cdot [g] = \left[\begin{pmatrix} \chi(\sigma) & 0 \\ 0 & 1 \end{pmatrix} g \right], \quad g \in GL_2(\mathbb{A}_f)$$

(In particular all cusps are def $/\mathbb{Q}_\infty$)

<u>Pf</u> (sketch): if c is a cusp, then $\sigma(c)$ characterized by

$$f(\sigma(c)) = \sigma(f(c)) \quad \text{whenever } f \in \mathcal{E}_{k,t}(\mathbb{Q})$$

↑ compute this using def^n $\square$

<u>Connected components</u>

$$(\text{Components of } X(U)) \xrightarrow{\sim} \hat{\mathbb{Z}}^\times / \det(U)$$

<u>Prop</u> If $x \in X_{\mathbb{Q}}(U)(\overline{\mathbb{Q}})$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then conn. cpt. of $\sigma(x)$ depends only on conn. cpt. of x, so $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on conn. cpts, + it acts as mult^n by $\chi(\sigma)$ on $\hat{\mathbb{Z}}^\times / \det(U)$.

Hence individual components def. over $(\mathbb{Q}_\infty)^{\chi^{-1}(\det U)}$ (a cyclotomic field)

<u>Pf</u> Take x to be a cusp. $\square$

<u>Other special pts</u>

K imag quad field $\mathbb{Q}(\sqrt{-d})$ $(d > 0)$

Embed $K^\times$ in $GL_2^+(\mathbb{Q})$ via

$$a + b\sqrt{-d} \longmapsto \begin{pmatrix} a & b \\ -db & a \end{pmatrix}$$

fixes $\frac{-1}{\sqrt{-d}} \in \mathcal{H}$.

$\rightsquigarrow$ get a map

$$K^\times \backslash \mathbb{A}^\times_{K,f} \xrightarrow{\iota} GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H}$$

and hence

$$K^\times \backslash \mathbb{A}^\times_{K,f} \longrightarrow Y(U) \quad \text{for any } U.$$

<u>Thm</u> Any point in the image is a $\overline{\mathbb{Q}}$-pt of $X_{\mathbb{Q}}$, defined over an abelian ext^ns of K.

Have an isomorphism

$$\mathrm{Gal}(K^{ab}/K) \xrightarrow[\sim]{\theta} \left(K^\times \backslash \mathbb{A}^\times_{K,f} \right)^\wedge$$

and $\boxed{\sigma \cdot \iota(x) = \iota(\theta(\sigma) x)}$

(Shimura reciprocity)

9.3 Classical viewpoint

$\Gamma < SL_2(\mathbb{Q})$ commensurable w. $SL_2\mathbb{Z}$ + congruence,
so $\Gamma = SL_2(\mathbb{Q}) \cap \overline{\Gamma}$,
$\overline{\Gamma}$ closure in $SL_2(\mathbb{A}_f)$

Thm (Shimura, again):
Let $f \in M_k(\Gamma) \cap \mathbb{Q}_\infty[[q]]$, $\sigma \in Gal(\mathbb{Q}_\infty/\mathbb{Q})$, $\gamma \in GL_2^+(\mathbb{Q})$
Then (i) $f|_\gamma \in M_k(\gamma^{-1}\Gamma\gamma) \cap \mathbb{Q}_\infty[[q]]$
(ii) $f^\sigma = \sum \sigma(a_n(f))\, q^n$ is modular of level
$\Gamma_\sigma = SL_2(\mathbb{Q}) \cap \begin{pmatrix}\chi(\sigma) & 0\\ 0 & 1\end{pmatrix}\overline{\Gamma}\begin{pmatrix}\chi(\sigma) & 0\\ 0 & 1\end{pmatrix}^{-1}$
(Restatement of Shimura's thm from §9.2.)

Corollary $\Gamma \leqslant SL_2\mathbb{Z}$, preimage of some $H \leqslant SL_2(\mathbb{Z}/N)$. Let $A = \{a \in (\mathbb{Z}/N)^\times : \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}$ normalizes $H\}$
$K_\Gamma = \mathbb{Q}(\zeta_N)^{\chi^{-1}(A)}$

Then $S_k(\Gamma)$ has a basis w. q-exp^{n}s in $K_\Gamma[[q]]$
(Easy corollary of (ii))

Eg. $S_k(\Gamma_0(N))$, $S_k(\Gamma_1(N))$, $S_k(\Gamma(N))$ have $\mathbb{Q}$-bases.

$\left(\begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix} \ltimes \overline{\Gamma}\right.$ is open in $\left.GL_2(\mathbb{A}_f)\right)$

This is not canonical, e.g. $SL_2(\mathbb{Z}/N)$ acts on $S_k(\Gamma(N))$ but doesn't preserve $S_k(\Gamma(N), \mathbb{Q})$

Better: $\exists$ action of $GL_2(\mathbb{Z}/N)$ on $S_k(\Gamma(N), \mathbb{Q}(\zeta_N))$ st
* $\gamma f = f|_k \gamma^{-1}$ if $\gamma \in SL_2$
* $\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} f = f^{\sigma_a}$ for $a \in (\mathbb{Z}/N)^\times$

Assume $K_\Gamma = \mathbb{Q}$. Then $X(\Gamma)$ is def/$\mathbb{Q}$

Prop (i) The cusp ∞ is a $\mathbb{Q}$-pt.
(ii) for $\gamma \in SL_2(\mathbb{Z})$, $\gamma\infty$ is def/$\mathbb{Q}(\zeta_N)$
and $\sigma(\gamma\infty) = \gamma'\infty$, $\gamma' = \begin{pmatrix}\chi(\sigma) & 0\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}\chi(\sigma) & 0\\ 0 & 1\end{pmatrix}^{-1}$
(γ' well-def in $\Gamma\backslash SL_2(\mathbb{Z})$)

Caveat These $\mathbb{Q}$-models of curves $X(\Gamma)$ are not canonical, so if $g \in GL_2^+(\mathbb{Q})$,
$g : X(\Gamma) \to X(g\Gamma g^{-1})$
not def/$\mathbb{Q}$ for above models