MA3G1: Theory of PDEs

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Abstract

The important and pervasive role played by pdes in both pure and applied mathematics is described in MA250 Introduction to Partial Differential Equations. In this module I will introduce methods for solving (or at least establishing the existence of a solution!) various types of pdes. Unlike odes, the domain on which a pde is to be solved plays an important role. In the second year course MA250, most pdes were solved on domains with symmetry (eg round disk or square) by using special methods (like separation of variables) which are not applicable on general domains. You will see in this module the essential role that much of the analysis you have been taught in the first two years plays in the general theory of pdes. You will also see how advanced topics in analysis, such as MA3G7 Functional Analysis I, grew out of an abstract formulation of pdes. Topics in this module include:

- Method of characteristics for first order PDEs.
- Fundamental solution of Laplace equation, Green's function.
- Harmonic functions and their properties, including compactness and regularity.
- Comparison and maximum principles.
- The Gaussian heat kernel, diffusion equations.
- Basics of wave equation (time permitting).

Aims: The aim of this course is to introduce students to general questions of existence, uniqueness and properties of solutions to partial differential equations.

Objectives: Students who have successfully taken this module should be aware of several different types of pdes, have a knowledge of some of the methods that are used for discussing existence and uniqueness of solutions to the Dirichlet problem for the Laplacian, have a knowledge of properties of harmonic functions, have a rudimentary knowledge of solutions of parabolic and wave equations.

Further reading. The contents of this course are basic and can be found in many textbooks. The bibliography lists some books for further reading.

Contents

Chapter 1

Introduction

1.1 What are partial differential equations

A *partial differential equation* is an equation which involves partial derivatives of some unknown function. PDEs are often used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics.

Definition 1 (PDE and classical solutions)**.** Suppose that $\Omega \subset \mathbb{R}^d$ is some open set. A *partial differential equation (PDE) of order k* is an equation of the form

$$F\Big[x, u(x), Du(x), D^2u(x), \cdots, D^k u(x)\Big] = 0. \tag{1.1}$$

In the expression above, $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d^2} \times \mathbb{R}^{d^k}$ is a given function; $u : \Omega \to \mathbb{R}$ is the unknown function and $D^k u$ contains all the partial derivatives of u of order k, see Appendix 5.3. A function $u \in C^k(\Omega)$ is said to be a classical solution to the PDE on the domain Ω if on substitution of u and its partial derivatives into (1.1) the relation is identically satisfied on Ω.

Traditionally, PDEs are often derived by using conservation laws and constitutive laws. Another method to derive PDEs, which have been developed recently, is to use operators and functionals. The derivations of basic equations (transport equation, Laplace equation, heat equation and wave equations) can be found in any textbook on the introduction of PDEs (e.g., the the lecture notes of the course MA250 Introduction to Partial Differential Equations). Therefore, we will not go into details on those in this module.

1.2 Examples of PDEs

PDEs are ubiquitous and appear in almost all fields in applied sciences. We list here just some popular examples. Some of these will be treated in the subsequent chapters.

Example 1 (The continuity/transport equation)**.** In general, a continuity equation describes the transport of some moveable quantity. For example in fluid dynamics, the continuity equation reads

$$\frac{\partial u}{\partial t}(x,t) + \operatorname{div}(u(t,x)v(t,x)) = 0, \tag{1.2}$$

where $u(x,t)$ is the fluid density at a point $x \in \mathbb{R}^3$ and at a time t, $v(x,t)$ is the flow velocity vector field and div denotes the divergence operator. Note that the above equation can be written as

$$\frac{\partial u}{\partial t}(x,t) + Du(t,x) \cdot v(t,x) + u(t,x) \operatorname{div}(v(t,x)) = 0,$$

Example 2 (The Laplace and Possion equations)**.** The *Laplace equation* is a second-order partial differential equation given by

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0. \tag{1.3}$$

We often want to solve this equation in some bounded domain $\Omega \subset \mathbb{R}^d$ and therefore, some boundary conditions need to be provided. Now let us give an example of application (or actually a derivation of this equation). We consider a solid body at thermal equilibrium and denotes by $u(x)$ its temperature. Let $\mathbf{Q}$ denotes the heat flux. Since the body is at equilibrium, there is no flux through the boundary $\partial\Omega$ for any smooth domain Ω

$$0 = \int_{\partial\Omega} \mathbf{Q} \cdot d\sigma = \int_{\Omega} \operatorname{div}(\mathbf{Q})\, dx,$$

where we have used the divergence theorem to obtain the second equality. This implies that $\operatorname{div} \mathbf{Q}$ must vanish identically

$$\operatorname{div} \mathbf{Q} = 0.$$

By Fourier's law of heat conduction, the heat flux is proportional to the temperature gradient, so that $\mathbf{Q} = k(x)Du(x)$, where k is the conductivity. If the body is homogeneous, then k is constant and we deduce that

$$0 = \operatorname{div} \mathbf{Q} = \operatorname{div}(kDu(x)) = k\Delta u(x).$$

Thus the temperature satisfies the Laplace equation.

In conclusion: the Laplace equation can be used to describe the temperature distribution of a solid body at thermal equilibrium.

The *Poisson equation* is the Laplace equation with an inhomogeneous right-hand side,

$$\Delta u(x) = f(x), \tag{1.4}$$

for some given function f. For instance, the Poisson equation appears in electrostatics. For a static electric field $\mathbf{E}$, Maxwell's equations reduce to

$$\operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{E} = \rho,$$

where ρ is the charge density (in appropriate units). By Helmholtz decomposition, the first equation implies that $\mathbf{E} = -Du$, for some scalar electric potential field u. By substituting this into the second equation, we obtain the Possion equation (1.4) with $f = -\rho$.

Example 3 (The heat/diffusion equation)**.** In example 2, if the (homogeneous) body is not in equilibrium, the the heat flux through the boundary of a region will equal the rate of change of internal energy of the material inside that region. If no work is done by the heat flow, then the rate of change of the internal energy is

$$\frac{dE}{dt} = \int_\Omega \frac{\partial u}{\partial t}(t,x)\,dx = \int_{\partial\Omega} \mathbf{Q}\cdot d\sigma = \int_\Omega \operatorname{div}(\mathbf{Q})\,dx = k\int_\Omega \Delta u(t,x)\,dx.$$

This implies that the temperature u satisfies the *heat equation*

$$\frac{\partial u}{\partial t}(t,x) = k\Delta u(t,x). \tag{1.5}$$

This equation also can be used to describe other diffusive processes such as a diffusion process (and called diffusion equation).

Example 4 (The wave equation)**.** The wave equation is a second-order linear partial differential equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = c^2\Delta u(t,x). \tag{1.6}$$

Here u is used to model, for example, the mechanical displacement of a wave. The wave equation arises in many fields like acoustics, electromagnetics, and fluid dynamics.

Example 5 (The Kramers equation)**.** The Kramer equation (or kinetic Fokker-Planck equation) describes the time evolution of the probability density of a particle moving under the influence of an external potential, a friction and a stochastic noise given by

$$\partial_t \rho + \frac{p}{m} \cdot \nabla_q \rho - \nabla V \cdot \nabla_q \rho = \gamma \operatorname{div}_p \left(\frac{p}{m} \rho + \beta^{-1} \nabla_p \rho \right),$$

where m is the mass of the particle, γ is the friction coefficient, β^{-1} is the temperature.

Another important equation of the same kind is the Boltzmann equation

$$\partial_t \rho + \frac{p}{m} \cdot \nabla_q \rho - \nabla V \cdot \nabla_q \rho = Q(\rho, \rho)$$

where the right hand side describes the effect of collisions between particles, which is a complicated formula.

Example 6 (A convection and nonlinear diffusion equation)**.** The following equation describes the time evolution of the probability density function, i.e., for each $t \in [0, T]$, $\rho(t)$ is a probability measure on $\mathbb{R}^d$, of some quantity,

$$\partial_t \rho = \operatorname{div} \Big(\rho \nabla [U'(\rho) + V] \Big),$$

where U, V are known as the internal and external energy functionals. In particular, when $U(\rho) = \rho \log \rho$, we obtain the Fokker-Planck equation, which is linear,

$$\partial_t \rho = \Delta \rho + \operatorname{div}(\rho \nabla V).$$

When $V = 0,\ U(\rho) = \frac{1}{m-1} \rho^m$, we obtain the porous medium equation

$$\partial_t \rho = \Delta \rho^m.$$

Example 7 (The Allen-Cahn equation)**.** The Allen-Cahn euqation is a reaction-diffusion equation and describes the process of phase separation

$$\partial_t \rho = D \Delta \rho - f'(\rho),$$

where f is some free energy functional. A typical example of f is $f(\rho) = \frac{1}{4}\rho^4 - \frac{1}{2}\rho^2$.

Example 8 (The Cahn-Hilliard equation)**.** A very closely related equation to the Allen-Cahn equation is the Cahn-Hilliard equation which is obtained by taking minus the Laplacian of the right-hand side of the former.

$$\partial_t \rho = -D \Delta^2 u + \Delta f'(\rho).$$

1.3 Well-posedness of a PDE

In all of the examples in the previous section, we need further information in order to solve the problem. We broadly refer to this information as the *data* for the problem.

Ex. 1 We can specify the density distribution at some initial time t_0. This is an example of *Cauchy* data (or an initial value problem).

Ex. 2 For the Laplace equation on a bounded domain, we need to prescribe the value of u on the boundary of the domain. This is an example of *Dirichlet* data (a Dirichlet problem). For the Poisson equation, we specify the function f and also the value of u on the boundary.

Ex. 3 For the heat equation, we specify the initial temperature of the body, as well as some boundary condition. For example, a Dirichlet boundary condition when the temperature at the boundary is prescribed or a Neumann boundary condition when the flux through the boundary is imposed.

An important aspect of the study of a PDE is about its *wellposedness*. A PDE problem, consisting of a PDE together with some data is well-posed if

a) There exists a solution to the problem (existence).

b) For given data, the solution is unique (uniqueness).

c) The solution depends continuously on the data.

These are called Hadamard's conditions. A problem is said to be *ill-posed* if any of the condition above fails to hold.

Notice that existence and uniqueness involves boundary conditions. While continuous dependence depends on considered metric/norm.

Wellposedness is important because for the vast majority of PDE problems that we encounter, it is not possible to write down a solution explicitly. However, if well-posedness is satisfied, we can often deduce properties of that solution directly from the PDE it satisfies without ever needing an explicit solution.

While the issue of existence and uniqueness of solutions of ordinary differential equations has a very satisfactory answer with the Picard-Lindelöf theorem, that is far from

the case for partial differential equations. In general, the answers to the above questions depend heavily on the PDE itself as well as the domain Ω. Definition 1.1 is often too general to provide any useful information. In this notes, we will be working with much simpler examples of PDEs having explicit forms, and nevertheless exhibit lots of interesting behaviour.

Example 9. Consider the following heat equation on $(0, 1)$.

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x). \end{cases}$$

This PDE is well-posed.

Example 10. Consider the following backwards heat equation

$$\begin{cases} u_t = -u_{xx} \\ u(0,t) = u(1,t), \\ u(x,0) = u_0(x). \end{cases}$$

This PDE is ill-posed.

In general, it is not straightforward to see whether a PDE is well-posed or not. More knowledge from subsequent chapters is needed.

1.4 Classification of PDEs

There are several ways to categorise PDEs. Following is a possibility.

Definition 2. i) We say that (1.1) is *linear* if F is a linear function of u and its derivatives, so that we can re-write (1.1) as (see Appendix 5.3 for the multi-index notation)

$$\sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x^\alpha} = f(x).$$

If $f = 0$, we say the equation is *homogeneous*.

ii) We say that (1.1) is *semilinear* if it is of the form

$$\sum_{|\alpha| = k} a_\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x^\alpha} + a_0\Big[x, u(x), Du(x), D^2u(x), \cdots, D^{k-1}u(x)\Big] = 0,$$

so that the highest order derivatives of u appear linearly, with coefficients depending only on x but not on u and its derivatives.

iii) We say that (1.1) is *quasilinear* if it is of the form

$$\sum_{|\alpha|=k} a_\alpha \Big[x, u(x), Du(x), D^2u(x), \cdots, D^{k-1}u(x)\Big] \frac{\partial^{|\alpha|}u}{\partial x^\alpha} + a_0 \Big[x, u(x), Du(x), D^2u(x), \cdots, D^{k-1}u(x)\Big] = 0,$$

so that the highest order derivatives of u appear linearly, with coefficients possibly depending only on the lower order derivatives of u.

iv) We say that (1.1) is *fully nonlinear* if is not linear, semilinear or quasilinear.

Exercise 1. Classify equations in the examples in Section 1.2.

Chapter 2

Quasilinear first order PDEs

In this chapter, we will study quasilinear first order PDEs. We will be able to apply the method of characteristic to solve these equations.

2.1 Introduction

For simplicity we will consider functions defined on a subset of $\mathbb{R}^2$. However, the methods in this chapter are applicable for functions defined on higher dimensional sets. In $\mathbb{R}^2$, a quasilinear first order PDE has the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{2.1}$$

where $a(x, y, u), b(x, y, u)$ and $c(x, y, u)$ are coefficients; u_x and u_y denote the partial derivatives of u with respect to x and y respectively. We often want to solve the above equation in some bounded domain $\Omega \subset \mathbb{R}^2$.

Example 11. Let us start with the most simplest quasilinear first order PDE. Let $\Omega \subset \mathbb{R}^2$, we consider the following equation

$$\frac{\partial u}{\partial x}(x, y) = 0 \quad \text{in} \quad \Omega.$$

Solutions to this equation depend on the domain Ω. For instance, if $\Omega = [0, 1] \times [0, 1]$ then the value of u on $\{0\} \times [0, 1]$ will determine u everywhere in Ω. In fact, suppose that $u(0, y) = h(y)$ with h continuous on $[0, 1]$, then using the fundamental theorem of calculus we have,

$$u(x, y) = u(0, y) + \int_0^x \frac{\partial u}{\partial s}(s, y)\, ds = h(y).$$

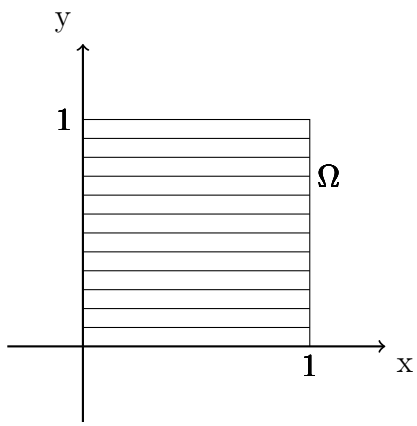

Figure 2.1: The domain $\Omega = [0,1] \times [0,1]$

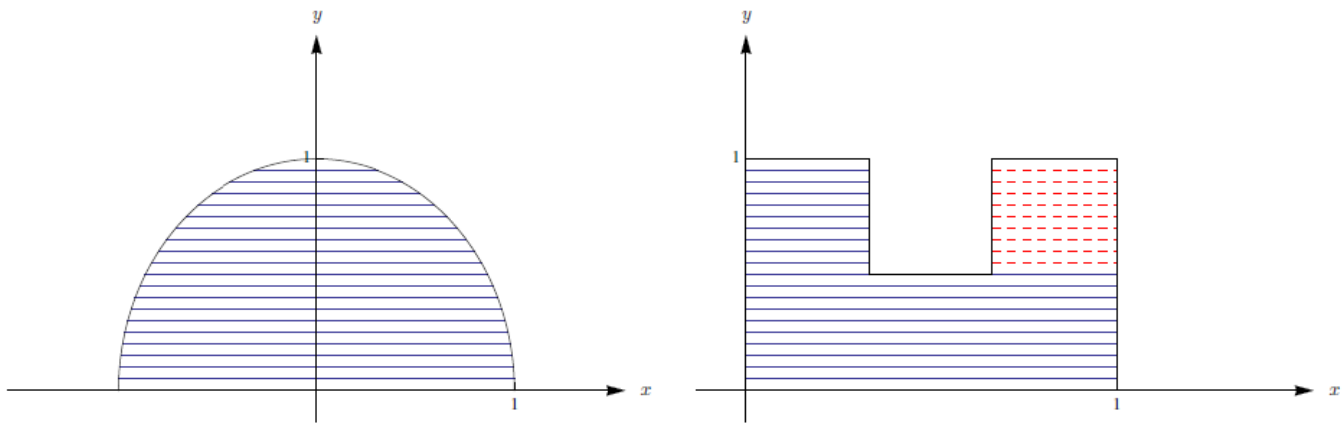

Figure 2.2: [L] A domain on which $u_x = 0$ is solvable with data on $\{0\} \times [0,1]$. [R] A domain on which $u_x = 0$ is not solvable with data on $\{0\} \times [0,1]$.

This means that u does not change if we move along a curve of constant y and we say that the value of u propagates along lines of constant y (Figure 2.1).

How about other domains? When is prescribing the value of u on $\{0\} \times [0,1]$ enough to determine u everywhere? Obviously for this approach to work, we must be able to connect every point in Ω to $\{0\} \times [0,1]$ by a horizontal line which lies in Ω. See Figure 2.1 for different situations. Note that while for a classical solution of a first order equation we would require $u \in C^1(\Omega)$, the procedure above in fact makes sense even if $h(y)$ is only continuous, so that u_y need not to be C^0. This suggests that we should consider a notion of solutions for some PDEs which is weaker than classical solutions.

2.2 The method of characteristic

Let us take another example.

Example 12 (Transport equation with constant speed)**.** Let $c \in \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be a given function. Consider the following PDE

$$cu_x + u_y = 0, \quad u(x, 0) = h(x). \tag{2.2}$$

Solution. The solution consists of three main steps

Step 1) Change the co-ordinate system $(x, y) \rightsquigarrow (s, t)$ by writing

$$\begin{cases} x = x(s, t) \\ y = y(s, t) \end{cases}$$

and define $z(s, t) := u(x(s, t), y(s, t))$.

Step 2) Solve the equation for z in the new co-ordinate system (s, t), which by construction will be easier to solve.

Step 3) Transform back to the original co-ordinate system

$$\begin{cases} s = s(x, y) \\ t = t(x, y) \end{cases}$$

and $u(x, y) = z(s(x, y), t(x, y))$.

By the chain rule, we have

$$\frac{dz}{dt} = \frac{dx}{dt} u_x + \frac{dy}{dt} u_y.$$

We want that the right-hand side of the expression above is equal to $cu_x + u_y$ that appears in the PDE. Therefore, we obtain the following system of ODEs

$$\frac{dx}{dt} = c$$
$$\frac{dy}{dt} = 1.$$

Solving this system, we obtain $x(s, t) = ct + A(s), \quad y(s, t) = t + B(s)$. To find $A(s), B(s)$ we impose that

$$(t = 0) \;\Rightarrow\; (y = 0), \quad (t = 0) \;\Rightarrow\; (x = s),$$

from which we deduce that $A(s) = s, B(s) = 0$, hence, $x(s,t) = ct + s$ and $y(s,t) = t$. In the new co-ordinate system, we now have

$$\frac{dz}{dt} = 0, \quad z(s,0) = h(s).$$

Solving this equation gives

$$z(s,t) = h(s).$$

We now transform back to (x,y) and $u(x,y)$. It is straightforward to find s,t in terms of x,y

$$s = x - cy, \quad t = y.$$

Therefore, $u(x,y) = z(s(x,y),t(x,y)) = h(x-cy)$. This is the solution of the transport equation with constant speed.

Some observations:

1) To establish the new co-ordinate system, we need to solve a system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= c, \quad x(s,0) = s, \\ \frac{dy}{dt} &= 1, \quad y(s,0) = 0, \\ \frac{dz}{dt} &= 0, \quad z(s,0) = h(s). \end{aligned}$$

These are called *the characteristic equations* associated to the PDE, and $x(s,t), y(s,t), z(s,t)$ are called the characteristic curves.

2) To transform back to the original co-ordinate system, we need to have the correspondence

$$\begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases} \rightsquigarrow \begin{cases} s = s(x,y) \\ t = t(x,y). \end{cases}$$

3) If $x - cy = k$, then $u(x,y) = u(k)$. In other words, u is constant along each line $x - cy = k$.

2.2.1 The characteristic equations

In the general case

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \quad u(f(x),g(x)) = h(x), \tag{2.3}$$

where f, g, h are given functions. The method of characteristics to solve the above equation above consists of three steps.

Step 1) Solve the following characteristic equations, which is a system of ODEs, for x, y, z as functions of s, t

$$\begin{aligned} \frac{dx}{dt} &= a(x,y,z), \quad x(s,0) = f(s), \\ \frac{dy}{dt} &= b(x,y,z), \quad y(s,0) = g(s), \\ \frac{dz}{dt} &= c(x,y,z), \quad z(s,0) = h(s). \end{aligned}$$

Step 2) Find the transformation

$$\begin{cases} s = s(x,y) \\ t = t(x,y). \end{cases}$$

Step 3) Find the solution $u(x,y) = z(s(x,y), t(x,y))$.

The characteristic projection is defined by

$$\sigma_s(t) = \big\{x(s,t), y(s,t)\big\}.$$

A shock is formed when two characteristic projections meet and assign different values of u, i.e., there exist $(s,t), (s',t')$ such that

$$x(s,t) = x(s',t'), \quad y(s,t) = y(s',t'),$$

but $z(s,t) \neq z(s',t')$.

2.2.2 Geometrical interpretation

We now give a geometrical interpretation (derivation) for the method of characteristics. Suppose that we are given a curve $\gamma(s) = (f(s), g(s)), s \in (\alpha, \beta)$. We will find a function u satisfying

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

such that for $s \in (a,b)$, we have $[u \circ \gamma](s) = u(f(s), g(s)) = h(s)$ for some given function $h : (\alpha, \beta) \to \mathbb{R}$. In other words, we specify the value of u along the curve γ.

The idea we shall pursue is to find the graph of u over some domain Ω by showing that we can foliate it by curves along which the value of u is determined by ODEs. Recall that the graph of u over Ω is the surface in $\mathbb{R}^3$ given by

$$\text{Graph}(u) = \{x, y, u(x, y) : (x, y) \in \Omega\}.$$

We know from the initial conditions that the curve Γ, given by $(f(s), g(s), h(s)), s \in (\alpha, \beta)$ must belong to Graph(u). We want to write

$$\bigcup_{s\in(\alpha,\beta)} C_s \subset \text{Graph}(u),$$

where

$$C_s = \{(x(s,t), y(s,t), z(s,t)) : t \in (-\varepsilon_s, \varepsilon_s)\}, \quad \varepsilon_s > 0,$$

with $x(s, 0) = f(s), y(s, 0) = g(s)$ and $z(s, 0) = h(s)$. In other words, through each line of the curve $(f(s), g(s), h(s))$ we would like to find another curve which lies in the graph of u.

Since the tangent vector to C_s, given by $\mathbf{t} := \left(\frac{dx}{dt}(s,t), \frac{dy}{dt}(s,t), \frac{dz}{dt}(s,t)\right)$ lies in the surface Graph(u), it must be orthogonal to any vector which is normal to Graph(u). Since the map

$$(x, y) \to \mathbf{u}(x, y) := (x, y, u(x, y))$$

is a parameterisation of the surface Graph(u), a vector normal to Graph(u) can be found by

$$\begin{aligned}
\mathbf{N} &:= \mathbf{u}_x \wedge \mathbf{u}_y \\
&= \begin{pmatrix} 1 \\ 0 \\ u_x \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ u_y \end{pmatrix} \\
&= \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}.
\end{aligned}$$

Therefore, C_s lies in the surface Graph(u) if and only if $\mathbf{t} \cdot \mathbb{N} = 0$,i.e.,

$$\frac{dx}{dt}u_x + \frac{dy}{dt}u_y - \frac{dz}{dt} = 0,$$

should hold at each point on C_s. Recall that u satisfies the PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u).$$

By comparing the two relationships above, we see that if $x(t;s), y(t;s), z(t;s)$ is the unique solution of the systems of ODEs

$$\begin{aligned} \frac{dx}{dt} &= a(x,y,z), \quad x(s,0) = f(s) \\ \frac{dy}{dt} &= b(x,y,z), \quad y(s,0) = g(s) \\ \frac{dz}{dt} &= c(x,y,z), \quad z(s,0) = h(s) \end{aligned} \tag{2.4}$$

then the curve C_s will lie in Graph(u). The Picard-Lindelöf theorem states that if a, b, c are sufficiently well behaved, then there exists a unique solution to (2.4) on some interval $t \in (-\varepsilon_s, \varepsilon_s)$ and that moreover the solution depends continuously on s. Later we will state more precisely these conditions on a, b, c such that we can solve (2.4).

The equation (2.4) are known as the *characteristic equations* and the corresponding solution curves $(x(t;s), y(t;s), z(t;s))$ are the *characteristic curves* or simply *characteristics*. The union of the characteristics forms a surface

$$\bigcup_{s\in(\alpha,\beta)} C_s = \{(x(s,t), y(s,t), z(s,t)) : t \in (-\varepsilon_s, \varepsilon_s), s \in (\alpha, \beta)\}$$

which is a portion of the graph of u that includes the curve Γ. We want to write this as a graph of the form $\{(x, y, u(x,y)) : (x,y) \in \Omega\}$. We can do this, provided we can invert the map $(s,t) \mapsto (x(s,t), y(s,t))$ to give $s(x,y), t(x,y)$. Then we have $u(x,y) = z(s(x,y); t(x,y))$.

Example 13. Solve the PDE

$$u_x + xu_y = u, \quad u(2,x) = h(x).$$

Solution. The characteristic equations are

$$\begin{aligned} \frac{dx}{dt} &= 1, \quad x(s,0) = 2, \\ \frac{dy}{dt} &= x, \quad y(s,0) = s, \\ \frac{dz}{dt} &= z, \quad z(s,0) = h(s). \end{aligned}$$

The first equation gives $x(s,t) = t+2$. Substituting this into the second equation we get $y(s,t) = \frac{t^2}{2} + 2t + s$. Finally, from the last equation, we have $z(s,t) = e^t h(s)$. By writing s, t in terms of x, y, we find

$$s = y - \frac{(x-2)^2}{2} - 2(x-2), \quad t = x - 2.$$

Therefore, $u(x,y) = z(s,t) = e^{x-2} h\left(y - \frac{(x-2)^2}{2} - 2(x-2)\right)$.

Note that the method of characteristic does not require that the PDEs are linear. We consider the following quasilinear equation.

Example 14 (Burger's equation)**.** Solve the following Burger's equation

$$uu_x + u_y = 0, \quad u(x,0) = h(x), \quad x, y \in \mathbb{R}, y \geq 0.$$

Solution. The characteristic equations reduce to

$$\begin{aligned} \frac{dx}{dt} &= z, \quad x(s,0) = s, \\ \frac{dy}{dt} &= 1, \quad y(s,0) = 0, \\ \frac{dz}{dt} &= 0, \quad z(s,0) = h(s). \end{aligned}$$

The last two equations are easily solved to get

$$y(s,t) = t, \quad z(s,t) = h(s).$$

Substituting these back to the first equation we get

$$x(s,t) = s + h(s)t.$$

In this case, we can not explicitly invert the relationship $(t,s) \mapsto (x,y)$. However, we have

$$s(x,y) = x - h(s(x,y))y, \quad t(x,y) = y.$$

Therefore,

$$u(x,y) = z(s(x,y), t(x,y)) = h(s(x,y)) = h(x - h(s(x,y))y) = h(x - u(x,y)y),$$

i.e., u satisfies an implicit equation

$$u(x,y) = h(x - u(x,y)y).$$

In a particular case when $h(x) = \sqrt{x^2+1}$, we have

$$u = \sqrt{(x-uy)^2+1},$$

which implies that

$$u(x,y) = \frac{xy - \sqrt{x^2-y^2+1}}{y^2-1}.$$

Note that this solution is only valid for $|y| < 1$. At this point, u blows up. To explore the blow-up phenomena in more generality, let us return to the characteristic equations for the Burger's equation

$$x(s,t) = h(s)t + s, \quad y(s,t) = t, \quad z(s,t) = h(s).$$

At fixed s the characteristic is simply a straight line parallel to the $x-y$ plane

$$y = \frac{x-s}{h(s)}, \quad z = h(s).$$

This means that $u(x,y) = z$ remains constant on each of the line $y = \frac{1}{h(s)}(x-s)$. The meeting point of two characteristic projections is determined by

$$\frac{x-s_1}{h(s_1)} = \frac{x-s_2}{h(s_2)} = \frac{s_2-s_1}{h(s_1)-h(s_2)}.$$

If h is increasing, then $\frac{s_2-s_1}{h(s_1)-h(s_2)} \leq 0$. In this case, the lines of constant u are "fanning out" and do not meet on $y \geq 0$.

If h is decreasing, then $\frac{s_2-s_1}{h(s_1)-h(s_2)} \geq 0$. The lines of constant u must meet on $y \geq 0$. At such a meeting point, the solution obtained by the method of characteristics is no longer admissible since the two characteristic projections are trying to assign two different values of u. This continuity is called a shock.

2.3 Compatibility of initial conditions

We recall that the initial condition is given by

$$u(f(x), g(x)) = h(x).$$

Differentiating this equation with respect to x, we find

$$f'(x)u_x(f(x),g(x)) + g'(x)u_y(f(x),g(x)) = h'(x). \tag{2.5}$$

On the other hand, from the PDE we have

$$a(f(x), g(x), h(x))u_x(f(x), g(x)) + b(f(x), g(x), h(x))u_y(f(x), g(x)) = c(f(x), g(x), h(x)). \tag{2.6}$$

Equations (3.9) and (2.6) forms a linear system to determine $u_x(f(x), g(x))$ and $u_y(f(x), g(x))$.

- If $a(f(x), g(x), h(x))g'(x) - b(f(x), g(x), h(x))f'(x) \neq 0$, then this system has a unique solution. In this case, we expect that the method of characteristics will work. We say that the initial conditions are *compatible* with the PDE.
- If $a(f(x), g(x), h(x))g'(x) - b(f(x), g(x), h(x))f'(x) = 0$, the initial conditions are incompatible with the PDE or they are *redundant*.

2.4 Local existence and uniqueness

Recall that we want to solve the PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad \text{on} \quad \Omega,$$

with initial data given along a curve $\gamma : x \mapsto (f(x), g(x)) \in \Omega$ such that $u(f(x), g(x)) = h(x)$. We will show that the compatibility of the initial conditions is the only obstruction (besides standard assumptions) to finding a solution in a neighbourhood of γ. Suppose that:

- $a, b, c \in C^1(\Omega \times \mathbb{R})$,
- $f, g, h \in C^1((\alpha, \beta))$ for some interval (α, β).
- The initial conditions are compatible with the PDE, i.e., $a(f(x), g(x), h(x))g'(x) - b(f(x), g(x), h(x))f'(x) \neq 0$ for all $x \in (\alpha, \beta)$.

The the characteristic equations

$$\begin{aligned}
\frac{dx}{dt} &= a(x, y, z), \quad x(s, 0) = f(s), \\
\frac{dy}{dt} &= b(x, y, z), \quad y(s, 0) = g(s), \\
\frac{dz}{dt} &= c(x, y, z), \quad z(s, 0) = h(s).
\end{aligned}$$

can be solved uniquely for each $s \in (\alpha, \beta)$ and for $t \in (-\varepsilon_s, \varepsilon_s)$, and the solution will be of class C^1. To find the solution in terms of (x, y) one now to invert the map $(s, t) \mapsto (x, y)$. Let's define $\Psi(s, t) = (x(s, t), y(s, t))$. Then Ψ is of class C^1 in a neighbourhood of γ and

$$D\Psi(s, 0) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} (s, 0) = \begin{pmatrix} f'(s) & a(f(s), g(s), h(s)) \\ g'(s) & b(f(s), g(s), h(s)) \end{pmatrix}.$$

According to the compatibility conditions, we have

$$\det D\Psi(s, 0) = a(f(s), g(s), h(s))g'(s) - b(f(s), g(s), h(s))f'(s) \neq 0, \quad \text{for all} \quad s \in (\alpha, \beta),$$

so that $D\Psi(s, 0)$ is invertible. By the inverse function theorem, Ψ^{-1} exists in a neighbourhood of $(f(s), g(s))$. Furthermore, we have

$$\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} (\Psi(s, t)) = (D\Psi^{-1})(\Psi(s, t)) = (D\Psi(s, t))^{-1} = \frac{1}{b\frac{\partial x}{\partial s} - a\frac{\partial y}{\partial s}} \begin{pmatrix} b & -a \\ -\frac{\partial y}{\partial s} & \frac{\partial x}{\partial s} \end{pmatrix}$$

By the chain rule (remembering that $u(x, y) = z(s(x, y), t(x, y))$)

$$\begin{aligned} u_x &= \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} \\ u_y &= \frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y} \end{aligned}$$

or equivalently, in a matrix form

$$\begin{pmatrix} u_x & u_y \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \frac{1}{b\frac{\partial x}{\partial s} - a\frac{\partial y}{\partial s}} \begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} \begin{pmatrix} b & -a \\ -\frac{\partial y}{\partial s} & \frac{\partial x}{\partial s} \end{pmatrix}$$

Hence

$$\begin{aligned} au_x + bu_y &= \begin{pmatrix} u_x & u_y \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{1}{b\frac{\partial x}{\partial s} - a\frac{\partial y}{\partial s}} \begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} \begin{pmatrix} b & -a \\ -\frac{\partial y}{\partial s} & \frac{\partial x}{\partial s} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{1}{b\frac{\partial x}{\partial s} - a\frac{\partial y}{\partial s}} \begin{pmatrix} \frac{\partial z}{\partial s} & c \end{pmatrix} \begin{pmatrix} 0 \\ b\frac{\partial x}{\partial s} - a\frac{\partial y}{\partial s} \end{pmatrix} \\ &= c, \end{aligned}$$

so the equation is indeed satisfied. Uniqueness follows from the fact that the characteristic through a point (x_0, y_0, z_0) is uniquely determined and moreover lies completely in

Graph(u) if (x_0, y_0, z_0) lies in Graph(u). Thus any solution would have to include all of the characteristics through $(f(s), g(s), h(s))$ and hence would have to locally agree with the solution we construct above.

We summarise this section by the following theorem

Theorem 1. *Suppose that $a, b, c \in C^1(\Omega \times \mathbb{R})$; $f, g, h \in C^1((\alpha, \beta))$; and that $s \mapsto \gamma(s) = (f(s), g(s))$ is injective. Suppose further that $a(f(x), g(x), h(x))g'(x) - b(f(x), g(x), h(x))f'(x) \neq 0$ for all $x \in (\alpha, \beta)$. Then, given $K \subset (\alpha, \beta)$ a closed (and hence compact) interval, there exists a neighbourhood U of $\gamma(K)$ such that the equation*

$$au_x + bu_y = c$$

has a unique solution u of class C^1 such that $u(\gamma(s)) = h(s)$.

Problem sheet 1

Exercise 2. Solve the problem

$$u_x + u_y = u, \quad u(x,0) = \cos x$$

Exercise 3. Solve the problem

$$u_x + xu_y = y, \quad u(0,y) = \cos y$$

Exercise 4. Consider the Burger's equation

$$uu_x + u_y = 0, \quad u(x,0) = h(x), \quad x, y \in \mathbb{R}, \quad y \geq 0$$

Suppose that $h \in C^1(\mathbb{R})$, h bounded and decreasing, $h' < 0$, and assume that h' is bounded. Show that no shock forms in the region

$$y < \frac{1}{\sup_{\mathbb{R}} |h'|},$$

and that for any $y > \frac{1}{\sup_{\mathbb{R}} |h'|}$ a shock has formed for some x.

Exercise 5. a) Solve the problem

$$u_x + 3x^2 u_y = 1, \quad u(x,0) = h(x), \quad (x,y) \in \mathbb{R}^2,$$

for $h \in C^1(\mathbb{R})$.

b) Show and explain why the problem

$$u_x + 3x^2 u_y = 1, \quad u(s,s^3) = 1, \quad (x,y) \in \mathbb{R}^2$$

has no solution $u \in C^1(\mathbb{R}^2)$.

c) Show and explain why the problem

$$u_x + 3x^2 u_y = 1, \quad u(s,s^3) = s - 1, \quad (x,y) \in \mathbb{R}^2$$

has infinitely many solutions $u \in C^1(\mathbb{R}^2)$.

Chapter 2

Laplace's equation and harmonic functions

In this chapter, we will study the Laplace's equation, which is an important equation both in physics and mathematics. Solutions of the Laplace's equation are called harmonic functions. We will study some important, both qualitative and quantitative, properties of harmonic functions such as the mean value property, smoothness and estimates of the derivatives, the maximum principle and Harnack's inequality.

2.1 Laplace's Equation

Let $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ be a C^2 function. The Laplacian operator is defined by

$$\Delta u(x) := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x).$$

The Laplace's equation is

$$\Delta u(x) = 0, \quad \text{for all} \quad x \in \Omega. \tag{2.1}$$

Remark 1 (Relations between the Laplacian, the gradient and the divergence operators)**.** Recall that the gradient of u, ∇u (or Du) is a vector field given by

$$\nabla u(x) = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T,$$

and that if $\mathbf{X} = (X_1, \ldots, X_n)^T$ is a vector field then the divergence of $\mathbf{X}$ is

$$\operatorname{div}(\mathbf{X}) = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}.$$

We have the following relation between the Laplacian, the gradient and the divergence operators

$$\Delta u = \operatorname{div}(\nabla u).$$

This relation will be used throughout for example when applying the divergence theorem.

□

Example 15. 1) If $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$ and $u(x) = \mathbf{a} \cdot \mathbf{x} + b$ (i.e., u is affine) then $\Delta u(x) = 0$.

2) If $u(x) = \sum_{i,j=1}^n a_{ij} x_i x_j$, with $a_{ij} = a_{ji}$ (i.e., the matrix $a = (a_{ij})_{i,j=1,\dots,n}$ is symmetric), then

$$(\nabla u)_i = 2 \sum_{i=1}^n a_{ij} x_j, \quad \text{and} \quad \Delta u(x) = 2 \sum_{i=1}^n a_{ii} = 2 \operatorname{Tr}(a).$$

So if a is trace-free (i.e., $\operatorname{Tr}(a) = 0$) then $\Delta u = 0$.

Example 16 (Laplacian operator of a radial function)**.** Suppose $u(x) = u(|x|) = u(r)$, where $r = |x|$. Then we have

$$\Delta u(|x|) = \Delta u(r) = u''(r) + \frac{n-1}{r} u'(r).$$

We will provide two proofs for this assertion. The first one is based on direct computations, the second one is hinged on a property that the Laplacian operator is rotation invariant.

The first proof (direct computations). Since $r = |x| = \sqrt{\sum_{i=1}^n x_i^2}$, we have $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$. By the chain rule we have

$$\begin{aligned} u_{x_i} &= \frac{du}{dr} \frac{\partial r}{\partial x_i} = u'(r) \frac{x_i}{r}. \\ u_{x_i x_i} &= u''(r) \frac{x_i}{r} \frac{x_i}{r} + u'(r) \frac{\partial}{\partial x_i} \left(\frac{x_i}{r} \right). \\ &= u''(r) \frac{x_i^2}{r^2} + u'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right). \end{aligned}$$

Therefore, recalling that $r^2 = \sum_{i=1}^n x_i^2$,

$$\begin{aligned} \Delta u(x) &= \sum_{i=1}^n u_{x_i x_i} = \sum_{i=1}^n \left[u''(r) \frac{x_i^2}{r^2} + u'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \right] \\ &= u''(r) + \frac{n-1}{r} u'(r). \end{aligned}$$

□

The second proof.

First observation: *the Laplacian is rotation invariant.* Let Q be a constant $n \times n$ matrix satisfying $QQ^T = Q^TQ = I$. Let $v(x) = u(Qx)$. By the chain rule, we have

$$v_{x_i}(x) = \sum_{j=1}^{n} u_{x_j}(Qx)Q_{ji},$$

and similarly

$$v_{x_ix_i} = \sum_{j,k=1}^{n} u_{x_jx_k}(Qx)Q_{ji}Q_{ki} = (Q^THQ)_{ii},$$

where $H = (\text{Hess}u)(Qx)$. Therefore

$$\Delta v(x) = \sum_{i=1}^{n} v_{x_ix_i} = \sum_{i=1}^{n}(Q^THQ)_{ii} = \text{Tr}(Q^THQ) = \text{Tr}(Q^TQH) = \text{Tr}(H) = \Delta u(Qx).$$

Here we use the fact that Q is orthogonal and the community of the trace $Tr(AB) = Tr(BA)$.

Note further that since Q is orthogonal, $|x|^2 = \langle x, x\rangle = \langle Qx, Qx\rangle = |Qx|^2$, it follows that $|x| = |Qx|$.

Now if u is radial, i.e., $u(x) = u(|x|) = u(|Qx|) = u(Qx)$, then $v(x) = u(Qx) = u(x)$, so that $\Delta v(x) = \Delta u(Qx) = \Delta u(x)$.

Second observation: By the divergence theorem, we have

$$\int_{\partial B_r} \nabla u \cdot n \, d\sigma = \int_{B_r} \text{div}(\nabla u)\, dx = \int_{B_r} \Delta u(x)\, dx. \tag{2.2}$$

Since u is radial, $\nabla u(x)\cdot n = u'(r)\frac{x}{|x|}\cdot n = u'(r)$. The first integral in (2.2) can be computed as

$$\int_{\partial B_r} \nabla u \cdot n \, d\sigma = u'(r)\int_{\partial B_r} d\sigma = u'(r)\sigma_{n-1}r^{n-1}, \tag{2.3}$$

where σ_{n-1} is the area of the unit $(n-1)-$sphere.

Now using the fact that $\Delta u(x)$ is radial, we can write the last integral in (2.2) as a radial integral

$$\int_{B_r} \Delta u(x)\, dx = \int_0^r \sigma_{n-1}s^{n-1}\Delta u(s)\, ds. \tag{2.4}$$

From (2.3) and (2.4) we obtain that for any $r_1 < r_2$

$$u'(r_2)r_2^{n-1} - u'(r_1)r_1^{n-1} = \int_{r_1}^{r_2} s^{n-1}\Delta u(s)\, ds.$$

This equality implies that $r^{n-1}\Delta u(r)$ is simply the derivative of $r^{n-1}u'(r)$, thus

$$\Delta u(r) = \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}u'(r)\right) = u''(r) + \frac{n-1}{r}u'(r).$$

□

2.2 Harmonic functions

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be open. A function $u \in C^2(\Omega)$ is said to be *harmonic* in Ω if

$$\Delta u(x) = 0$$

for all $x \in \Omega$. In other words, harmonic functions solve Laplace's equation.

Example 17. 1) We saw from Example 15 that affine functions $u(x) = \mathbf{a} \cdot x + b$, for $\mathbf{a} \in \mathbb{R}^n, b \in R$, and $u(x) = x_1^2 + \ldots + x_{n-1}^2 - (n-1)x_n^2$ are harmonic in $\mathbb{R}^n$.

2) In polar coordinates, the function $r^k \sin(k\theta)$ is harmonic in $\mathbb{R}^2$ and the function $r^{-k} \sin(k\theta)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ for $k \in \mathbb{N}$.

3) The function $e^x \sin y$ is harmonic in $\mathbb{R}^2$.

When Ω is a domain in $\mathbb{R}^2$, an important class of harmonic functions come from *holomorphic functions*.

Example 18. Suppose $\Omega \subset \mathbb{R}^n \simeq \mathbb{C}$ and that $f(z)$ is holomorphic in Ω, i.e., the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in \Omega$. Suppose that $f(x+iy) = u(x,y) + iv(x,y)$. Then u, v are harmonic function in Ω. In fact, by the Cauchy-Riemann equations, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

Therefore, $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$. Similarly, $\Delta v(x,y) = 0$.

2.3 Mean value property for harmonic functions

Let us motivate this section by looking at a one-dimensional example. Suppose that $u : (a,b) \to \mathbb{R}$ is harmonic, i.e., $u''(x) = 0$ in (a,b). It follows that $u(x) = cx + d$ for some constants $c, d \in \mathbb{R}$. Let $x_0 \in (a,b)$ and $r > 0$ is sufficiently small such that $[x_0 - r, x_0 + r] \subset (a,b)$. Let us compute

$$\begin{aligned} \frac{1}{2}[u(x_0 - r) + u(x_0 + r)] &= \frac{1}{2}[c(x_0 - r) + d + c(x_0 + r) + d] \\ &= cx_0 + d, \end{aligned}$$

and

$$\begin{aligned}\frac{1}{2r}\int_{x_0-r}^{x_0+r} u(y)\,dy &= \frac{1}{2r}\int_{x_0-r}^{x_0+r} (cy+d)\,dy \\ &= \frac{1}{2r}\left(c\frac{y^2}{2}+dy\right)\Big|_{x_0-r}^{x_0+r} \\ &= cx_0+d.\end{aligned}$$

Therefore,

$$u(x_0) = \frac{1}{2}[u(x_0-r)+u(x_0+r)] = \frac{1}{2r}\int_{x_0-r}^{x_0+r} u(y)\,dy.$$

This property holds true for harmonic function in any dimensional space.

Theorem 2 (Mean value property)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is open and that $u \in C^2(\Omega)$ is harmonic. Then if $\overline{B_r(x_0)} \subset \Omega$ then*

$$u(x_0) = \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} u(y)\,d\sigma(y) \tag{2.5}$$

$$= \frac{1}{|(B_r(x_0))|}\int_{B_r(x_0)} u(y)\,dy. \tag{2.6}$$

Proof. We first prove (2.5). We perform two changes of variables, successively setting $y = x_0 + z$ and $z = r\zeta$, which shifts $B_r(x_0)$ to be centred at the origin and then scales it to have unit radius. We obtain

$$\int_{\partial B_r(x_0)} u(y)\,d\sigma(y) = \int_{\partial B_r(0)} u(x_0+z)\,d\sigma(z) = r^{n-1}\int_{\partial B_1(0)} u(x_0+r\zeta)\,d\sigma(\zeta)$$

Therefore, recalling that $|\partial B_r(x_0)| = r^{n-1}\sigma_{n-1}$, where σ_{n-1} is the area of the $(n-1)$-unit sphere,

$$\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} u(y)\,d\sigma(y) = \frac{1}{\sigma_{n-1}}\int_{\partial B_1(0)} u(x_0+r\zeta)\,d\sigma(\zeta). \tag{2.7}$$

Let $F(r)$ be defined as the right-hand side of the above equality. Let $v(\zeta) := u(x_0+r\zeta)$, then $\nabla v(\zeta) = r\nabla u(x_0+r\zeta)$ and $\Delta v(z) = r^2\Delta u(x_0+r\zeta) = 0$. We will show, by computing

the derivative of $F(r)$, that $F(r)$ is indeed a constant function.

$$\begin{aligned}
\frac{dF(r)}{dr} &= \frac{1}{\sigma_{n-1}}\frac{d}{dr}\int_{\partial B_1(0)} u(x_0+r\zeta)\,d\sigma(\zeta) \\
&= \frac{1}{\sigma_{n-1}}\int_{\partial B_1(0)} \frac{d}{dr}u(x_0+r\zeta)\,d\sigma(\zeta) \\
&= \frac{1}{\sigma_{n-1}}\int_{\partial B_1(0)} \zeta\cdot\nabla u(x_0+r\zeta)\,d\sigma(\zeta) \\
&= \frac{1}{\sigma_{n-1}}\frac{1}{r}\int_{\partial B_1(0)} \zeta\cdot\nabla v(\zeta)\,d\sigma(\zeta) \\
&\overset{(*)}{=} \frac{1}{\sigma_{n-1}}\frac{1}{r}\int_{B_1(0)} \Delta v(\zeta)\,d\zeta \\
&= 0,
\end{aligned}$$

where we have used the divergence theorem to obtain the equality $(*)$ above. Hence by the continuity of u, we have

$$F(r) = \lim_{r\to 0} F(r) = \frac{1}{\sigma_{n-1}}\int_{\partial B_1(0)} u(x_0)\,d\sigma(\zeta) = u(x_0). \tag{2.8}$$

From (2.7) and (2.8), we obtain (2.5).

The equality (2.6) follows from (2.5). Indeed, we have

$$\begin{aligned}
\int_{B_r(x_0)} u(y)\,dy &= \int_0^r ds \int_{\partial B_s(x_0)} u(y)\,d\sigma(y) = \int_0^r |\partial B_s(x_0)| u(x_0)\,ds \\
&= u(x_0)\int_0^r |\partial B_s(x_0)|\,ds = u(x_0)|B_r(x_0)|
\end{aligned}$$

□

Remark 2. The converse results are also true, namely that if $u \in C(\Omega)$ satisfies the mean value property (either (2.5) or (2.6)) then $u \in C^2(\Omega)$ and u is harmonic in Ω. Note that it requires only that $u \in C(\Omega)$. See Exercises. □

2.4 Smoothness and estimates on derivatives of harmonic functions

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be open and let u be harmonic in Ω. Then $u \in C^\infty(\Omega)$ and furthermore*

$$|D^\alpha u(x)| \le \left(\frac{n|\alpha|}{d(x)}\right)^{|\alpha|} \sup_{y\in\Omega} |u(y)|, \tag{2.9}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Proof. We first prove that $u \in C^\infty(\Omega)$. Let ϕ be satisfy the following property

(i) $\phi \in C^\infty(\mathbb{R}^n)$,

(ii) ϕ is radial,

(iii) ϕ is supported in B_r for some $r > 0$,

(iv) $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$.

We will show that $u(x) = (u * \phi)(x)$, where the convolution is defined by

$$(u * \phi)(x) = \int_{\mathbb{R}^n} u(y)\phi(x-y)\, dy.$$

Without loss of generality, set $x = 0$, and we take r small enough so that $B_r \subset \Omega$. Now we have

$$\begin{aligned}
(u * \phi)(0) &= \int_{\mathbb{R}^n} u(y)\phi(-y)\, dy \\
&= \int_{B_r} u(y)\phi(-y)\, dy \\
&= \int_{B_r} u(y)\phi(y)\, dy \\
&= \int_0^r \left(\int_{\partial B_s} u(y)\, dS(y) \right) \phi(s)\, ds \\
&= \int_0^r |\partial B_s| u(0)\phi(s)\, ds \\
&= u(0) \int_0^r |\partial B_s| \phi(s)\, ds \\
&= u(0) \int_0^r \int_{\partial B_s} \phi(s) dS\, ds \\
&= u(0) \int_{B_r} \phi(y)\, dy \\
&= u(0).
\end{aligned}$$

So $u(x) = (u * \phi)(x)$. Since $u * \phi \in C^\infty(\Omega)$, it follows that $u \in C^\infty(\Omega)$ as claimed.

Step 1. Next we will prove the estimate (2.9) by induction on $|\alpha|$. For $|\alpha| = 1$, we need to prove that

$$|D_i u(x)| \leq \frac{n}{d(x)} \sup_{y \in \Omega} |u(y)|. \tag{2.10}$$

Take $r < d(x)$ then $\overline{B_r(x)} \subset \Omega$. According to the mean value property

$$D_i u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} D_i u(y)\, dy = \frac{1}{|B_r(x)|} \int_{\partial B_r(x)} u(y) n_i(y)\, dS(y),$$

Since $|n_i| = 1$, we can estimate

$$|D_i u(x)| \leq \frac{|\partial B_r(x)|}{|B_r(x)|} \sup_{y \in B_r(x)} |u(y)| = \frac{r^{n-1}\sigma_{n-1}}{r^n \omega_n} \sup_{y \in B_r(x)} |u(y)| = \frac{n}{r} \sup_{y \in B_r(x)} |u(y)|.$$

Since this is true for any $r < d(x)$ and $\overline{B_r(x)} \subset \Omega$, it follows that

$$|D_i u(x)| \leq \frac{n}{d(x)} \sup_{y \in \Omega} |u(y)|.$$

Step 2. Suppose that the statement is true for $|\alpha| \leq N$.

Step 3. We need to prove that it is also true for $|\alpha| = N + 1$. Write $D^\alpha u = D_i D^\beta u$, where β is a multi-index such that $|\beta| = |\alpha| - 1 = N$. Now we fix some $\lambda \in (0, 1)$. We apply the technique in Step 1 to $r = \lambda d(x)$, to obtain

$$|D^\alpha u(x)| = |D_i D^\beta u(x)| \leq \frac{n}{\lambda d(x)} \sup_{y \in B_{\lambda d(x)}(x)} |D^\beta u(y)|.$$

By the triangle inequality if $y \in B_{\lambda d(x)}(x)$, then $d(y) \geq (1 - y)d(x)$, so applying the induction assumption we have

$$|D^\beta(y)| \leq \left(\frac{n|\beta|}{d(y)}\right)^{|\beta|} \sup_{z \in \Omega} |u(z)| \leq \left(\frac{n|\beta|}{(1-\lambda)d(x)}\right)^{|\beta|} \sup_{z \in \Omega} |u(z)|,$$

which implies that

$$\sup_{y \in B_{\lambda d(x)}} |u(y)| \leq \left(\frac{n|\beta|}{(1-\lambda)d(x)}\right)^{|\beta|} \sup_{z \in \Omega} |u(z)|,$$

and that (recalling that $|\alpha| = |\beta| + 1$)

$$|D^\alpha u(x)| \leq \frac{n}{\lambda d(x)} \left(\frac{n|\beta|}{(1-\lambda)d(x)}\right)^{|\beta|} \sup_{z \in \Omega} |u(z)| = \left[\left(\frac{n}{d(x)}\right)^{|\alpha|} |\beta|^{|\beta|} \sup_{z \in \Omega} |u(z)|\right] \frac{1}{\lambda(1-\lambda)^{|\beta|}}. \tag{2.11}$$

To obtain the sharpest upper bound, we now minimizes the function $\lambda \mapsto \frac{1}{\lambda(1-\lambda)^{|\beta|}}$, which is equivalent to maximises the function

$$\lambda \mapsto g(\lambda) := \lambda(1-\lambda)^{|\beta|},$$

to obtain the optimal λ. The optimal λ satisfies the equation

$$\begin{aligned} 0 = \frac{dg(\lambda)}{d\lambda} &= (1-\lambda)^{|\beta|} - \lambda|\beta|(1-\lambda)^{|\beta|-1} \\ &= (1-\lambda)^{|\beta|-1}[1 - \lambda - |\beta|\lambda] = (1-\lambda)^{|\beta|-1})(1 - |\alpha|\lambda), \end{aligned}$$

which implies that $\lambda = \frac{1}{|\alpha|}$, which is in the interval $(0,1)$ as required. Substituting this value to $g(\lambda)$, we get (recalling that $|\alpha| = |\beta| + 1$)

$$g(\lambda)\Big|_{\lambda=\frac{1}{|\alpha|}} = \frac{1}{|\alpha|}\left(1 - \frac{1}{|\alpha|}\right)^{|\beta|} = \frac{|\beta|^{|\beta|}}{|\alpha|^{|\alpha|}}$$

Putting this back into (2.11) we obtain

$$|D^\alpha u(x)| \leq \frac{n^{|\alpha|}|\alpha|^{|\alpha|}}{d(x)^{|\alpha|}} \sup_\Omega |u|,$$

which is the reduction assumption for $|\alpha| = N + 1$. Hence the result holds true for any N. □

Proposition 1 has two interesting consequences in the following theorems.

Theorem 3. *Harmonic functions are analytic.*

Proof. Let $x_0, x \in \Omega$ and suppose that the segment $[x_0, x] = tx + (1-t)x_0, t \in [0,1]$ is contained in Ω. Without loss of generality we assume that $x_0 = 0$. Let u be a harmonic function in Ω. By Taylor's theorem we have

$$u(x) = P_{m-1}(x) + R_m(x),$$

where $P_{m-1}(x)$ is a polynomial of degree $m-1$ and

$$R_m(x) = \sum_{|\alpha|=m} \frac{D^\alpha u(\xi)}{\alpha!} x^\alpha,$$

for some $\xi \in [x_0, x]$. To show that u is analytic we need to show that $R_m(x) \to 0$ as $m \to \infty$ uniformly on some small disc around x_0. By Proposition 1, we have

$$\begin{aligned} |R_m(x)| &\leq \left(\frac{nm}{d(\xi)}\right)^m \sup_\Omega |u| \sum_{|\alpha|=m} \frac{|x^\alpha|}{\alpha!} \\ &= \frac{1}{m!}\left(\frac{nm}{d(\xi)}\right)^m \sup_\Omega |u| \sum_{|\alpha|=m} \frac{m!}{\alpha_1! \dots \alpha_n!} |x_1|^{\alpha_1} \dots |x_n|^{\alpha_n}. \end{aligned}$$

Set $||x||_1 := \sum_{i=1}^n |x_i|$, then by the multinomial theorem we have

$$||x||_1^m = \sum_{|\alpha|=m} \frac{m!}{\alpha_1! \dots \alpha_n!} |x_1|^{\alpha_1} \dots |x_n|^{\alpha_n}.$$

Therefore, we obtain that

$$|R_m(x)| \leq \frac{||x||_1^m}{m!}\left(\frac{nm}{d(\xi)}\right)^m \sup_\Omega |u|. \tag{2.12}$$

We now use a weak version of the Stirling's approximation, which holds true for all positive integer m,

$$\frac{m^m}{m!} \leq e^m.$$

We get

$$|R_m(x)| \leq \left(\frac{ne||x||_1}{d(\xi)}\right)^m \sup_\Omega |u|.$$

Taking $|x| = |x - x_0| < \frac{1}{2}d(x_0)$, by the triangular inequality, we have $d(\xi) \leq \frac{1}{2}d(x_0)$, so that

$$|R_m(x)| \leq \left(\frac{2ne||x||_1}{d(x_0)}\right)^m \sup_\Omega |u|. \tag{2.13}$$

Finally, if we take x such that $||x||_1 < \frac{d(x_0)}{2ne}$, then $\frac{2ne||x||_1}{d(x_0)} < 1$ and $\left(\frac{2ne||x||_1}{d(x_0)}\right)^m \to 0$ as $m \to \infty$, that implies that $|R_m(x)| \to 0$ as $m \to \infty$. □

Theorem 4 (Liouville)**.** *Suppose u is a bounded harmonic function in $\mathbb{R}^n$, then u is a constant.*

Proof. Suppose $|u| \leq C$ in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $r > 0$. Since u is harmonic in $B_r(x)$, applying Proposition 1 in $B_r(x)$, we have

$$|D_i u(x)| \leq \frac{n}{r} \sup_{B_r(x)} |u| \leq \frac{nC}{r}.$$

Since r is arbitrary and C is independent of r, the above estimate implies that $|\nabla u(x)| = 0$. So u is constant. □

2.5 The maximum principle

2.5.1 Subharmonic and superharmonic functions

Definition 4. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$. We say u is *subharmonic* (respectively *superhamornic*) in Ω iff $\Delta u \geq 0$ (respectively $\Delta u \leq 0$).

Note that a function $u \in C^2(\Omega)$ is harmonic if and only if it is both subharmonic and superharmonic.

Example 19. 1) In one dimension, u is subharmonic iff $u'' > 0$, which means that u' is increasing and u is convex. Similarly u is superharmonic iff u is concave.

2) If $u = x^T Ax$ for some constant, symmetric matrix A. Then $\Delta u = 2\mathrm{Tr}(A)$, so u is subharmonic (respectively, superharmonic) iff $\mathrm{Tr}(A) \geq 0$ (respectively, $\mathrm{Tr}(A) \leq 0$).

2.5.2 Some properties of subharmonic and superharmonic functions

Recall that harmonic functions satisfy the mean value property, which means that for any ball $\overline{B_r(x_0)} \subset \Omega$, we have

$$u(x_0) = \frac{1}{|\partial B_r x_0|} \int_{\partial B_r(x_0)} u(y) d\sigma(y) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(y)\, dy.$$

Recall that to prove this in Theorem 2, we considered the function $F(r) = \frac{1}{|\partial B_r x_0|} \int_{\partial B_r(x_0)} u(y) d\sigma(y)$ and showed

(i) $F(r) \to u(x_0)$ as $r \to 0$. This relied on the continuity of u,

(ii) For all $r > 0$ we have

$$F'(r) = \frac{1}{r\sigma_{n-1}} \int_{B_1(0)} \Delta u(x_0 + r\zeta)\, d\zeta. \tag{2.14}$$

It follows from (2.14) that if u is subharmonic (respectively, superharmonic) then $F'(r) \geq 0$ (respectively, $F'(r) \leq 0$).

Proposition 2. *If u is subharmonic in Ω and $\overline{B_r(x_0)} \subset \Omega$, then*

$$u(x_0) \leq \frac{1}{|\partial B_r x_0|} \int_{\partial B_r(x_0)} u(y) d\sigma(y) \quad \textit{and} \quad u(x_0) \leq \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(y)\, dy. \tag{2.15}$$

If u is superharmonic then the reverse inequalities hold.

It follows from this Proposition and Theorem 2 that the value of a subharmonic (respectively, superharmonic) function at the center of a ball is less (respectively, greater) than or equal to the value of a harmonic function with the same values on the boundary. Thus, the graphs of subharmonic functions lie below the graphs of harmonic functions and the graphs of superharmonic functions lie above, which explains the terminology.

2.5.3 The maximum principle

An important feature of subharmonic functions is the following theorem.

Theorem 5 (Weak maximum principle)**.** *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be subharmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u. \tag{2.16}$$

If, rather, u is superharmonic in Ω then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{2.17}$$

Proof. Suppose u is subharmonic. We consider two cases.

Case 1: $\Delta u > 0$ in Ω. Suppose that there exists $x_0 \in \Omega$ such that u attains a local maximum at x_0, then we must have $\nabla u(x_0) = 0$ and $\operatorname{Hess} u(x_0)$ is negative definite, i.e., $\xi^T \operatorname{Hess} u(x_0)\xi \leq 0$ for all $\xi \in \mathbb{R}^n, \xi \neq 0$. It implies that

$$\Delta u(x_0) = \operatorname{Tr}(\operatorname{Hess} u(x_0)) = \sum_{i=1}^{n} e_i^T \operatorname{Hess} u(x_0) e_i \leq 0,$$

which is a contradiction. Therefore, u can not have maxima in the interior. Thus, the maximum of u in $\overline{\Omega}$ must be achieved on the boundary Ω, i.e., (2.16) holds.

Case 2: $\Delta u \geq 0$ in Ω. Define $u_\varepsilon(x) := u(x) + \varepsilon|x|^2$. Then $\Delta u_\varepsilon(x) = \Delta u(x) + 2\varepsilon n > 0$, so that we can apply Case 1 above for $u_\varepsilon(x)$ and obtain

$$\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon.$$

Since Ω is bounded, $|x|^2$ is a bounded function in Ω, and $u_\varepsilon \to u$ uniformly on $\bar{\Omega}$ as $\varepsilon \to 0$. It follows that

$$\max_{\overline{\Omega}} u = \lim_{\varepsilon \to 0} \max_{\overline{\Omega}} u_\varepsilon = \lim_{\varepsilon \to 0} \max_{\partial\Omega} u_\varepsilon = \max_{\partial\Omega} u.$$

The statement (2.17) for superhamornic function is obtained by applying (2.16) for $-u$, which is subharmonic, and using the fact that $\max_A(-u) = -\min_A u$. □

Since a harmonic function is both subharmonic and superhamornic, we obtain the following corollary.

Corollary 1. *1) Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is harmonic. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \quad \textit{and} \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

This implies

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

2) (The comparison principle) If $\Omega \subset \mathbb{R}^n$ is open and bounded and if $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying $\Delta u \geq \Delta v$ in Ω and $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.

The comparison follows by applying the weak maximum principle to the function $w := u - v$, which is subharmonic and satisfies $w \leq 0$ on $\partial\Omega$, hence $w \leq 0$ (i.e, $u \leq v$) in Ω.

Note that the weak maximum principle states that the that the maximum of a subharmonic function is to be found on the boundary, but may re-occur in the interior as well. The following theorem provides a stronger statement.

Theorem 6 (Strong maximum principle)**.** *Let Ω be open and connected (possibly unbounded), and let u be subharmonic in Ω. If u attains a global maximum value in Ω, then u is constant in Ω.*

Proof. Define $M := \max_{\Omega} u$, and $A := u^{-1}\{M\} = \{x \in \Omega : u(x) = M\}$. By the assumption of the theorem, A is non-empty. Since u is continuous and $\{M\}$ is a closed set in R, it follows that A is relatively closed in Ω (i.e., there exists a closed $F \subset \mathbb{R}^n$ such that $A = F \cap \Omega$). We now show that A is also open. Indeed, let $x \in \Omega$ and let r be sufficiently small that $\overline{B_r(x)} \subset \Omega$. By the mean value property for subharmonic functions in Proposition 2, we have

$$0 = u(x) - M \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} (u(y) - M)\, dy.$$

Since $M = \max_{\Omega} u$, we have $u(y) - M \leq 0$ for all $y \in B_r(x)$. This implies that

$$0 = u(x) - M \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} (u(y) - M)\, dy \leq 0.$$

Since u is continuous, this happens only if $u = M$ in $B_r(x)$. So that $B_r(x) \subset A$, which implies that A is open. Since Ω is connected and $A \neq \emptyset$ is both open and closed, it follows that $A = \Omega$, and hence $u \equiv M$ in Ω. This finishes the proof. $\square$

Again, since a harmonic function is both subharmonic and superharmonic, we get a stronger result.

Corollary 2 (Strong maximum principle for harmonic functions)**.** *Let Ω be open and connected and $u : \Omega \to \mathbb{R}$ a harmonic function. Then if u attains either a global maximum or global minimum in Ω then u is constant in Ω.*

2.6 Harnack's inequality

The next result we shall derive is Harnack's inequality for harmonic functions. To state this theorem, we need to define the notion of a compactly contained subset. Let Ω, Ω' be two open subsets of $\mathbb{R}^n$, then we say that Ω' is compactly contained in Ω and write $\Omega' \Subset \Omega$ if there exists a compact set K such that $\Omega' \subset K \subset \Omega$.

Theorem 7 (Harnack's inequality for harmonic functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $u : \Omega \to \mathbb{R}$ be a non-negative harmonic function. Then for any connected $\Omega' \Subset \Omega$ there exists a constant C depending only on Ω, Ω', n such that*

$$\sup_{\Omega'} \leq C \inf_{\Omega'} u.$$

Proof. Let $r > 0$ satisfy that $\inf_{\Omega'} d(x) > 4r$ where $d(x) = \mathrm{dist}(\mathrm{x}, \partial\Omega)$. Let $x, y \in \Omega'$ be such that $|x - y| \leq r$. Applying the mean value property for u we have

$$u(x) = \frac{1}{|B_{2r}(x)|} \int_{B_{2r}(x)} u(z)\, dz = \frac{1}{2^n r^n \omega_n} \int_{B_{2r}(x)} u(z)\, dz \geq \frac{1}{2^n r^n \omega_n} \int_{B_r(y)} u(z)\, dz,$$

where $\omega_n = |B_1(0)|$ and we have used that $u \geq 0$ and $B_r(y) \subset B_{2r}(x)$ to obtain the above inequality. Now using the mean value property again, we have

$$\frac{1}{2^n r^n \omega_n} \int_{B_r(y)} u(z)\, dz = \frac{1}{2^n} \frac{1}{|B_r(y)|} \int_{B_r(y)} u(z)\, dz = \frac{1}{2^n} u(y).$$

It follows from these two inequalities that

$$\frac{1}{2^n} u(y) \leq u(x).$$

By interchanging x and y, we obtain $u(x) \leq 2^n u(y)$, so that for all $|x - y| \leq r$ we have

$$\frac{1}{2^n} u(y) \leq u(x) \leq 2^n u(y). \tag{2.18}$$

Now we will expand this estimate to any two points x_0, x_1 in Ω'. Since Ω' is open and connected, it is path connected. Hence there exists a curve $\gamma : [0, 1] \to \Omega'$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Since $\Omega' \subset K$ for some compact subset K, we can cover Ω' with at most N open balls of radius r, where N depends only on Ω, Ω'. From these balls, we select $M \leq N$ balls $B_i, i = 1, \ldots, M$ such that

$$x_0 \in B_1, \quad x_1 \in B_M, \quad \gamma([0, 1]) \subset \bigcup_{i=1}^{M} B_i,$$

and $B_i \cap B_{i+1} \neq \emptyset, \quad i = 1, \ldots, M-1.$

Applying the estimate (2.18) successively on these balls, we get

$$u(x_1) \leq 2^{Mn} u(x_0) \leq 2^{Nn} u(x_0).$$

Since this inequality holds for any two points $x_0, x_1 \in \Omega'$, it follows that

$$\sup_{\Omega'} u \leq 2^{nN} \inf_{\Omega'} u,$$

which is the claim. □

Harnack's inequality is used to prove Harnack's theorem about the convergence of sequences of harmonic functions. Harnack's inequality can also be used to show the interior regularity of weak solutions of partial differential equations.

Corollary 3. *Let Ω be connected and let u_k be a sequence of harmonic functions in Ω such that $u_k \leq u_{k+1}$ for all k. If there exists $x_0 \in \Omega$ such that $u_k(x_0)$ converges (to a finite value) then u_k converges to some harmonic function u uniformly on the compact sets of Ω.*

Problem Sheet 2

Exercise 6 (Laplacian operator in the cylindrical and spherical coordinates)**.**

1) Consider the cylindrical coordinates in $\mathbb{R}^3$ defined by

$$\begin{cases} x_1(r,\varphi,z) = r\cos\varphi, \\ x_2(r,\varphi,z) = r\sin\varphi, \qquad r\in[0,\infty),\ \varphi\in[0,2\pi),\ z\in\mathbb{R}, \\ x_3(r,\varphi,z) = z. \end{cases}$$

Prove that in these coordinates

$$\Delta u(r,\varphi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2}.$$

2) Consider the spherical coordinates in $\mathbb{R}^3$ defined by

$$\begin{cases} x_1(r,\theta,\varphi) = r\sin\theta\cos\varphi, \\ x_2(r,\theta,\varphi) = r\sin\theta\sin\varphi, \qquad r\in[0,\infty),\ \theta\in[0,\pi),\ \varphi\in[0,2\pi), \\ x_3(r,\varphi,z) = z. \end{cases}$$

Prove that in these coordinates

$$\Delta u(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right).$$

[**Hint: write gradients and use integration by parts**]

3) Show that if u, v are C^2 functions then

$$\Delta(uv)(x) = v(x)\Delta u(x) + u(x)\Delta v(x) + 2\nabla u(x)\cdot\nabla v(x).$$

Exercise 7.

1) Find the general form of radial harmonic functions in $\mathbb{R}^n \setminus \{0\}$. Which of these extend to $\mathbb{R}^n$ as smooth functions?

2) Find all radial solutions of
$$\Delta u(x) = \frac{1}{(1+|x|^2)^2}$$
in $\mathbb{R}^2 \setminus \{0\}$. Which of these solutions extend smoothly to all of $\mathbb{R}^2$ (i.e., near the origin).

3) Let (z, φ) be polar coordinates in $\mathbb{R}^2$, and consider the set $\Omega = \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$. Show that the functions $\log r$, φ are harmonic in Ω.

Exercise 8 (Converse of the mean value property of harmonic functions in 1D)**.**

1) Suppose $u : (a, b) \to \mathbb{R}$ is continuous and satisfies
$$u(x_0) = \frac{1}{2}[u(x_0 - r) + u(x_0 + r)], \quad \forall x_0, r \text{ such that } [x_0 - r, x_0 + r] \subset (a, b).$$
Show that u is harmonic (i.e., affine).

2) Prove the same if $u : (a, b) \to \mathbb{R}$ is continuous and satisfies
$$u(x_0) = \frac{1}{2r} \int_{x_0 - r}^{x_0 + r} u(x)\, dx, \quad \forall x_0, r \text{ such that } [x_0 - r, x_0 + r] \subset (a, b).$$

Exercise 9. 1) Prove that if $u \in C^2(\Omega)$ satisfies the mean value property (either in the spherical or bulk form), then $\Delta u = 0$ in Ω.

2) Suppose $\Omega \subset \mathbb{R}^n$ is open and such that $\Omega^0 = \Omega \cap \{x_n = 0\} \neq \emptyset$. Let $\Omega^+ := \Omega \cap \{x_n > 0\}$, and define $\tilde{\Omega} = \Omega^+ \cup \Omega^0 \cup \Omega^-$, where
$$\Omega^- = \{x = (x', x_n) \in \mathbb{R}^n \quad \text{such that} \quad (x, -x_n) \in \Omega^+\}.$$
Suppose that $u \in C^2(\overline{\Omega^+})$ is such that $\frac{\partial u}{\partial x_n} = 0$ on Ω^0, and define $\tilde{u} : \tilde{\Omega} \to \mathbb{R}$ by
$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega^+ \cup \Omega^0, \\ u(x', -x_n), & x = (x', x_n) \in \Omega^-. \end{cases}$$
Prove that $\tilde{u} \in C^2(\tilde{\Omega})$ and that $\tilde{u}$ is harmonic in $\tilde{\Omega}$. See the figure below for illustration of the sets $\Omega, \Omega^0, \Omega^\pm$.

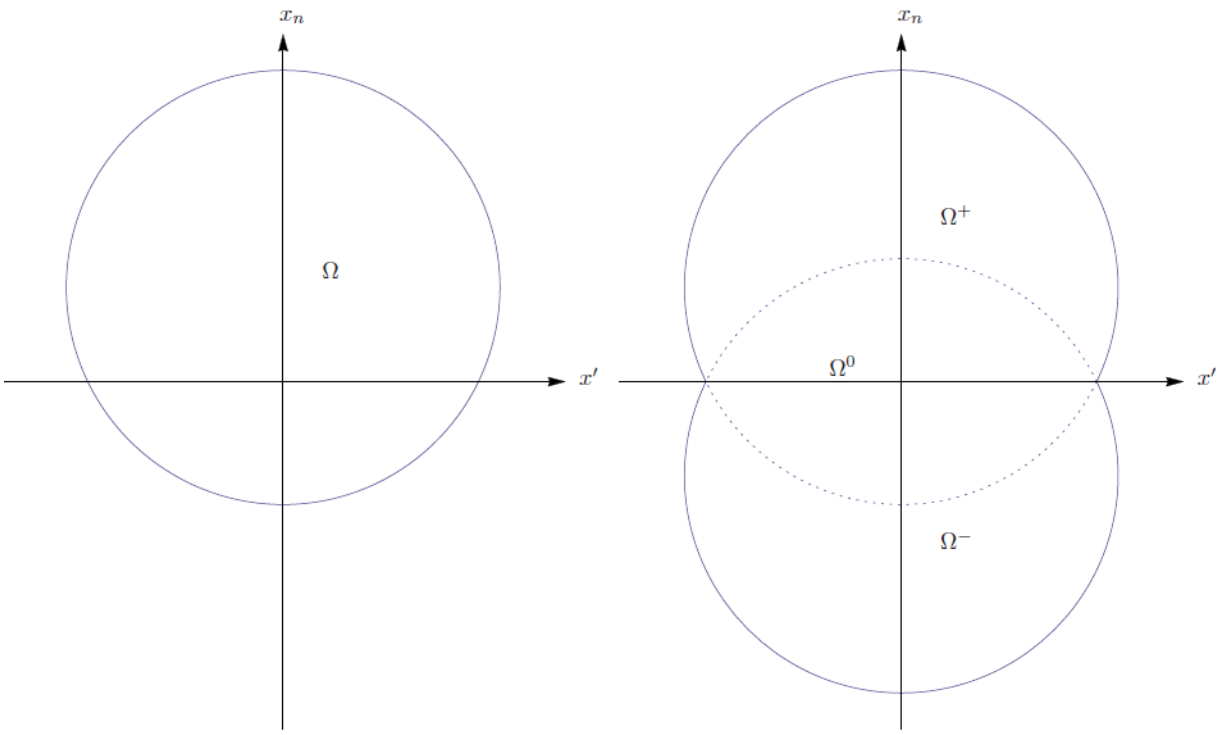

Figure 2.1: Illustration of the sets $\Omega, \Omega^0, \Omega^\pm$

3) Let $\Omega, \Omega^0, \Omega^\pm, \tilde{\Omega}$ be as above (see also the figure), and suppose that $v \in C^2(\Omega)$ satisfies $v = 0$ on Ω^0. Define $\tilde{v} : \tilde{\Omega} \to \mathbb{R}$ by

$$\tilde{v}(x) = \begin{cases} v(x), & x \in \Omega^+ \cup \Omega^0, \\ -v(x', -x_n), & x = (x', x_n) \in \Omega^-. \end{cases}$$

Prove that $\tilde{v} \in C^2(\tilde{\Omega})$ and that $\tilde{v}$ is harmonic in $\tilde{\Omega}$.

Exercise 10. Recall the Azelà-Ascoli theorem: Let $K \subset \mathbb{R}^n$ be compact and let $f_i : K \to \mathbb{R}$ be a sequence of functions that are uniformly bounded and equicontinuous. Then there exists a sequence f_{i_k} which converges uniformly. [Recall that f_i are equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon$ whenever $|x - y| < \delta$].

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u_i : \Omega \to \mathbb{R}$ be a sequence of harmonic functions which are uniformly bounded. Prove that for any multi-index α and for any compact $K \subset \Omega$ there exists a subsequence u_{i_k} such that $D^\alpha u_{i_k}$ converges uniformly in K.

Exercise 11. 1) Suppose $u : \Omega \to \mathbb{R}$ is harmonic and that $F : \mathbb{R} \to \mathbb{R}$ is a convex function of class C^2. Show that $F(u(x))$ is subharmonic.

2) Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and that

$$\int_{\mathbb{R}^n} u^2(x)\, dx = M < \infty.$$

Prove that $u \equiv 0$ in $\mathbb{R}^n$.

3) If u is harmonic in $\Omega \subset \mathbb{R}^n$ and if $\overline{B_r(x_0)} \subset \Omega$, prove that

$$\frac{d}{dr} \fint_{\partial B_r(x_0)} u(y)^2 d\sigma(y) = \frac{2r}{n} \fint_{B_r(x_0)} |\nabla u|^2(x)\, dx.$$

Chapter 3

The Dirichlet problem for harmonic functions

In this chapter we will study the Direchlet problem for harmonic functions in a bounded domain. We will establish a representation formula based on the Green's function, find the Green's function and solve the problem in a ball, and finally use the Perron's method to solve the problem in a general domain.

3.0.1 Introduction

In this chapter, we will study the following Dirichlet problem for harmonic functions.

The Dirichlet problem: Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $g \in C(\partial\Omega)$ be a given function. Find a function $u \in C^2(\Omega) \cap C^0(\partial\Omega)$ such that

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega, \\ u = g, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

The schedule to solve this problem will be the following.

Step 1 Establish a representation formula for a solution to (3.1) based on Green's function.

Step 2 Find the Green's function and solve the problem (3.1) in a ball,

Step 3 Solve the problem (3.1) for a general domain using the Perron's method.

To start with, we have the following lemma

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, let $g_1, g_2 \in C(\partial\Omega)$ be given. Suppose that u_1, u_2 solve the following problem for $i = 1, 2$ respectively*

$$\begin{cases} \Delta u_i = 0, & \text{in} \quad \Omega, \\ u_i = g_i, & \text{on} \quad \partial\Omega. \end{cases}$$

Then

$$\sup_{\overline{\Omega}} |u_1 - u_2| = \sup_{\overline{\Omega}} |g_1 - g_2|. \tag{3.2}$$

As a consequence, a solution to (3.1)*, if it exists, is unique and depends continuously on the data.*

Proof. This is a direct consequence of the maximum principle for $w := u_1 - u_2$, which satisfies that

$$\begin{cases} \Delta w = 0, & \text{in} \quad \Omega, \\ w = g_1 - g_2, & \text{on} \quad \partial\Omega. \end{cases}$$

□

In view of this Lemma, the main task is now to find a solution to (3.1). We first recall the Green's identities that will be used throughout this chapter. The starting point for these identities is the divergence theorem.

$$\int_\Omega \operatorname{div} \overrightarrow{F}\, dx = \int_{\partial\Omega} \overrightarrow{F} \cdot \overrightarrow{n}\, dS.$$

Given $f, g \in C^2(\overline{\Omega})$. Applying the divergence theorem above for $\overrightarrow{F} = f\nabla g$, we obtain

$$\int_\Omega \operatorname{div}(f\nabla g)\, dx = \int_{\partial\Omega} f\nabla g \cdot \overrightarrow{n}\, dS.$$

Note that

$$\operatorname{div}(f\nabla g) = f\Delta g + \nabla g \cdot \nabla f, \quad \nabla g \cdot \overrightarrow{n} = \frac{\partial g}{\partial \overrightarrow{n}},$$

so that we can write the equality above as

$$\int_\Omega f\Delta g\, dx = -\int_\Omega \nabla f \cdot \nabla g\, dx + \int_{\partial\Omega} f \frac{\partial g}{\partial \overrightarrow{n}}\, dS. \tag{3.3}$$

This relation is known as the first Green's identity.

By interchanging f and g in the first Green's identity, we get

$$\int_\Omega g\Delta f\,dx = -\int_\Omega \nabla g\cdot\nabla f\,dx + \int_{\partial\Omega} g\frac{\partial f}{\partial \overrightarrow{n}}\,dS.$$

This together with the first Green's identity implies the following Green's second identity

$$\int_\Omega (f\Delta g - g\Delta f)\,dx = \int_{\partial\Omega}\Big(f\frac{\partial g}{\partial \overrightarrow{n}} - g\frac{\partial f}{\partial \overrightarrow{n}}\Big)\,dS. \tag{3.4}$$

Define the function

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & \text{if} \quad n=2,\\ \frac{1}{(2-n)\sigma_{n-1}}|x|^{2-n}, & \text{if} \quad n=3, \end{cases} \tag{3.5}$$

where $\sigma_{n-1} = |\partial B_1(0)|$. Then $\Phi \in C^\infty \mathbb{R}^n \setminus \{0\}$. For any $x \neq 0$, we have

$$\begin{aligned} \partial_{x_i}\Phi(x) &= \frac{1}{\sigma_{n-1}} x_i|x|^{-n},\\ \partial^2_{x_ix_i}\Phi(x) &= \frac{1}{\sigma_{n-1}}\left[\frac{-nx_i^2}{|x|^{n+2}} + |x|^{-n}\right]. \end{aligned}$$

Therefore,

$$\Delta\Phi(x) = \sum_{i=1}^n \partial^2_{x_ix_i}\Phi(x) = \sum_{i=1}^n \frac{1}{\sigma_{n-1}}\left[\frac{-nx_i^2}{|x|^{n+2}} + |x|^{-n}\right] = 0.$$

So

$$\Delta\Phi(x) = 0, \quad \forall x\neq 0, \quad \lim_{x\to 0}\Phi(x) = -\infty. \tag{3.6}$$

3.0.2 Representation formula based on the Green's function

We first provide a motivation for the Green's function. Recall that we want to solve

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega,\\ u = g, & \text{in} \quad \partial\Omega. \end{cases}$$

Suppose that for each $x \in \Omega$, we can find a function $G(x,y)$ such that

$$\begin{cases} \Delta_y G(x,y) = \delta_x, & \text{in} \quad \Omega,\\ G(x,y) = 0, & \text{in} \quad \partial\Omega. \end{cases}$$

Then formally for a solution u to (3.1), using the second Green's identity, we have

$$\begin{aligned}
u(x) &= \int_\Omega u(y)\delta_x \, dy \\
&= \int_\Omega u(y)\Delta_y G(x,y)\, dy \\
&= \int_{\partial\Omega} \Big(u(y)\frac{\partial G}{\partial \overrightarrow{n}}(x,y) - G(x,y)\frac{\partial u}{\partial \overrightarrow{n}}(y)\Big)\, dS(y) + \int_\Omega G(x,y)\Delta u(y)\, dy \\
&= \int_\Omega \frac{\partial G}{\partial \overrightarrow{n}}(x,y) g(y)\, dS(y).
\end{aligned}$$

Therefore, we can represent u in terms of G and g. This suggests that to solve the problem (3.1), one should find the function $G(x,y)$. We observe that $\Phi(x-y)$ satisfies that $\Delta_y \Phi(y-x) = \delta_x$, but it does not satisfy the boundary condition. The idea that we shall follow is that we will construct $G(x,y)$ from $\Phi(y-x)$ taking into account the boundary conditions.

To do so, let $u \in C^2(\overline{\Omega})$ and consider the following integral

$$\int_\Omega \Phi(y-x)\Delta u(y)\, dy.$$

We want to integrate by parts this integral. However, note that $\Phi(y-x)$ has a singularity at $x = y$. To integrate this, therefore, we proceed as follows. Take $x \in \Omega$ and $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(x) \subset \Omega$. Define $V_\varepsilon := \Omega \setminus B_\varepsilon(x)$. By the Green's second identity, we have

$$\begin{aligned}
\int_{V_\varepsilon} \Phi(y-x)\Delta u(y)\, dy &= \int_{V_\varepsilon} \Delta_y \Phi(y-x) u(y)\, dy + \int_{\partial V_\varepsilon} \Phi(y-x)\frac{\partial u}{\partial \overrightarrow{n}}(y)\, dS(y) \\
&\quad - \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x)\, dS(y) \\
&= \int_{\partial V_\varepsilon} \Phi(y-x)\frac{\partial u}{\partial \overrightarrow{n}}(y)\, dS(y) - \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x)\, dS(y),
\end{aligned}$$

where we have used that $\Delta_y \Phi(y-x) = 0$. Next, we will pass to the limit $\varepsilon \to 0$ this relation. The LHS is straightforward by the dominated convergence theorem

$$\lim_{\varepsilon\to 0} \int_{V_\varepsilon} \Phi(y-x)\Delta u(y)\, dy = \int_\Omega \Phi(y-x)\Delta u(y)\, dy.$$

For the boundary terms, we will show later that

Claim 1

$$\lim_{\varepsilon\to 0}\left[-\int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x)\, dS(y)\right] = -\int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x)\, dS(y) + u(x). \tag{3.7}$$

Claim 2

$$\lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon}\Phi(y-x)\frac{\partial u}{\partial \overrightarrow{n}}(y)\,dS(y)=\int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial \overrightarrow{n}}(y)\,dS(y) \tag{3.8}$$

Assuming these claims at the moment, then we find that for any $u\in C^2(\overline{\Omega})$, we have

$$u(x)=\int_\Omega \Phi(y-x)\Delta u(y)\,dy+\int_{\partial\Omega}u(y)\frac{\partial\Phi}{\partial\overrightarrow{n}}(y-x)\,dS(y)-\int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial\overrightarrow{n}}(y)\,dS(y). \tag{3.9}$$

Note that in the Direchlet problem, we know Δu in Ω and u on $\partial\Omega$, but we do not know $\frac{\partial u}{\partial\overrightarrow{n}}$ on $\partial\Omega$.

To overcome this, for each $x\in\Omega$, we introduce a *corrector function* $h^x(y)$ such that

$$\begin{cases}\Delta_y h^x(y)=0, & y\in\Omega,\\ h^x(y)=\Phi(y-x), & y\in\partial\Omega.\end{cases}$$

Then using the Green's second identity, we have

$$\begin{aligned}\int_\Omega h^x(y)\Delta u(y)\,dy&=\int_\Omega \Delta_y h^x(y)u(y)\,dy+\int_{\partial\Omega}\left[h^x(y)\frac{\partial u}{\partial\overrightarrow{n}}(y)-u(y)\frac{\partial h^x}{\partial\overrightarrow{n}}(y)\right]dS(y).\\ &=\int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial\overrightarrow{n}}(y)\,dS(y)-\int_{\partial\Omega}u(y)\frac{\partial h^x}{\partial\overrightarrow{n}}(y)\,dS(y).\end{aligned}$$

Hence,

$$\int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial\overrightarrow{n}}(y)\,dS(y)=\int_{\partial\Omega}u(y)\frac{\partial h^x}{\partial\overrightarrow{n}}(y)\,dS(y)+\int_\Omega h^x(y)\Delta u(y)\,dy.$$

Substituting this equality back to (3.9), we obtain that

$$u(x)=\int_\Omega[\Phi(y-x)-h^x(y)]\Delta u(y)\,dy+\int_{\partial\Omega}\left(\frac{\partial\Phi}{\partial\overrightarrow{n}}(y-x)-\frac{\partial h^x}{\partial\overrightarrow{n}}(y)\right)u(y)\,dS(y). \tag{3.10}$$

By defining $G(x,y):=\Phi(y-x)-h^x(y)$, then we get that for any $u\in C^2(\overline{\Omega})$

$$u(x)=\int_\Omega G(x,y)\Delta u(y)\,dy+\int_{\partial\Omega}\frac{\partial G}{\partial\overrightarrow{n}}(y-x)\,u(y)\,dS(y). \tag{3.11}$$

The function $G(x,y)$ is called the Green's function for Ω. In particular, if u is harmonic then

$$u(x)=\int_{\partial\Omega}\frac{\partial G}{\partial\overrightarrow{n}}(y-x)\,u(y)\,dS(y).$$

We now prove the two claims.

Proof of Claim 1. We have

$$-\int_{\partial V_\varepsilon}u(y)\frac{\partial\Phi}{\partial\overrightarrow{n}}(y-x)\,dS(y)=-\int_{\partial\Omega}u(y)\frac{\partial\Phi}{\partial\overrightarrow{n}}(y-x)\,dS(y)+\int_{\partial B_\varepsilon(x)}u(y)\frac{\partial\Phi}{\partial\overrightarrow{n}}(y-x)\,dS(y).$$

The first term on the right-hand side does not depend on ε, so it is unchanged when passing $\varepsilon \to 0$. We consider the second term. We have

$$\frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x) = \nabla_y \Phi(y-x) \times \overrightarrow{n}(y),$$

where $\overrightarrow{n}(y)$ is the outward normal vector on $\partial_\varepsilon(x)$. By direct computations

$$\begin{aligned}
&\nabla_y \Phi(y-x) = \frac{1}{(2-n)\sigma_{n-1}} \nabla_y |y-x|^{2-n} = \frac{1}{\sigma_{n-1}} \frac{y-x}{|y-x|^n}, \\
&\overrightarrow{n}(y) = \frac{y-x}{|y-x|},
\end{aligned}$$

so that

$$\frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x) = \frac{1}{\sigma_{n-1}} \frac{y-x}{|y-x|^n} \frac{y-x}{|y-x|} = \frac{1}{\sigma_{n-1}} \frac{1}{|y-x|^{n-1}}.$$

Therefore,

$$\begin{aligned}
\int_{\partial B_\varepsilon(x)} u(y) \frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x)\, dS(y) &= \frac{1}{\sigma_{n-1}} \int_{\partial B_\varepsilon(x)} \frac{1}{|y-x|^{n-1}} u(y)\, dS(y) \\
&= \frac{1}{\sigma_{n-1}\varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} u(y)\, dS(y) \\
&= \frac{1}{|\partial B_\varepsilon(x)|} \int_{\partial B_\varepsilon(x)} u(y)\, dS(y).
\end{aligned}$$

Since u is continuous, we have that

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} u(y) \frac{\partial \Phi}{\partial \overrightarrow{n}}(y-x)\, dS(y) = \lim_{\varepsilon \to 0} \frac{1}{|\partial B_\varepsilon(x)|} \int_{\partial B_\varepsilon(x)} u(y)\, dS(y) = u(x),$$

from which the assertion of the claim 1 is followed.

Proof of the claim 2. Similarly as in the proof of the claim 1, we have

$$\int_{\partial V_\varepsilon} \Phi(y-x) \frac{\partial u}{\partial \overrightarrow{n}}(y)\, dS(y) = \int_{\partial \Omega} \Phi(y-x) \frac{\partial u}{\partial \overrightarrow{n}}(y)\, dS(y) - \int_{\partial B_\varepsilon(x)} \Phi(y-x) \frac{\partial u}{\partial \overrightarrow{n}}(y)\, dS(y).$$

We now show the second term vanishes as $\varepsilon \to 0$. In deed, using the explicit formula for Φ for $n \geq 3$ (the case $n = 2$ is similar), we have

$$\begin{aligned}
\left| \int_{\partial B_\varepsilon(x)} \Phi(y-x) \frac{\partial u}{\partial \overrightarrow{n}}(y)\, dS(y) \right| &\leq \frac{1}{(n-2)\sigma_{n-1}} \int_{\partial B_\varepsilon(x)} \frac{1}{|y-x|^{n-2}} \left| \frac{\partial u}{\partial \overrightarrow{n}}(y) \right| dS(y) \\
&\leq \left\| \frac{\partial u}{\partial \overrightarrow{n}} \right\|_{L^\infty(B_\varepsilon(x))} \frac{1}{(n-2)\sigma_{n-1}\varepsilon^{n-2}} \int_{\partial B_\varepsilon(x)} dS(y) \\
&\leq C\varepsilon,
\end{aligned}$$

where to obtain the last step, we have used that

$$\int_{\partial B_\varepsilon(x)} dS(y) = \sigma_{n-1}\varepsilon^{n-1}.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} \Phi(y-x)\frac{\partial u}{\partial \overrightarrow{n}}(y)\,dS(y) = 0,$$

and the claim 2 is followed.

To summarise, we have proved the following.

Theorem 8. *If $u \in C^(\overline{\Omega})$ is a solution of*

$$\begin{cases} \Delta u = f, & in \quad \Omega, \\ u = g, & in \quad \Omega, \end{cases} \tag{3.12}$$

where f, g are continuous, then for $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y)f(y)\,dy + \int_{\partial\Omega} \frac{\partial G}{\partial \overrightarrow{n}}(x,y)g(y)\,dS(y). \tag{3.13}$$

In particular, if $f \equiv 0$, then

$$u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \overrightarrow{n}}(x,y)g(y)\,dS(y). \tag{3.14}$$

3.1 The Green function for a ball

Let $\Omega = B_R(0)$. Recall that to solve the Dirichlet problem in Ω we need to find, for each $x \in \Omega$, the corrector function h^x that satisfies

$$\begin{cases} \Delta_y h^x(y) = 0, & \text{in} \quad \Omega, \\ h^x(y) = \Phi(y-x), & \text{on} \quad \partial\Omega. \end{cases} \tag{3.15}$$

We will solve this problem by *the method of image charge.* Think of (3.15) as that we want to find the electrostatic potential h^x inside a spherical conductor of radius R de to the point charge located at x.

We consider the image of x under the inversion in ∂B_R,

$$x^* = \frac{R^2}{|x|^2}x,$$

which in particular, implies that

$$|x^*| = \frac{R^2}{|x|}, \quad x^* \cdot x = R^2.$$

Then, if $y \in \partial B_R$, we have

$$\begin{aligned}
|y-x|^2 &= |y|^2 - 2x\cdot y + |x|^2 \\
&= R^2 - 2x\cdot y + |x|^2 \\
&= \frac{|x|^2}{R^2}\left(\frac{R^4}{|x|^2} - 2\frac{xR^2}{|x|^2}\cdot y + R^2\right) \\
&= \frac{|x|^2}{R^2}\left(|x^*|^2 - 2x^*\cdot y + |y|^2\right) \\
&= \frac{|x|^2}{R^2}|x^*-y|^2.
\end{aligned}$$

So we obtain that

$$|y-x| = \frac{|x|}{R}|y-x^*| \qquad \forall y \in \partial B_R.$$

Let

$$h^x(y) := \begin{cases} \frac{1}{2\pi}\log\left(\frac{|x|}{R}|y-x^*|\right), & n=2, \\ \frac{1}{(2-n)\sigma_{n-1}}\left(\frac{|x|}{R}\right)^{2-n}|y-x^*|^{2-n}, & n\geq 3. \end{cases}$$

Note that since $|x^*|\,|x| = R^2$ and $x \in B_R$, it follows that $x^* \notin B_R$, so that $y \neq x^*$. Therefore $y \mapsto h^x(y)$ is harmonic in B_R. In addition, by the computation above, it holds that $h^x(y) = \Phi(y-x)$ on ∂B_R. Therefore, $h^x(y)$ is the unique corrector function.

Hence, the Green's function for the ball is given by

$$G_{B_R}(x,y)\begin{cases} \frac{1}{2\pi}\log\left(\frac{R}{|x|}\frac{|y-x|}{|y-x^*|}\right), & n=2, \\ \frac{1}{(2-n)\sigma_{n-1}}\left[|y-x|^{2-n} - \left(\frac{|x|}{R}\right)^{2-n}|y-x^*|^{2-n}\right], & n\geq 3. \end{cases}$$

By the representation formula (3.14), we have

$$u(x) = \int_{\partial\Omega}\frac{\partial G}{\partial \overrightarrow{n}}(x,y)g(y)\,dS(y).$$

Next, we will compute $\frac{\partial G}{\partial \overrightarrow{n}}(x,y)$. We have

$$\frac{\partial G}{\partial \overrightarrow{n}}(x,y) = \nabla_y G(x,y)\cdot\overrightarrow{n}.$$

Here

$$\begin{aligned}
\overrightarrow{n} &= \frac{y}{R}, \\
\nabla_y G(x,y) &= \frac{1}{\sigma_{n-1}}\left[\frac{y-x}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{2-n}\frac{y-x^*}{|y-x^*|^n}\right].
\end{aligned}$$

Note that this formula is true for $n \geq 2$. Therefore, we obtain

$$\begin{aligned}
\frac{\partial G}{\partial \overrightarrow{n}}(x,y) &= \frac{1}{R\sigma_{n-1}}\left[\frac{y-x}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{2-n}\frac{y-x^*}{|y-x^*|^n}\right]\cdot y \\
&= \frac{1}{R\sigma_{n-1}}\left[\frac{|y|^2 - x\cdot y}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{2-n}\frac{|y|^2 - x^*\cdot y}{|y-x^*|^n}\right] \\
&= \frac{1}{R\sigma_{n-1}}\left[\frac{|y|^2}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{-n}\frac{|x|^2}{|y-x^*|^n} - x\cdot y\Big(\frac{1}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{-n}\frac{1}{|y-x^*|^n}\Big)\right] \\
&= \frac{1}{R\sigma_{n-1}}\frac{R^2-|x|^2}{|y-x|^n}.
\end{aligned}$$

Set

$$K_R(x,y) := \frac{1}{R\sigma_{n-1}}\frac{R^2-|x|^2}{|y-x|^n}. \tag{3.16}$$

This function is called the Poisson kernel. We have proved that

Proposition 3. *Let $u \in C^2(\overline{B_R})$ be harmonic in B_R. Then for $x \in B_R$, one has*

$$u(x) = \int_{\partial B_R} K_R(x,y)u(y)\,dS(y) = \frac{1}{R\sigma_{n-1}}\int_{\partial B_R}\frac{R^2-|x|^2}{|y-x|^n}u(y)\,dS(y). \tag{3.17}$$

The formula above is called the Poisson's integral formula.

Note that this proposition only tells us that if a solution exists, then it is given in the interior by the Poisson's integral formula. We need to show that for suitable g, the Poisson's integral formula indeed defines us a harmonic function.

We will need some properties of the Poisson's kernel $K_R(x,y)$.

Lemma 2. *The function $K_R(x,y)$ satisfies the following properties*

(i) For any $y \in \partial B_R$, the map

$$\begin{aligned}
K_R^y : B_R &\to \mathbb{R} \\
x &\mapsto K_R(x,y)
\end{aligned}$$

is harmonic.

(ii) If $y \in \partial B_R$ and $x \in B_R$ then $K_R(x,y) > 0$.

(iii) For all $x \in B_R$, we have

$$\int_{\partial B_R} K_R(x,y)\,dS(y) = 1.$$

(iv) For $y \neq z$ with $y, z \in \partial B_R$, we have

$$\lim_{\substack{x \to z \\ x \in B_R}} K_R(x, y) = 0$$

and this convergence is uniform in y for $|y - z| > \delta > 0$.

Proof. (i) We notice that for each $i = 1, \ldots, n$, the function $\frac{x_i}{|x|^n}$ is harmonic on $\mathbb{R}^n \setminus \{0\}$. Indeed, since

$$\frac{x_i}{|x|^n} = \frac{\partial}{\partial x_i}\left(\frac{|x|^{2-n}}{2-n}\right)$$

so

$$\Delta \frac{x_i}{|x|^n} = \Delta\left[\frac{\partial}{\partial x_i}\left(\frac{|x|^{2-n}}{2-n}\right)\right] = \frac{\partial}{\partial x_i}\left[\Delta\left(\frac{|x|^{2-n}}{2-n}\right)\right] = 0.$$

It follows that $|x - y|^{2-n}$ and $\frac{(x-y)_i}{|x-y|^n}$ are both harmonic. So that

$$R\sigma_{n-1}K_R(x, y) = \frac{R^2 - |x|^2}{|x - y|^n} = \frac{2|y|^2 - |x - y|^2 - 2x \cdot y}{|x - y|^n} = -|x - y|^{2-n} + 2y \cdot \frac{y - x}{|y - x|^n}$$

is also a harmonic function.

(ii) This is obvious since $|x| < R$ for $x \in B_R$.

(iii) Applying the Poisson's integral formula for $u \equiv 1$ yields the result.

(iv) $\lim\limits_{\substack{x \to z \\ x \in B_R}} \frac{R^2 - |x|^2}{|y-x|^n} = \frac{0}{|y-z|^n} = 0$. For $|x - z| < \frac{\delta}{2}$, then $|y - x| \geq |y - z| - |x - z| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}$. Therefore,

$$\sup_{|y-z|>\delta} \frac{R^2 - |x|^2}{|y - x|^n} \leq \frac{R^2 - |x|^2}{\delta/2} \to 0.$$

□

Theorem 9. *Let $g \in C^0(\partial B_R)$ and let us define the function $u : B_R \to \mathbb{R}$ by*

$$u(x) = \int_{B_R} K_R(x, y) g(y)\, dS(y).$$

Then u is harmonic in B_R, $u \in C^0(\overline{B_R})$ and for $x_0 \in \partial B_R$, we have

$$\lim_{\substack{x \to x_0 \\ x \in B_R}} u(x) = g(x_0).$$

Proof. Since $K_R(x, y)$ is harmonic in x, it is also in C^∞. Define

$$I(x, y) := K_R(x, y) g(y).$$

Let U be compactly contained in B_R. Then $I : U \times \partial B_R \to \mathbb{R}$ is infinitely differentiable in x and the derivative $D_x^\alpha I(x,y)$ are continuous maps from $U \times \partial B_R$ to $\mathbb{R}$. Moreover, since ∂B_R is compact, according to the results of the Appendix, we have that u is C^∞ and that for each $x \in U$ we have

$$D^\alpha u(x) = \int_{\partial B_R} D_x^\alpha K_R(x,y) g(y)\, dS(y).$$

Since any $x \in B_R$ belongs to some open set $U \Subset \Omega$, it follows that u is smooth and

$$\Delta u(x) = \int_{\partial B_R} \Delta_x K_R(x,y) g(y)\, dS(y) = 0.$$

Now we show the continuity at the boundary. Let $x_0 \in \partial B_R$. By Property (iii) in 2, we have

$$u(x) - g(x_0) = \int_{\partial B_R} K_R(x,y)(g(y) - g(x_0))\, dS(y) = I_1 + I_2,$$

where

$$I_1 := \int_{\partial B_R \cap B_\delta(x_0)} K_R(x,y)(g(y) - g(x_0))\, dS(y),$$

and

$$I_2 := \int_{\partial B_R \setminus B_\delta(x_0)} K_R(x,y)(g(y) - g(x_0))\, dS(y),$$

for some $\delta > 0$. Since $g \in C^0(\partial B_R)$, for a given $\varepsilon > 0$, we can take $\delta > 0$ small enough that $|g(y) - g(x_0)| \le \varepsilon$ for $y \in \partial B_R \cap B_\delta(x_0)$. Therefore, the term I_1 can be estimated by

$$\begin{aligned} |I_1| &\le \int_{\partial B_R \cap B_\delta(x_0)} K_R(x,y)|g(y) - g(x_0)|\, dS(y) \\ &\le \varepsilon \int_{\partial B_R \cap B_\delta(x_0)} K_R(x,y)\, dS(y) \\ &\le \varepsilon \int_{\partial B_R} K_R(x,y)\, dS(y) = \varepsilon. \end{aligned} \tag{3.18}$$

Since g is continuous (and hence bounded) on ∂B_R from part (iv) of Lemma 2, we have that

$$K_R(x,y)(g(y) - g(x_0)) \to 0 \quad \text{as} \quad x \to x_0$$

uniformly in $y \in \partial B_R \setminus B_\delta(x_0)$. Thus as $x \to x_0$, we have $I_2 \to 0$.

Together, with the estimate (3.18), we have that $\lim_{x \to x_0} u(x) = g(x_0)$ as claimed. $\square$

As consequences of the Poisson's integral formula, we obtain the following strengthened versions of the Harnack's inequality and the Liouville theorem.

Theorem 10 (Harnack's inequality for the ball)**.** *Let $u \geq 0$ be harmonic in B_R. Then for any $x \in B_R$ we have the following estimate*

$$\frac{\left(1-\frac{|x|}{R}\right)u(0)}{\left(1+\frac{|x|}{R}\right)^{n-1}} \leq u(x) \leq \frac{\left(1+\frac{|x|}{R}\right)u(0)}{\left(1-\frac{|x|}{R}\right)^{n-1}}.$$

Proof. According to the Poisson's integral formula we have

$$u(x) = \frac{1}{R\sigma_{n-1}} \int_{\partial B_R} \frac{R^2-|x|^2}{|y-x|^n} u(y)\, dS(y).$$

By the triangle inequality, we have for any $y \in \partial B_R$

$$R - |x| \leq |y-x| \leq |x| + R$$

So that

$$\frac{R^2-|x|^2}{\sigma_{n-1}R(R+|x|)^n} \int_{\partial B_R} u(y)\, dS(y) \leq u(x) \leq \frac{R^2-|x|^2}{\sigma_{n-1}R(R-|x|)^n} \int_{\partial B_R} u(y)\, dS(y)$$

By the mean value property

$$\int_{\partial B_R} u(y)\, dS(y) = |B_R| u(0) = R^{n-1}\sigma_{n-1}u(0).$$

Hence

$$\frac{R^2-|x|^2}{\sigma_{n-1}R(R+|x|)^n} R^{n-1}\sigma_{n-1}u(0) \leq u(x) \leq \frac{R^2-|x|^2}{\sigma_{n-1}R(R-|x|)^n} R^{n-1}\sigma_{n-1}u(0)$$

Simplifying this equality, we obtain

$$\frac{\left(1-\frac{|x|}{R}\right)u(0)}{\left(1+\frac{|x|}{R}\right)^{n-1}} \leq u(x) \leq \frac{\left(1+\frac{|x|}{R}\right)u(0)}{\left(1-\frac{|x|}{R}\right)^{n-1}}$$

as claimed. □

Theorem 11 (Improved Liouville theorem)**.** *Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded from below. Then u must be a constant.*

Proof. Without loss of generality, we assume u is harmonic in $\mathbb{R}^n$ and $u \geq 0$. Take $x \in \mathbb{R}^n$, according to Theorem 10 , for any $R > 0$, we have

$$\frac{\left(1-\frac{|x|}{R}\right)u(0)}{\left(1+\frac{|x|}{R}\right)^{n-1}} \leq u(x) \leq \frac{\left(1+\frac{|x|}{R}\right)u(0)}{\left(1-\frac{|x|}{R}\right)^{n-1}}$$

Sending $R \to \infty$, we find that $u(x) = u(0), \forall x \in \mathbb{R}^n$. □

3.2 C^0-subharmonic functions

Suppose $\Omega \subset \mathbb{R}^n$.

Definition 5. We say that a function $u \in C^0(\Omega)$ is C^0-subharmonic in Ω if for any $B_\rho(x) \Subset \Omega$, we have the implication

$$\left.\begin{aligned} &h \in C^2(B_\rho(x)) \cap C^0(\overline{B_\rho(x)}) \\ &\Delta h = 0, \quad \text{in} \quad B_\rho(x), \\ &u \le h, \quad \text{on} \quad \partial B_\rho(x) \end{aligned}\right\} \Longrightarrow u \le h \quad \text{in} \quad B_\rho(x).$$

Note that in this definition, there is no requirement that $u \in C^2(\Omega)$.

Lemma 3. *If u is C^0-subharmonic, then for any $B_\rho(x) \Subset \Omega$, we have*

$$u(x) \le \fint_{\partial B_\rho(x)} u(y)\, dS(y) \quad \textit{and} \quad u(x) \le \fint_{B_\rho(x)} u(y)\, dy,$$

where $\fint$ denotes the average integral. And conversely, if $u \in C^0$ and satisfies either the inequalities above for any ball $B_\rho(x) \Subset \Omega$ then u is C^0-subharmonic.

Proof. By Theorem 9, there exists a function h solving

$$\begin{cases} h \in C^2(B_\rho(x)) \cap C^0(\overline{B_\rho(x)}) \\ \Delta h = 0, \quad \text{in} \quad B_\rho(x), \\ u = h, \quad \text{on} \quad \partial B_\rho(x) \end{cases}$$

Since u is C^0-subharmonic, we have

$$u(x) \le h(x) = \fint_{\partial B_\rho(x)} h(y)\, dS(y) = \fint_{\partial B_\rho(x)} u(y)\, dS(y),$$

where the first equality follows from the mean value property for h.

The second estimate of the Lemma is similar.

Now suppose that

$$\left.\begin{aligned} &h \in C^2(B_\rho(x)) \cap C^0(\overline{B_\rho(x)}) \\ &\Delta h = 0, \quad \text{in} \quad B_\rho(x), \\ &u \le h, \quad \text{on} \quad \partial B_\rho(x) \end{aligned}\right\}$$

Then

$$(u-h)(x) \le \fint_{B_\rho(x)} (u-h)(y)\, dy,$$

i.e., $u - h$ satisfies the mean value property. Recalling that to prove the strong maximum principle, we only need the mean value property and connectedness of the domain. Therefore, it implies that $u - h$ satisfies the strong maximum principle. Hence,

$$u - h \leq \sup_{B_\rho(x)} (u - h) \leq 0 \quad \text{in} \quad B_\rho(x)$$

i.e., the implication in the definition holds. □

Proposition 4 (Properties of C^0-subharmonic functions)**.**

(i) If u is C^0-subharmonic function then the strong maximum principle holds. Moreover, we have comparison principle with harmonic functions on every connected set $\Omega' \Subset \Omega$, i.e., in the definition 5, one can replace the ball $B_\rho(x)$ by Ω'.

(ii) Let u be C^0-subharmonic in Ω and let $B_\rho(x) \Subset \Omega$. We define $\bar{u}$ to be the solution of the problem

$$\begin{cases} \Delta \bar{u} = 0, & \text{in} \quad B_\rho(x), \\ \bar{u} = u, & \text{on} \quad \partial B_\rho(x). \end{cases}$$

and then define the function U by

$$U(y) = \begin{cases} \bar{u}(y), & \text{if} \quad y \in B_\rho(x), \\ u(y), & \text{if} \quad y \in \Omega \setminus B_\rho(x). \end{cases}$$

Then U is C^0-subharmonic in Ω.

The function U is called the harmonic lifting of u in $B_\rho(x)$.

(iii) Let u, v be C^0-subharmonic in Ω and such that $u \leq v$. Suppose $B_\rho(x) \Subset \Omega$ and let U, V respectively be the harmonic liftings of u and v in $B_\rho(x)$. Then $U \leq V$.

(iv) If $u_1, \ldots, u_N$ are C^0-subharmonic in Ω, then

$$u(x) := \max u_1(x), \ldots, u_N(x)$$

is also C^0-subharmonic in Ω.

Proof. 1. From the proof of Theorem 6, the strong maximum principle follows from the mean value property. According to Lemma 3 a C^0-subharmonic function satisfies the mean value property, so that it satisfies a strong maximum principle.

2. Let $B_r(z) \Subset \Omega$ and suppose that

$$\left.\begin{aligned} &h \in C^2(B_r(z)) \cap C^0(\overline{B_r(z)}) \\ &\Delta h = 0, \quad \text{in} \quad B_r(z), \\ &U \leq h, \quad \text{on} \quad \partial B_r(z). \end{aligned}\right\}$$

First of all, note that since $\bar{u} \geq u$ in $B_\rho(x)$, then $U \geq u$ in in $B_\rho(x)$ and hence in Ω. Hence $u \leq U \leq h$ in $\partial B_r(z)$, it follows that $u \leq h$ in $B_r(z)$. Therefore, $U \leq h$ in $B_r(z) \setminus B_\rho(x)$. In addition, since $\bar{u} = u$ on $\partial B_\rho(x)$, it follows that $\bar{u} \leq u \leq h$ in $\partial(B_r(x) \cap B_\rho(x))$. Since $\bar{u}$ is harmonic in $\partial(B_r(x) \cap B_\rho(x))$, we obtain that $U = \bar{u} \leq h$ in $B_r(x) \cap B_\rho(x)$. So $U \leq h$ in $B_r(z$, which means that U is C^0-subharmonic function in Ω.

3. On the set $\Omega \setminus B_\rho(x)$, we have $U = u \leq v = V$. In addition, we have

$$\begin{aligned} &\Delta \bar{u} = 0, \quad \Delta \bar{v} = 0, \quad \text{in} \quad B_\rho(x), \\ &\bar{u} = u, \quad \bar{v} = v \quad \text{on} \quad \partial B_\rho(x) \end{aligned}$$

Since $u \leq v$ on $\partial B_\rho(x)$, by the comparison principle, we have that $\bar{u} \leq \bar{v}$ in $B_\rho(x)$, so that also $U \leq V$ in $B_\rho(x)$.

In conclusion, we get $U \leq V$ in Ω.

4. For two functions $f, g \in C^0(\Omega)$ it holds that

$$\max\{f, g\} = \frac{1}{2}[f + g + |f - g|],$$

which implies that $\max\{f, g\} \in C^0(\Omega)$. Repeating this argument, we have that if $u_1, \ldots, u_N \in C^0(\Omega)$ then $u = \max\{u_1, \ldots, u_N\} \in C^0(\Omega)$. Let h be the function satisfies the conditions in the definition 5. Then $u_i \leq h$ in $\partial B_\rho(x)$ and hence $u_i \leq h$ in $B_\rho(x)$ for all $i = 1, \ldots, N$ since u_i are C^0-subharmonic. Therefore $u \leq h$ in $B_\rho(x)$, so u is also C^0-subharmonic.

□

3.3 Perron's method for the Dirichlet problem on a general domain

We are now ready to introduce Perron's method to solve the Dirichlet problem on a general domain

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega \\ u = g, & \text{on} \quad \partial\Omega. \end{cases}$$

Let $g \in C^0(\Omega)$, define

$$\underline{S}_g := \{v : \Omega \to \mathbb{R} \ \text{ such that } \ v \ \text{ is } C^0\text{-subharmonic in } \Omega \ \text{ and } \ v \leq g \ \text{ on } \ \partial\Omega\}.$$

Theorem 12. *Suppose that Ω is open and bounded and that $g \in C^0(\partial\Omega)$. Define*

$$u(x) := \sup_{v \in \underline{S}_g} \{v(x)\}.$$

Then u is harmonic in Ω.

Proof. Step 1) First of all $\underline{S}_g \neq \emptyset$ since $\inf_{\partial\Omega} g \in \underline{S}_g$. Moreover, for each $v \in \underline{S}_g$ then $v \leq \sup_{\partial\Omega} g$. So that $\underline{S}_g$ is non-empty, bounded above. Therefore, for each x, $u(x)$ is well-defined, finite number.

Step 2) Now pick $y \in \Omega$. By definition of u, there exist $v_i \in \underline{S}_g$ such that $v_i(y) \to u(y)$ as $i \to \infty$. Define

$$\tilde{v}_i(x) = \max\{\inf_{\partial\Omega} g, v_1(x), \ldots, v_i(x)\}.$$

Then

- $\inf_{\partial\Omega} g \leq \tilde{v}_1 \leq \tilde{v}_2 \leq \ldots \leq \tilde{v}_i \leq \ldots \leq \sup_{\partial\Omega} g$.
- $\tilde{v}_i$ are subharmonic.

Step 3) Take ρ sufficiently small such that $B_\rho(y) \Subset \Omega$. Let V_i be the harmonic lifting of $\tilde{v}_i$ in $B_\rho(y)$. Then

- $V_i \in \underline{S}_g, V_i(y) \to u(y)$,
- V_i are harmonic in $B_\rho(y)$,
- $\inf_{\partial\Omega} g \leq V_1 \leq V_2 \leq \ldots \leq V_i \leq \ldots \leq \sup_{\partial\Omega} g$.

Step 4) By Harnack's theorem 3, $V_i \to V$ uniformly in $B_{\rho'(y)}$ for any $\rho' < \rho$, where V is harmonic. Since $V_i \in \underline{S}_g$, it follows that $V \leq u$ in Ω.

Step 5) We now show that $V = u$ in $B_\rho(y)$. Suppose this is not true. Then there exists $z \in B_\rho(y)$ such that $V(z) < u(z)$. By definition of u, there exists $\tilde{v} \in \underline{S}_g$ such that $V(z) < \tilde{v}(z)$. We define

$$w_i = \max\{\tilde{v}, V_i\},$$

and let W_i be the harmonic lifting of w_i in $B_\rho(y)$. By similar arguments as in the previous steps, there exists a new harmonic function W such that

- $V \leq W \leq u$ in $B_\rho(y)$,
- $V(y) = W(y) = u(y)$,
- $W(z) \geq \tilde{v}(z)$.

The function $W - V$ is harmonic in $B_\rho(y)$ satisfying

$$W - V \geq 0 \ \text{ in } \ B_\rho(y) \ \text{ and } \ (W - V)(y) = 0.$$

By the strong maximum principle it follows that $V = W$ in $B_\rho(y)$. However, this is a contradiction since

$$V(z) < \tilde{v}(z) \leq W(z).$$

Therefore, $V = u$ in $B_\rho(y)$.

Step 6) Since y is arbitrary, we conclude that $V = u$ in Ω, thus u is harmonic in Ω.

□

The function u constructed by Theorem 12 is called Perron's solution. It is a good candidate for a solution of the Dirichlet problem (3.1). It remains to show that it satisfies the boundary condition. To achieve this, we need to make some assumptions on the domain Ω.

Definition 6. We say that Ω satisfies the *barrier postulate* if for each $y \in \partial\Omega$, there exists a function (called a barrier function) $Q_y(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfying

(i) $Q_y(x)$ is superharmonic in Ω,

(ii) $Q_y(x) > 0$ in $\bar{\Omega} \setminus \{y\}$ and $Q_y(y) = 0$.

Theorem 13. *Suppose that Ω is open, bounded and satisfies the barrier postulate and that $g \in C^0(\partial\Omega)$. Let u be the Perron's solution. Then for any $y \in \partial\Omega$, we have*

$$\lim_{\substack{x\in\Omega \\ x\to y}} u(x) = g(y).$$

Proof. Let $\varepsilon > 0$ and set $M = \sup_{\partial\Omega} |g|$. Pick $y \in \partial\Omega$ and let $Q_y(x)$ be the barrier function. By the continuity of g and the positivity of $Q_y(x)$ in $\bar{\Omega} \setminus \{y\}$, there exist δ, K depending on ε such that

$$|g(y) - g(z)| < \varepsilon \quad \text{for} \quad z \in \partial\Omega, \quad |y - z| < \delta, \tag{3.19}$$

and

$$KQ_y(x) > 2M \quad \text{for} \quad z \in \partial\Omega, \quad |y - z| \geq \delta. \tag{3.20}$$

We consider the functions

$$\underline{u}_{y,\varepsilon}(x) = g(y) - \varepsilon - KQ_y(x),$$
$$\bar{u}_{y,\varepsilon}(x) = g(y) + \varepsilon + KQ_y(x).$$

Since $Q_y(x)$ is superharmonic, $\underline{u}_{y,\varepsilon}(x)$ is subharmonic. If $z \in \partial\Omega$, then from (3.19) and (3.20), we have

$$\underline{u}_{y,\varepsilon}(z) - g(z) = g(y) - g(z) - \varepsilon - KQ_y(x) \leq 0,$$

which implies that $\underline{u}_{y,\varepsilon} \in \underline{S}_g$. Similarly, we get $\bar{u}_{y,\varepsilon} \in \bar{S}_g$. Hence

$$\underline{u}_{y,\varepsilon} \leq u(x) \leq \bar{u}_{y,\varepsilon}(x) \quad \forall x \in \Omega,$$

so that

$$g(y) - \varepsilon - KQ_y(x) \leq u(x) \leq g(y) + \varepsilon + KQ_y(x) \quad \forall x \in \Omega.$$

This implies that

$$-\varepsilon - KQ_y(x) \leq u(x) - g(y) \leq \varepsilon + KQ_y(x),$$

i.e.,

$$|u(x) - g(y)| \leq \varepsilon + KQ_y(x).$$

Since $Q_y \in C^0(\bar{\Omega}$ and $Q_y(y) = 0$, there exists δ' such that if $|x-y| < \delta'$ then $Q_y(x) \leq K^{-1}\varepsilon$ and

$$|u(x) - g(y)| \leq 2\varepsilon,$$

which implies that $\lim_{x\to y} u(x) = g(y)$ as claimed. $\square$

Corollary 4. *Under the assumptions as in Theorem 13, the Dirichlet problem* (3.1) *is well-posed.*

To conclude this section, we provide a criterion that a barrier function exists on Ω for the point y.

Lemma 4. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded, and that $y \in \partial\Omega$. If there exists $B_r(z)$ such that $B_r(z) \cap \bar{\Omega} = \{y\}$, then there exists a barrier function on Ω for y.*

Proof. Consider the function

$$Q_y(x) = \begin{cases} r^{2-n} - |x-z|^{2-n}, & n \geq 3, \\ \log\left(\frac{|x-z|}{r}\right), & n = 2. \end{cases}$$

This function is a barrier function on Ω for y. $\square$

Problem Sheet 3

Exercise 12 (The Kelvin transform)**.** Given $a > 0$ and consider the following map

$$\begin{aligned} T_a : \quad & \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} \\ & x \mapsto a^2 \frac{x}{|x|^2} \end{aligned}$$

which is known as *inversion with respect to the sphere of radius a.*

a) Show that the map is *conformal*, i.e., it preserves the angles between vectors

$$\frac{(DT_a(x)[v], DT_a(x)[w])}{||DT_a(x)[v]||\,||DT_a(x)[w]||} = \frac{(v,w)}{||v||\,||w||} \tag{3.21}$$

for all $v, w \in \mathbb{R}^n, x \in \mathbb{R}^n \setminus \{0\}$.

b) Show that in two dimensions T_a sends lines and circles into lines and circles.

c) If T_a is as above, define

$$v_a(x) = \left(\frac{a}{|x|}\right)^{n-2} u(T_a(x)).$$

Show that

$$\Delta v_a(x) = \left(\frac{a}{|x|}\right)^{n+2} \Delta u(T_a(x)).$$

What can we say if u is harmonic? v_a is called the *the Kelvin transform* of u.

Exercise 13 (Hopf Lemma. See Section 6.4.2 in Evans's book with $Lu = \Delta u$)**.** Suppose $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain (which implies the interior ball condition in Evans'book). Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is subharmonic in Ω and suppose there exists $x_0 \in \partial\Omega$ with $u(x_0) > u(x)$ for all $x \in \Omega$. Then

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the outer unit normal vector.

Exercise 14 (Symmetry of the Green's function)**.** Let $\Omega \subset \mathbb{R}^n$ be open, bounded and of class C^1. Let $G(x, y)$ be the Green's function for Ω. Prove that G is symmetric, i.e., for all $x, y \in \Omega, x \neq y$, we have

$$G(x, y) = G(y, x)$$

Exercise 15. Using the method of image charges, find the Green's function of the half space

$$\Omega_1 := \{x \in \mathbb{R}^n : x_n > 0\}$$

and of the half ball in $\mathbb{R}^3$:

$$\Omega_2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z > 0\}.$$

Exercise 16. a) Find an example of a C^0-subharmonic function on $(0, 1)$ that is not everywhere differentiable.

b) Find an example of a C^1-subharmonic function on $(0, 1)$ which is C^1-regular but not everywhere of class C^2.

Chapter 4

The theory of distributions and Poisson's equation

In this chapter, we will solve Poisson's equation: given $f : \mathbb{R}^n \to \mathbb{R}$, solve

$$\Delta u = f \tag{4.1}$$

for some $u : \mathbb{R}^n \to \mathbb{R}$. The approach will be via *the fundamental solution* which is the solution to the equation where the right-hand side is replaced by the so-called *Dirac distribution* (Dirac delta function). The solution to Poisson's equation then will be determined by a *convolution* of the fundamental solution with the function f. The main tool of this chapter will be *the theory of distributions*. This theory plays an important role in mathematics and is of independent interest.

4.1 The theory of distributions

As a motivation, let us consider the following simple problem: solve

$$x^2 = a.$$

We knew that if $a \geq 0$, this equation has real solutions $x \in \mathbb{R}$. However, if $a < 0$, it has no real solution. However, if we *extend* the set of real numbers $\mathbb{R}$ to the set of complex numbers $\mathbb{C}$, then it has solutions $z \in \mathbb{C}$ even if $a < 0$. To extend $\mathbb{R}$ to $\mathbb{C}$ we not only enlarge $\mathbb{R}$ to a bigger set $\mathbb{R} \subset \mathbb{C}$, but also generalise the operations on $\mathbb{R}$, such as addition, subtraction and multiplication, to $\mathbb{C}$. After extension to the complex numbers,

the problem of solving an algebraic equation (not only a quadratic one as the one above) is in some ways simpler than the original problem: the number of complex roots in an algebraic equation of order k is always k (counting multiplicities) while over the real line the number can be any thing between 0 and k. Having found the complex roots, we then can consider separately the problem of showing that some of them are in fact real.

Turing back to Poisson's equation (4.1), the aim of this chapter is to treat the situation where u and f need not be of class C^2. Generally, we want to make sense of the PDE equation of order k

$$Lu := \sum_{|\alpha|\leq k} a_\alpha D^\alpha u = f,$$

where $a_\alpha \in C^\infty(\Omega)$ for an open $\Omega \subset \mathbb{R}^n$, in the situation where u and f need not be of class $C^k(\Omega)$. To do so, similarly as in the motivating example above, we need to extend the class of smooth functions to a larger set and generalise the operations on smooth functions to elements of this set. Let us denote by $\mathcal{D}'(\Omega)$ the space to which our generalised solution u and the right-hand side function f should belong. What properties do we require for $\mathcal{D}'(\Omega)$ so that we can at least make sense of the PDE (4.1)? The list of desirable properties should contain

1) The smooth functions $C^\infty(\Omega)$ should be contained in $\mathcal{D}'(\Omega)$ in such a way that we can recover them, i.e., the inclusion map $\iota : C^\infty(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ should be injective.

2) We need to be able to add elements of $\mathcal{D}'(\Omega)$.

3) We need to be able to multiply elements of $\mathcal{D}'(\Omega)$ by elements of $C^\infty(\Omega)$.

4) We need to be able to differentiate elements of $\mathcal{D}'(\Omega)$. Moreover, the generalised notion of differentiation should agree with the classical one when restricted to smooth functions.

5) We need some topology on $\mathcal{D}'(\Omega)$ such that the above operations are continuous.

Before introducing $\mathcal{D}'(\Omega)$, we need to introduce another space, the space of test functions $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$. A topology on $\mathcal{D}(\Omega)$ is defined as follows.

Definition 7 (Convergence in $\mathcal{D}(\Omega)$)**.** A sequence $(\phi)_{j\in\mathbb{N}}, \phi_j \in \mathcal{D}(\Omega) = C_c^\infty(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $\mathrm{supp}(\phi_j), \mathrm{supp}(\phi) \subset K$ for all

$j \in \mathbb{N}$ and that

$$D^\alpha \phi_j \to D^\alpha \phi \quad \text{as} \quad j \to \infty$$

uniformly in K for any multi-index α.

The space $\mathcal{D}(\Omega)$ equipped with this topology becomes a topological vector space over the real number, i.e, a vector space together with a topology such that addition of vectors and scalar multiplication are continuous functions. For any topological vector space V, one can define *the continuous dual space* which consists of continuous linear maps

$$V' = \{\omega : V \to \mathbb{R} | \ \ \omega \ \text{ linear, continuous}\}.$$

The space $\mathcal{D}'(\Omega)$ that we are seeking for will be the continuous dual space of $\mathcal{D}(\Omega)$.

Definition 8. A distribution $T \in \mathcal{D}'(\Omega)$ is a linear functional on the space of test functions

$$T : \mathcal{D}(\Omega) \to \mathbb{R}$$

which is continuous with respect to the topology on $\mathcal{D}(\Omega)$ defined in Definition 7. In other words, if $\phi_j \to \phi$ in the sense of Definition 7 then

$$T\phi_j \to T\phi \quad \text{as} \quad j \to \infty.$$

We consider two important examples

Example 20.

1. If $f : \Omega \to \mathbb{R}$ is continuous, the distribution T_f associated to f is define by

$$T_f(\phi) := \int_\Omega f(x)\phi(x)\, dx \quad \forall \phi \in C_c^\infty(\Omega).$$

In one-dimension, we can allow f to have a finite number of points of finite discontinuity. Because of this, a distribution is also called generalised function.

2. (The Dirac delta) For $x \in \Omega$, the Dirac distribution δ_x is defined via

$$\delta_x \phi = \phi(x).$$

We now show that the wish-list properties above are satisfied.

Proposition 5. *Suppose that $f \in C^0(\Omega)$. Let $\rho_{k,y}, k = 1, 2, \ldots,$ be a family of mollifiers concentrating at y, i.e., $\rho_{k,y} \in C_c^\infty(\Omega), \rho_{k,y} \geq 0$ and*

$$\int_\Omega \rho_{k,y}(x)\, dx = 1, \quad \text{supp}(\rho_{k,y}) \subset B_{1/k}(y).$$

Then

$$f(y) = \lim_{k\to\infty} T_f(\rho_{k,y}).$$

As a consequence, if $f, g \in C^0(\Omega)$ *and* $T_f = T_g$ *then* $f = g$.

Proof. Since $\int_\Omega \rho_{k,y}(x)\,dx = 1$, we have that

$$f(y) - T_f(\rho_{k,y}) = \int_\Omega (f(y) - f(x))\,\rho_{k,y}(x)\,dx.$$

Since f is continuous, given $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|f(y) - f(x)| < \varepsilon$ for $x \in B_{1/k}(y)$ whenever $k \geq K$. For such a k, we have

$$|f(y) - T_f(\rho_{k,y})| \leq \int_\Omega |f(y) - f(x)|\,\rho_{k,y}(x)\,dx \leq \varepsilon \int_\Omega \rho_{k,y}(x)\,dx = \varepsilon,$$

which implies that $f(y) = \lim_{k\to\infty} T_f(\rho_{k,y})$.

Now suppose that $f, g \in C^0(\Omega)$ and $T_f = T_g$. For any $y \in \Omega$, let $\rho_{k,y}$ be the mollifiers concentrating at y. Then

$$f(y) = \lim_{k\to\infty} T_f(\rho_{k,y}) = \lim_{k\to\infty} T_g(\rho_{k,y}) = g(y),$$

so that $f = g$ as claimed. □

Next, we extend the operations on functions to distributions as follows.

1) Given $T_1, T_2 \in \mathcal{D}'(\Omega)$. The distribution $T_1 + T_2$ is defined by

$$(T_1 + T_2)(\phi) := T_1(\phi) + T_2(\phi), \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

2) Given $a \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, the distribution aT is defined by

$$(aT)(\phi) := T(a\phi), \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

Note that the right-hand side makes sense since $a\phi \in \mathcal{D}(\mathbb{R}^n)$.

3) First, note that if $f \in C^\infty(\Omega)$, then $D_i f \in C^\infty(\Omega)$ and

$$T_{D_i f}(\phi) = \int_\Omega D_i f(x)\,\phi(x)\,dx = -\int_\Omega f(x)\,D_i\phi(x)\,dx = -T_f(D_i\phi),$$

Note that in the above computations, there are no boundary terms because ϕ vanishes at the infinity. Motivated by this, we define the derivatives of a distribution as follows. For any multi-index α, we define

$$(D^\alpha T)(\phi) := (-1)^{|\alpha|} T(D^\alpha \phi).$$

Let us do some examples

Example 21. 1) $T = \delta_x$. Then

$$(D_iT)(\phi) = -T(D_i\phi) = -\int_\Omega \delta_x D_i\phi(y)\,dy = -(D_i\phi)(x), \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

2) Consider the Heaviside step function

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases} \qquad (x \in \mathbb{R}).$$

Note that this function is not differentiable at $x = 0$. We consider the distribution T_H by

$$T_H(\phi) = \int_{\mathbb{R}} H(x)\phi(x)\,dx, \quad \text{for} \quad \phi \in \mathcal{D}(\mathbb{R}^n).$$

Then we compute

$$\begin{aligned}(D_xT_H)(\phi) &= -T_H(D_x\phi) \\ &= -\int_{\mathbb{R}} H(x)D_x\phi(x)\,dx \\ &= -\int_0^\infty D_x\phi(x)\,dx \\ &= \phi(0) = \delta_0(\phi).\end{aligned}$$

Therefore, $D_xT_H = \delta_0$.

So the theory of distributions provides us some sort of meaning to the derivative of a function whose classical derivatives do note exist.

Theorem 14. *Suppose that $f \in C^0(\Omega)$ and let T_f be the distribution associated to f. Suppose further that for any multi-index α, with $|\alpha| \leq k$, there exist $g_\alpha \in C^0(\Omega)$ such that*

$$D^\alpha T_f = T_{g_\alpha}$$

where D^α is the distributional derivative. Then $f \in C^k(\Omega)$ and $D^\alpha f = g_\alpha$ in the classical sense.

This theorem states that the distributional derivatives agree with the classical ones when they are both defined.

We have addressed the four wish-lists that we asked in the introduction. It is straightforward to check that all the operations that we have defined are indeed continuous with respect to the topology that we have introduced.

The advantage of introducing distributions is expressed in the following result, which is followed from Theorem 14 and all the properties we have defined.

Proposition 6. *Suppose that $f \in C^0(\Omega)$ and that $T \in \mathcal{D}'(\mathbb{R}^n)$ satisfies the distributional equation*

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha T = T_f,$$

where $a_\alpha \in C^\infty(\Omega)$ and that there exist functions $u_\alpha \in C^0(\Omega)$ such that $D^\alpha T = u_\alpha$. Then $u := u_0$ is in $C^k(\Omega)$ and is a classical solution of the PDE

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha u = f. \tag{4.2}$$

This proposition provides us a method to solve the PDE (4.2) that consists of two steps.

1) Lift up the PDE (4.2) to the distributional PDE

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha T = T_f$$

and solve this distributional equation.

2) Check whether a distributional solution is a classical one.

In the next section, we will study how to solve the distributional equation above.

4.2 Convolutions and the fundamental solution

Recalling that for $f \in C^0(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$, the convolution $f * \phi$ is again a function defined by

$$(f * \phi)(x) := \int_{\mathbb{R}^n} f(y)\phi(x-y)\, dy.$$

How to extend this definition to $T * \phi$ where $T \in \mathcal{D}'(\mathbb{R}^n), \phi \in C_c^\infty(\mathbb{R}^n)$? To do so we define

$$\begin{aligned} \tau_x\phi : \mathbb{R}^n &\to \mathbb{R} \\ y &\mapsto \phi(x-y). \end{aligned}$$

If $\phi \in C_c^\infty(\mathbb{R}^n)$ then so is $\tau_x\phi$. Now we can write

$$(f * \phi)(x) = T_f(\tau_x\phi).$$

The advantage of writing this way is that we can extend it to distributions: let $T \in \mathcal{D}'(\Omega), \phi \in \mathcal{D}(\mathbb{R}^n)$, we define

$$(T * \phi)(x) := T(\tau_x\phi).$$

Before stating some important properties of convolutions, we need to introduce the concept of support of a distribution.

Definition 9 (Support of a distribution)**.** A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is supported in the closed set $K \subset \Omega$ if

$$T\phi = 0 \quad \forall \phi \in C_c^\infty(\Omega \setminus K),$$

i.e, T vanishes when applying to any test function that vanishes on K. The support of T is defined by

$$\text{suppT} = \bigcap \{\text{K : T supported in K}\}.$$

Example if $T = \delta_x$ then suppT = {x}.

Proposition 7 (Properties of convolutions)**.** *Suppose $T, T_i \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then*

(i) If $T_1 * \phi = T_2 * \phi, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$ then $T_1 = T_2$.

(ii) $T * \phi \in C^\infty(\mathbb{R}^n)$ and

$$D^\alpha(T * \phi) = T * D^\alpha\phi = D^\alpha T * \phi.$$

(iii) If T has compact support, then $T * \phi$ also has compact support; in particular $T * \phi$ is a test function.

Proof. (i) We have $(\tau_x\phi)(y) = \phi(x - y)$, so that $\phi(-y) = (\tau_0\phi)(y)$, or equivalently, $\phi = \tau_0\tilde{\phi}$, where $\tilde{\phi}(y) := \phi(-y)$. Then we have

$$T_1(\phi) = T_1(\tau_0\tilde{\phi}) = (T_1 * \tilde{\phi})(0) = (T_2 * \tilde{\phi})(0) = T_2(\phi),$$

which means that $T_1 = T_2$ as desired.

(ii) We first show that $D_i(T * \phi) = T * D_i\phi$. In fact, let e_i be the i-th unit vector, we calculate

$$\begin{aligned}\frac{(T*\phi)(x+\varepsilon e_i)-(T*\phi)(x)}{\varepsilon} &= \frac{1}{\varepsilon}\Big[T(\tau_{x+\varepsilon e_i}\phi)-T(\tau_x\phi)\Big]\\ &= T\Big[\frac{1}{\varepsilon}(\tau_{x+\varepsilon e_i}\phi-\tau_x\phi)\Big],\end{aligned}$$

where we have used the linearity of T. Since $\phi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\tau_{x+he_i}\phi-\tau_x\phi)(y) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\phi(x-y+\varepsilon e_i)-\phi(x-y))\\ &= \frac{\partial}{\partial x_i}\phi(x-y) = (\tau_x D_i\phi)(y),\end{aligned}$$

with convergence in the topology of $\mathcal{D}(\mathbb{R}^n)$. By the continuity of distributions, we have

$$\lim_{\varepsilon\to 0}\frac{(T*\phi)(x+\varepsilon e_i)-(T*\phi)(x)}{\varepsilon} = T(\tau_x D_i\phi) = (T*D_i\phi)(x),$$

i.e., $D_i(T*\phi) = T*D_i\phi$. By repeating this arguments for higher derivatives, we establish the first equality. To get the second equality, we calculate

$$\begin{aligned}D^\alpha(\tau_x\phi)(y) &= \frac{\partial^{|\alpha|}}{\partial y^\alpha}\phi(x-y)\\ &= (-1)^{|\alpha|}(D^\alpha\phi)(x-y)\\ &= (-1)^{|\alpha|}(\tau_x D^\alpha\phi)(y),\end{aligned}$$

so that $D^\alpha(\tau_x\phi) = (-1)^{|\alpha|}(\tau_x D^\alpha\phi)$.

Therefore

$$\begin{aligned}(D^\alpha T*\phi)(x) &= D^\alpha T[\tau_x\phi]\\ &= (-1)^{|\alpha|}T(D^\alpha\tau_x\phi)\\ &= (-1)^{|\alpha|}T((-1)^{|\alpha|}\tau_x D^\alpha\phi)\\ &= T(\tau_x D^\alpha\phi)\\ &= T*D^\alpha\phi(x).\end{aligned}$$

Hence $D^\alpha T*\phi = T*D^\alpha\phi$.

(iii) Suppose without loss of generality that T and ϕ are supported in $B_R(0)$ for some large R. Let $x \in \mathbb{R}^n$ be such that $|x| > 2R$. Then if $y \in B_R(0)$, by the triangle inequality, $|y-x| > R$. So that $(\tau_x\phi)(y) = \phi(x-y) = 0$, i.e., $\tau_x\phi \in C_c^\infty(\mathbb{R}^n \setminus B_R(0))$. By the

assumption that T is supported in $B_R(0)$, we have that $T(\tau_x\phi) = (T * \phi)(x) = 0$, hence $T * \phi(x) = 0$ for all $|x| > 2R$, i.e., $T * \phi$ has compact support.

□

We have extended addition, multiplication, derivatives and convolution of a distribution with a test function. Next, we want to extend the convolution of two distributions: given $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$, how to define $T_1 * T_2$? We recall that if $f, g, h \in C_c^0(\mathbb{R}^n)$, then

$$(f * g) * h = f * (g * h).$$

In deed,

$$\begin{aligned}
[(f * g) * h](x) &= \int_{\mathbb{R}^n} (f * g)(y) h(x - y)\, dy \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(z) g(y - z)\, dz \right] h(x - y)\, dy \\
&= \int_{\mathbb{R}^n} f(z) \left[\int_{\mathbb{R}^n} g(y - z) h(x - y)\, dy \right] dz \\
&= \int_{\mathbb{R}^n} f(z) \left[\int_{\mathbb{R}^n} g(\tilde{y}) h(x - \tilde{y} - z)\, d\tilde{y} \right] dz \\
&= \int_{\mathbb{R}^n} f(z) (g * h)(x - z)\, dz \\
&= (f * (g * h))(x).
\end{aligned}$$

Motivated by this we define

Definition 10. Suppose that $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$ and that T_2 has compact support. The convolution $T_1 * T_2$ is defined to be the unique distribution that satisfies

$$(T_1 * T_2) * \phi = T_1 * (T_2 * \phi), \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

Note that in the definition above, the requirement that T_2 has compact support ensures that $T_2 * \phi$ is a test function according to the last property in Proposition 7.

Theorem 15. *Suppose that $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$ and that T_2 has compact support. Then*

$$D^\alpha(T_1 * T_2) = T_1 * D^\alpha T_2 = D^\alpha T_1 * T_2.$$

Proof. We have

$$\begin{aligned} D^\alpha(T_1 * T_2) * \phi &= (T_1 * T_2) * D^\alpha \phi \\ &= T_1 * (T_2 * D^\alpha \phi) \\ &= T_1 * (D^\alpha T_2 * \phi) \\ &= (T_1 * D^\alpha T_2) * \phi \end{aligned}$$

so that $D^\alpha(T_1 * T_2) = (T_1 * D^\alpha T_2)$. Similarly,

$$\begin{aligned} (T_1 * D^\alpha T_2) * \phi &= T_1 * (D^\alpha T_2 * \phi) \\ &= T_1 * D^\alpha (T_2 * \phi) \\ &= D^\alpha T_1 * (T_2 * \phi) \\ &= (D^\alpha T_1 * T_2) * \phi, \end{aligned}$$

so that $(T_1 * D^\alpha T_2) = (D^\alpha T_1 * T_2)$ and we are done. □

Example 22. (i) If $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $\delta_0 * \phi = \phi$.

(ii) If $T \in \mathcal{D}'(\mathbb{R}^n)$ has compact support then $\delta_0 * T = T$.

Definition 11 (Definition of the fundamental solution)**.** We say that a distribution G is a fundamental solution of the partial differential operator

$$L := \sum_{|\alpha| \leq k} a_\alpha D^\alpha$$

where a_α are constants, if it satisfies the distributional equation

$$LG = \delta_0.$$

Lemma 5. *Suppose that $G \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of L and that T_0 is a distribution with compact support. Then the distribution $G * T_0$ solves the distributional equation*

$$LT = \sum_{|\alpha| \leq k} a_\alpha D^\alpha T = T_0.$$

Proof. We have

$$\begin{aligned} L(G * T_0) &= \sum_{|\alpha| \le k} a_\alpha D^\alpha (G * T_0) \\ &= \sum_{|\alpha| \le k} a_\alpha (D^\alpha G * T_0) \\ &= \Big(\sum_{|\alpha| \le k} a_\alpha D^\alpha G \Big) * T_0 \\ &= (LG) * T_0 \\ &= \delta_0 * T_0 = T_0. \end{aligned}$$

□

4.3 Poisson's equation

We will apply the abstract framework in the previous section to solve Poisson's equation: given $f : \mathbb{R}^n \to \mathbb{R}$ has compact support. Find u such that

$$\Delta u = f \quad \text{in} \quad \mathbb{R}^n.$$

First of all, we notice that, for $n \ge 3$, classical solutions to this problem are unique (if they exist) if we assume that $u \to 0$ as $|x| \to \infty$. Indeed, suppose that u_1, u_2 satisfy both Poisson's equation and the decay condition. Then $w = u_1 - u_2$ is harmonic in $\mathbb{R}^n$ and is bounded. By Liouville's theorem, w must be a constant. Since $w \to 0$ as $|x| \to \infty$, the constant must be 0. Hence, $u_1 = u_2$.

Next, we will show that provided $\rho \in C_c^2(\mathbb{R}^n)$, a classical solution satisfying the decay condition indeed exists. The procedure follows from the abstract framework

1. We first show that a distributional solution exists using the fundamental solution.

2. Then we show that the distributional solution arises from a C^2 function. This function will be the classical solution to Poisson's equation.

4.3.1 The fundamental solution

Recall that in Chapter 3, we have defined the function

$$\Phi(x) = \begin{cases} \frac{1}{2\pi} \log |x|, & n = 2, \\ \frac{1}{(2-n)\sigma_{n-1}} |x|^{2-n}, & n \geq 3. \end{cases}$$

Furthermore, for any $\phi \in C_c^\infty(\mathbb{R}^n)$ we have that

$$\phi(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \Phi(x-y) \Delta\phi(y)\, dy.$$

By changing variable $z = x - y$, the above identity can be written as

$$\phi(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(z) \Delta\phi(x-z)\, dz = (\mathcal{G} * \Delta\phi)(x),$$

where the distribution $\mathcal{G}$ is defined by

$$\mathcal{G}\phi = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(z) \Delta\phi(z)\, dz.$$

So we have $\phi = \mathcal{G} * \Delta\phi = \Delta\mathcal{G} * \phi$, which means that $\Delta\mathcal{G} = \delta_0$, i.e., $\mathcal{G}$ is the fundamental solution to Poisson's equation.

As a consequence, if $f \in C_c^0(\mathbb{R}^n)$, then

$$\Delta(\mathcal{G} * T_f) = T_f$$

i.e., $\mathcal{G} * T_f$ is a solution to the distributional equation $\Delta T = T_f$.

Lemma 6. *Suppose $f \in C_c^k(\mathbb{R}^n)$. Then*

$$\mathcal{G} * T_f = T_u,$$

where

$$u(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \Phi(x-y)\, f(y)\, dy$$

and $u \in C^k(\mathbb{R}^n)$.

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$ and $R > 0$ sufficiently large that $R > |x|$ and that

$\mathrm{supp}(\phi) \cup \mathrm{supp(f)} \subset \mathrm{B_R}(0)$. We calculate

$$\begin{aligned}
[(\mathcal{G} * T_f) * \phi](x) &= [G * (T_f * \phi)](x) \\
&= \mathcal{G}[\tau_x(T_f * \phi)] \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)[\tau_x(T_f * \phi)](y)\, dy \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)(T_f * \phi)(x-y)\, dy \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y)\Big[\int_{\mathbb{R}^n} f(z)\phi(x-y-z)\, dz\Big]\, dy \\
&= \lim_{\varepsilon \to 0} \int_{B_{3R}(0) \setminus B_\varepsilon(0)} \Phi(y)\Big[\int_{B_R(0)} f(z)\phi(x-y-z)\, dz\Big]\, dy \\
&= \lim_{\varepsilon \to 0} \int_{B_{3R}(0) \setminus B_\varepsilon(0)} \Phi(y)\Big[\int_{B_{4R}(0)} f(w-y)\phi(x-w)\, dw\Big]\, dy \\
&= \lim_{\varepsilon \to 0} \int_{B_{4R}(0)} \Big[\int_{B_{3R}(0) \setminus B_\varepsilon(0)} \Phi(y) f(w-y)\, dy\Big]\phi(x-w)\, dw \\
&= \lim_{\varepsilon \to 0} \int_{B_{2R}(0)} \Big[\int_{\mathbb{R}^n \setminus B_\varepsilon(w)} \Phi(w-z) f(z)\, dz\Big]\phi(x-w)\, dw
\end{aligned}$$

Define

$$u_\varepsilon(w) = \int_{\mathbb{R}^n \setminus B_\varepsilon(w)} \Phi(w-z) f(z)\, dz.$$

Then due to the property of Φ, we have that $u_\varepsilon \to u$ uniformly in w, so that we can take the limit inside the integral above and obtain

$$\begin{aligned}
[(\mathcal{G} * T_f) * \phi](x) &= \int_{B_{2R}(0)} \Big[\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(w)} \Phi(w-z) f(z)\, dz\Big]\phi(x-w)\, dw \\
&= \int_{B_{2R}(0)} u(w)\phi(x-w)\, dw \\
&= (T_u * \phi)(x),
\end{aligned}$$

so that $\mathcal{G} * T_f = T_u$.

Next we show that $u \in C^k(\mathbb{R}^n)$. Since $f \in C_c^k(\mathbb{R}^n)$, we have

$$u_\varepsilon(x) = \int_{\mathbb{R}^n \setminus B_\varepsilon(w)} \Phi(w-z) f(z)\, dz = \int_{B_R(0) \setminus B_\varepsilon(0)} \Phi(y) f(x-y)\, dy.$$

If $|\alpha| \le k$, then

$$\begin{aligned}
D^\alpha u_\varepsilon &= \int_{B_{2R}(0) \setminus B_\varepsilon(0)} \Phi(y) D^\alpha f(x-y)\, dy \\
&= \int_{B_R(0) \setminus B_\varepsilon(0)} \Phi(x-y) D^\alpha f(y)\, dy \\
&= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(x-y) D^\alpha f(y)\, dy
\end{aligned}$$

which converges uniformly to some limit by our previous results. Since u_ε and $D^\alpha u_\varepsilon$ converge uniformly for $|\alpha| \leq k$, we conclude that $u \in C^k(\mathbb{R}^n)$. $\square$

Now we are ready to state the following theorem on the existence of a classical solution satisfying the decay condition.

Theorem 16. *If $f \in C_c^2(\mathbb{R}^n)$ then*

$$u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \Phi(x-y)\, f(y)\, dy$$

is a classical to Poisson's equation. Furthermore, if $n \geq 3$ then $u \to 0$ as $|x| \to \infty$.

Proof. By Lemma 6 $u \in C^2(\mathbb{R}^n)$ and T_u is a distributional solution of Poisson's equation. By Proposition 6, u is a classical solution of Poisson's equation. Now let $n \geq 3$. Let $R > 0$ be such that $\mathrm{supp(f)} \subset \mathrm{B_R}(0)$. For $|x| > 2R$ and $y \in B_R(0)$ then $|y| \leq \frac{|x|}{2}$, so that $|x-y| > \frac{|x|}{2}$. We then can estimate

$$|u(x)| = \left| \int_{B_R(0)} \Phi(x-y)\, f(y)\, dy \right| \leq \frac{\mathrm{Vol(B_R(0))}}{(n-2)\sigma_{n-1}} \left(\frac{2}{|x|} \right)^{n-2} \sup_{\mathbb{R}^n} |f|,$$

which implies that $u(x) \to 0$ as $|x| \to \infty$.

4.3.2 Poisson's equation in a bounded domain

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and satisfies the barrier postulate. Poisson's equation in Ω is: given $f \in C_c^2(\Omega)$ and $g \in C^0(\partial\Omega)$, find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = g, & \text{in } \partial\Omega. \end{cases} \tag{4.3}$$

This problem can be solved in two steps as follows. Each step is a problem that we know how to solve.

Step 1) Find $w \in C^2(\mathbb{R}^n)$ solve $\Delta w = f$ in $\mathbb{R}^n$.

Step 2) Find $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = g - w, & \text{in } \partial\Omega. \end{cases}$$

Then $u = w + v$ solve (4.3). Indeed, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and

$$\Delta u = \Delta w + \Delta v = f \quad \text{in } \Omega, u = v + w = g - w + w = g \quad \text{on } \partial\Omega.$$

□

Problem Sheet 4

Exercise 17. Consider the following domain of $\mathbb{R}^2$, defined using polar coordinates (r, φ):

$$\Omega_0 := \{(r, \varphi) : \theta < \varphi < 2\pi - \theta; \quad 0 < r\}$$

where $\theta \in (0, \pi)$.

a) Determine, with justification, for which values of θ the exterior sphere condition is satisfied for *every point* of $\partial\Omega_0$. (Recall: Suppose that Ω is open and bounded and that $y \in \partial\Omega$. If there exists $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$ then we say that the exterior sphere condition holds at y.)

b) Show that a barrier function exists at all points of $\partial\Omega_0$, *even when the exterior sphere conditions does not hold.* [**Hint**: Consider the expression for the Laplacian on $\mathbb{R}^2$ in polar coordinates.]

c) An open domain Ω of $\mathbb{R}^2$ is said to satisfy the exterior cone condition at a point $\xi \in \partial\Omega$ if there exists a cone C with vertex ξ and a ball $B_r(\xi)$ such that

$$\overline{\Omega} \cap B_r(\xi) \cap C = \{\xi\}$$

Prove using the idea for b) that if the exterior cone condition is satisfied at ξ the a barrier function exists for ξ.

Exercise 18. a) Show that if $f_1, f_2 \in C^0(\Omega)$ and $a \in C^\infty(\Omega)$ then

$$aT_{f_1} + T_{f_2} = T_{af_1+f_2}.$$

b) Show that if $f \in C^k(\Omega)$ then

$$D^\alpha T_f = T_{D^\alpha f}$$

for $|\alpha| \leq k$.

c) Deduce that if $f \in C^k(\Omega)$ then

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha T_f = T_{Lf},$$

where

$$Lf = \sum_{|\alpha| \leq k} a_\alpha D^\alpha f.$$

Exercise 19. a) Show that if $f, g, h \in C_c^0(\mathbb{R}^n)$ then

$$T_{f*g} = T_f * T_g.$$

b) Show that convolution is linear in both of its arguments, i.e, if $T_i \in \mathcal{D}'(\mathbb{R}^n)$ and T_3, T_4 have compact support then

$$(T_1 + aT_2) * T_3 = T_1 * T_3 + aT_2 * T_3,$$

and

$$T_1 * (T_3 + aT_4) = T_1 * T_3 + aT_1 * T_4$$

where $a \in \mathbb{R}$ is a constant.

Chapter 5

The heat equation

In this chapter we will study the heat equation: given $u_0 : \mathbb{R}^n \to \mathbb{R}$. We want to find a function $u : \mathbb{R}^n \times [0, \infty)$ that satisfies the following *heat equation*

$$\partial_t u(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^n, t > 0, \tag{5.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{5.2}$$

Note that Δ is the Laplacian operator with respect to the spatial variable only. The heat equation describes the evolution of the temperature of a homogeneous body given the temperature profile at the initial time.

The procedure to solve this equation will be similar as in the previous chapter: first, we construct the fundamental solution, and then taking the convolution of the fundamental solution with the initial data.

5.1 The fundamental solution of the heat equation

We first have two observations.

Lemma 7. *If $u(x,t)$ satisfies (5.1) then so does $v(x,t) := u(\lambda x, \lambda^2 t)$ for any $\lambda > 0$.*

Proof. By the chain rule we have

$$\partial_t v(x,t) = \lambda^2 \partial_t u(\lambda x, \lambda^2 t) \text{ and } \Delta v(x,t) = \lambda^2 \Delta u(\lambda x, \lambda^2 t).$$

Thus $\partial_t v(x,t) = \lambda^2 [\partial_t u - \Delta u](\lambda x, \lambda^2 t) = 0$ as desired. $\square$

Lemma 8. *If u satisfies* (5.1) *and Du vanishes at the infinity then for any $t > 0$*

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx.$$

Proof. We calculate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\,dx &= \int_{\mathbb{R}^n} \partial_t u(x,t)\,dx \\ &= \int_{\mathbb{R}^n} \Delta u(x,t)\,dx \\ &= \int_{\mathbb{R}^n} \mathrm{div(Du(x,t))\,dx} \\ &= 0,\end{aligned}$$

where in the last equality we have applied the divergence theorem and the assumption that Du vanishes at the infinity. □

Motivated by these two observations, we will seek a *self-similar* solution of the heat equation of the form

$$u(x,t) = h(t)f(\xi) \quad \text{where} \quad \xi = \frac{|x|^2}{t} \in \mathbb{R},$$

for some function h and f, such that $\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx = 1$. The last identity is assumed without loss of generality.

We first find h. We have

$$\begin{aligned}1 = \int_{\mathbb{R}^n} u(x,t)\,dx &= h(t)\int_{\mathbb{R}^n} f\Big(\frac{|x|^2}{t}\Big)\,dx \\ &= h(t)\int_{S^{n-1}}\int_0^\infty f\Big(\frac{r^2}{t}\Big) r^{n-1}\,dw\,dr.\end{aligned}$$

By changing of variable $y = \frac{r^2}{t}$, we have $r = (yt)^{\frac{1}{2}}$ and $dr = \frac{t}{2r}\,dy = \frac{t^{\frac{1}{2}}}{2y^{\frac{1}{2}}}\,dy$. Substituting this back into the calculations above, we get

$$1 = t^{\frac{n}{2}} h(t)\frac{\sigma_{n-1}}{2}\int_0^\infty f(y)y^{\frac{n}{2}-1}\,dy.$$

This implies that

$$h(t) = t^{-\frac{n}{2}} \quad \text{and} \quad 1 = \frac{\sigma_{n-1}}{2}\int_0^\infty f(y)y^{\frac{n}{2}-1}\,dy.$$

It remains to find f. We have

$$\partial_t u = \partial_t \left(t^{-\frac{n}{2}} f\Big(\frac{|x|^2}{2}\Big)\right) = -\frac{n}{2} t^{-\frac{n}{2}-1} f(\xi) - t^{-\frac{n}{2}} \frac{|x|^2}{t^2} f'(\xi) = -t^{-\frac{n}{2}-1}\left[\frac{n}{2} f(\xi) + \xi f'(\xi)\right]$$

$$D_i u = 2t^{-\frac{n}{2}} \frac{x_i}{t} f'(\xi),$$

$$D_{ii}^2 u = \frac{2t^{\frac{n}{2}}}{t} f'(\xi) + 4t^{-\frac{n}{2}} \frac{x_i^2}{t^2} f''(\xi)$$

$$\Delta u = \sum_{i=1}^{n} D_{ii}^2 u = t^{-\frac{n}{2}-1}\left(2nf'(\xi) + 4\xi f''(\xi)\right).$$

Therefore, we obtain the following equation for $f(\xi)$

$$4\xi f''(\xi) + \xi f'(\xi) + 2nf'(\xi) + \frac{n}{2} f(\xi) = 0.$$

Set $g(\xi) = f'(\xi) + \frac{1}{4} f(\xi)$, the above equation can be written as

$$\xi g'(\xi) + \frac{n}{2} g(\xi) = 0,$$

which implies that

$$g(\xi) = A\xi^{-\frac{n}{2}}$$

for some constant A. Therefore,

$$f'(\xi) + \frac{1}{4} f(\xi) = A\xi^{-\frac{n}{2}},$$

i.e.,

$$A = \xi^{\frac{n}{2}}(f'(\xi) + \frac{1}{4} f(\xi)).$$

Assume that f and f' decay fast enough at the infinity, we obtain that the right-hand side vanishes as $|\xi| \to \infty$. Thus $A = 0$ and so

$$f'(\xi) + \frac{1}{4} f(\xi) = 0.$$

By multiplying with the integrating factor $e^{\frac{\xi}{4}}$, we obtain

$$\frac{d}{d\xi}\Big(e^{\frac{\xi}{4}} f(\xi)\Big) = 0$$

Solving this equation, we find

$$f(\xi) = Be^{-\frac{\xi}{4}}. \tag{5.3}$$

We now find B. We have

$$\begin{aligned}
1 &= B\frac{\sigma_{n-1}}{2}\int_0^\infty e^{-\frac{\xi}{4}}\xi^{\frac{n}{2}-1}\,d\xi \\
&= B\sigma_{n-1}4^{\frac{n}{2}}\int_0^\infty e^{-s^2}s^{n-1}\,ds \quad \text{(change of variable } \ \xi = 4s^2) \\
&= B4^{\frac{n}{2}}\int_{\mathbb{R}^n} e^{-|x|^2}\,dx \\
&= B\left(2\int_0^\infty e^{-x^2}\,dx\right)^n = B(2\sqrt{\pi})^{\frac{n}{2}}.
\end{aligned}$$

so $B = \frac{1}{(2\sqrt{\pi})^n}$. Conclusion

$$u(x,t) = t^{-\frac{n}{2}}\frac{1}{(2\pi)^n}e^{-\frac{|x|^2}{4t}} = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}.$$

To summarise, we have proved the following result

Proposition 8. *The function*

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

is a solution of the equation (5.1) *and satisfies*

$$\int_{\mathbb{R}^n}\Gamma(x,t)\,dx = 1, \quad \forall t > 0.$$

The function Γ is called the *heat kernel* or the *fundamental solution of the heat equation.* It plays a similar role to the Green's function for the Laplace operator, which can be seen in the following theorem.

Theorem 17. *Let $u_0 \in C_c^0(\mathbb{R}^n)$. Define $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by*

$$u(x,t) := \int_{\mathbb{R}^n}\Gamma(x-y,t)u_0(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}u_0(y)\,dy. \tag{5.4}$$

Then u solves the heat equation (5.1)-(5.2)*. That is, u satisfies*

$$\partial_t u(x,t) = \Delta u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty),$$

and

$$\lim_{t\to 0} u(x,t) = u_0(x) \quad \forall x \in \mathbb{R}^n.$$

Proof. According to Proposition 8, $\Gamma(x-y,t)$ satisfies

$$\partial_t\Gamma(x-y,t) = \Delta\Gamma(x-y,t).$$

Furthermore, as a function of x and t, $\Gamma(x-y,t)$ is smooth in $\mathbb{R}^n \times (0,\infty)$. Suppose that u_0 is supported in $B_R(0)$. Then

$$u(x,t) = \int_{B_R(0)} \Gamma(x-y,t)u_0(y)\,dy.$$

This implies that $u(x,t)$ is also smooth in $\mathbb{R}^n \times (0,\infty)$ and we can differentiate under the integral and obtain that

$$(\partial_t u - \Delta u)(x,t) = \int_{B_R(0)} (\partial_t - \Delta_x)\Gamma(x-y,t)u_0(y)\,dy = 0 \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty).$$

We now verify the initial condition. Since u_0 is continuous and compactly supported, it is uniformly continuous. Therefore, given any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, z \in \mathbb{R}^n$ with $|x-z| < \delta$ then $|u_0(x) - u_0(z)| < \varepsilon$. Since

$$\int_{\mathbb{R}^n} \Gamma(x-y,t)\,dy = 1,$$

we can write

$$u(x,t) - u_0(x) = \int_{\mathbb{R}^n} \Gamma(x-y,t)(u_0(y) - u_0(x))\,dy$$

and estimate

$$\begin{aligned}
|u(x,t) - u_0(x)| &\le \int_{\mathbb{R}^n} \Gamma(x-y,t)|u_0(y) - u_0(x)|\,dy \\
&= \int_{B_\delta(x)} \Gamma(x-y,t)|u_0(y) - u_0(x)|\,dy + \int_{\mathbb{R}^n \setminus B_\delta(x)} \Gamma(x-y,t)|u_0(y) - u_0(x)|\,dy. \\
&:= I_1 + I_2.
\end{aligned}$$

Since for $y \in B_\delta(x)$, we have $|u_0(y) - u_0(x)| < \varepsilon$, the term I_1 can be estimated

$$I_1 \le \varepsilon \int_{B_\delta} \Gamma(x-y,t)\,dy \le \varepsilon \int_{\mathbb{R}^n} \Gamma(x-y,t)\,dy = \varepsilon.$$

The term I_2 can be estimated as follows

$$\begin{aligned}
I_2 &\le 2\sup_{\mathbb{R}^n} |u_0| \int_{B_\delta(x)} \Gamma(x-y,t)\,dy \\
&= 2\sup_{\mathbb{R}^n} |u_0| \int_{B_\delta(0)} \Gamma(y,t)\,dy \\
&= 2\sup_{\mathbb{R}^n} |u_0| \frac{\sigma_{n-1}}{(4\pi t)^{\frac{n}{2}}} \int_\delta^\infty e^{-\frac{s^2}{4t}} s^{n-1}\,ds \\
&= 2\sup_{\mathbb{R}^n} |u_0| \frac{\sigma_{n-1}}{(\pi)^{\frac{n}{2}}} \int_{\frac{\delta}{2\sqrt{t}}}^\infty e^{-z^2} z^{n-1}\,dz \quad \to 0 \quad \text{as} \quad t \to 0.
\end{aligned}$$

Therefore, there exists $\delta' > 0$ depending on δ (thus on ε but not on x) such that $I_2 < \varepsilon$ for $t < \delta'$. In conclusion, we have shown that for any $\varepsilon > 0$ there exists $\delta' > 0$ such that for $t < \delta'$ and for any x, we have

$$|u(x,t) - u_0(x)| < 2\varepsilon.$$

So $\lim_{t\to 0} u(x,t) = u_0(x)$ as claimed. □

We now show that the solution of the heat equation is always bounded.

Theorem 18. *Let $u \in C_c^0(\mathbb{R}^n)$ and let $u(x,t)$ be defined as in* (5.4)*. Then*

$$|u(x,t)| \leq \max_{\mathbb{R}^n} |u_0| \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Proof. This estimate is straightforward:

$$|u(x,t)| \leq \int_{\mathbb{R}^n} \Gamma(x-y,t)|u_0(y)|\,dy \leq \max_{\mathbb{R}^n} |u_0| \int_{\mathbb{R}^n} \Gamma(x-y,t)\,dy = \max_{\mathbb{R}^n} |u_0|.$$

□

Furthermore, we now show that the solution of the heat equation decays in time.

Theorem 19. *Let $u \in C_c^0(\mathbb{R}^n)$ and let $u(x,t)$ be defined as in* (5.4)*. Then there exists a constant C depending on n and u_0 such that*

$$|u(x,t)| \leq \frac{C}{t^{\frac{n}{2}}} \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Proof. Suppose that u_0 is supported in $B_R(0)$. Since $\Gamma(x-y,t) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}}$, we can estimate

$$\begin{aligned} |u(x,t)| &\leq \int_{B_R(0)} \Gamma(x-y,t)|u_0(y)|\,dy \\ &\leq \max_{\mathbb{R}^n} |u_0| \int_{B_R(0)} \Gamma(x-y,t)\,dy \\ &\leq \max_{\mathbb{R}^n} |u_0| \frac{1}{(4\pi t)^{\frac{n}{2}}} \mathrm{Vol(B_R(0))} = \frac{\mathrm{C}}{\mathrm{t}^{\frac{\mathrm{n}}{2}}}, \end{aligned}$$

where $C = \max_{\mathbb{R}^n} |u_0| \frac{1}{(4\pi)^{\frac{n}{2}}} \mathrm{Vol(B_R(0)}$. □

Theorem 17 only tells us that the function u defined by (5.4) is a solution to the heat equation, but does not ensure it uniqueness. To obtain a statement about the uniqueness, we will need to use the maximum principle in the next sections.

5.2 The maximum principle for the heat equation on a bounded domain

We will show that the heat equation also satisfies a similar maximum principle as harmonic functions.

Definition 12. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $T > 0$ be given. The *space-time domain* Ω_T is defined by

$$\Omega_T = \Omega \times (0, T).$$

The *parabolic boundary* of Ω_T is defined by

$$\partial_p \Omega_T = \partial(\Omega_T) \setminus (\Omega \times \{T\}).$$

We denote by $C^{2,1}(\Omega_T)$ the set of functions that are continuously differentiable twice in space and once in time. For functions $u \in C^{2,1}(\Omega_T)$, we define the *heat operator* L by

$$(Lu)(x,t) := \partial_t u(x,t) - \Delta u(x,t).$$

We have the following maximum principle.

Theorem 20 (The maximum principle for the heat operator on bounded domains)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded, $T > 0$ and that $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ satisfies*

$$Lu \leq 0 \quad \textit{in} \quad \Omega_T.$$

Then

$$\sup_{\Omega_T} u = \sup_{\partial_p \Omega_T} u.$$

If instead $Lu \geq 0$ in Ω_T, then

$$\inf_{\Omega_T} u = \inf_{\partial_p \Omega_T} u.$$

Proof. Suppose that $Lu \leq 0$. Let $0 < \varepsilon < T$ be arbitrary and let us define the function

$$\begin{aligned} v : \overline{\Omega_{T-\varepsilon}} &\to \mathbb{R} \\ (x,t) &\mapsto u(x,t) - \varepsilon t. \end{aligned}$$

The function v satisfies

$$v(x,t) \leq u(x,t) \leq v(x,t) + \varepsilon T, \quad \text{and} \tag{5.5}$$

$$v_t - \Delta v \leq -\varepsilon < 0 \quad \text{in} \quad \Omega \times (0, T-\varepsilon]. \tag{5.6}$$

Since v is continuous, it attains a global maximum at some point $(x_0, t_0) \in \overline{\Omega}_{T-\varepsilon}$.

1. If $(x_0, t_0) \in \Omega_{T-\varepsilon}$ then we must have that $\partial_t v(x_0, t_0) = 0$ and $\mathrm{Hess_x v(x_0, t_0)} \leq 0$. So

   $$(v_t - \Delta v)(x_0, t_0) = -\Delta v(x_0, t_0) \geq 0,$$

   which contradicts (5.6).

2. If $(x_0, t_0) \in \Omega \times \{T-\varepsilon\}$ then we must have that $v_t(x_0, t_0) \geq 0$ (since v must decrease if we fix x and allow t to decrease) and $\mathrm{Hess_x v(x_0, t_0)} \leq 0$ (since the function $v(T-\varepsilon, x)$ must have a local max at x_0). Therefore,

   $$(v_t - \Delta v)(x_0, t_0) \geq 0,$$

   again contradicts (5.6).

Therefore, we have that $(x_0, t_0) \in \partial_p \Omega_{T-\varepsilon}$ and hence

$$\sup_{\overline{\Omega}_{T-\varepsilon}} v = \sup_{\partial_p \Omega_{T-\varepsilon}} v \leq \sup_{\partial_p \Omega_{T-\varepsilon}} u \leq \sup_{\partial_p \Omega_T} u.$$

On the other hand, since $u \leq v + \varepsilon T$, we have

$$\sup_{\overline{\Omega}_{T-\varepsilon}} u \leq \sup_{\overline{\Omega}_{T-\varepsilon}} v + \varepsilon T \leq \varepsilon T + \sup_{\partial_p \Omega_T} u. \tag{5.7}$$

Since u is uniformly continuous on $\overline{\Omega}_T$, we have that

$$\lim_{\varepsilon \to 0} \sup_{\overline{\Omega}_{T-\varepsilon}} u = \sup_{\overline{\Omega}_T} u.$$

Thus taking $\varepsilon \to 0$ in (5.7), we obtain that

$$\sup_{\overline{\Omega}_T} u = \lim_{\varepsilon \to 0} \sup_{\overline{\Omega}_{T-\varepsilon}} u \leq \lim_{\varepsilon \to 0} (\varepsilon T + \sup_{\partial_p \Omega_T} u) = \sup_{\partial_p \Omega_T} u \leq \sup_{\overline{\Omega}_T} u. \tag{5.8}$$

Therefore, all of the inequalities in (5.8) can be replaced with inequalities. In particular, we get

$$\sup_{\partial_p \Omega_T} u = \sup_{\overline{\Omega}_T} u$$

as desired.

To obtain the result for $Lu \geq 0$, we note that $-u$ satisfies $L(-u) \leq 0$ and apply the first part of the theorem. $\square$

Corollary 5 (The maximum principle for solutions of the heat equation on bounded domain)**.** *Suppose $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ solves the heat equation in Ω_T. Then*

$$\max_{\overline{\Omega}_T} |u| = \max_{\partial_p \Omega_T} |u|.$$

Corollary 6. *Let $f \in C^0(\Omega_T)$ and $g \in C^0(\partial_p \Omega_T)$. Then there is at most one solution $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ of the boundary value problem*

$$\begin{cases} u_t - \Delta u = f \quad in \quad \Omega_T, \\ u = g \quad on \quad \partial_p \Omega_T, \end{cases}$$

Proof. Suppose there are two solutions u_1 and u_2. Then $u = u_1 - u_2$ satisfies the equation

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega_T, \\ u = 0 \quad \text{on} \quad \partial_p \Omega_T, \end{cases}$$

Apply Corollary 5 we obtain that

$$\max_{\overline{\Omega}_T} |u| = \max_{\partial_p \Omega_T} |u| = 0,$$

which implies that $u \equiv 0$, i.e., $u_1 = u_2$ in $\overline{\Omega}_T$. □

Note that unlike the Dirichlet problem for harmonic functions, we are *not* free to prescribe the value of u on *all* of $\partial_p \Omega_T$. The restriction is because of the maximum principle: we can not specify the boundary of u such that u achieves a value on $\Omega \times \{T\}$ that is strictly greater than the value it achieves on $\partial_p \Omega_T$.

5.3 The maximum principle for the heat operator on the whole space

In this section, we show a maximum principle for the heat operator in the whole space. We need to make an assumption on the behaviour of u at the infinity. Denote by $S_T = \mathbb{R}^n \times (0, T)$ the space-time domain in this case.

Theorem 21 (The maximum principle for the heat operator on S_T)**.** *Let $u \in C^{2,1}(S_T) \cap C^0(\overline{S}_T)$ satisfy $Lu \leq 0$ in S_T. In addition, assume that there exist constants $C, \alpha > 0$ such that*

$$u(x,t) \leq C e^{\alpha |x|^2}, \quad \forall x \in \mathbb{R}^n, 0 \leq t \leq T.$$

Then

$$\sup_{S_T} u = \sup_{x\in\mathbb{R}^n} u(x,0).$$

If instead, $Lu \geq 0$ in S_T and there exist constants $C, \alpha > 0$ such that

$$u(x,t) \geq -Ce^{\alpha|x|^2}, \quad \forall x \in \mathbb{R}^n, 0 \leq t \leq T.$$

Then

$$\inf_{S_T} u = \inf_{x\in\mathbb{R}^n} u(x,0).$$

Proof. We will prove the minimum principle. The maximum principle then follows by changing u to $-u$. Suppose that $Lu \geq 0$ in S_T and there exist constants $C, \alpha > 0$ such that

$$u(x,t) \geq -Ce^{\alpha|x|^2}, \quad \forall x \in \mathbb{R}^n, 0 \leq t \leq T.$$

We need to show that

$$\inf_{S_T} u = \inf_{x\in\mathbb{R}^n} u(x,0).$$

If $u(x,0)$ is not bounded from below then the assertion trivially holds. Therefore, we can assume that $u(x,0)$ is bounded from below and define $I := \inf_{x\in\mathbb{R}^n} u(x,0)$. Let $\beta > \alpha$ such that $T_1 := \frac{1}{8\beta} < T$. The idea of the proof is as follows. We first show that $u \geq I$ in S_{T_1} and then by dividing the time interval $[0,T]$ into finitely many intervals of length less than T_1 and successively applying the result of the first step.

We define

$$v(x,t) := \frac{e^{\beta\frac{|x|^2}{1-4\beta t}}}{(1-4\beta t)^{\frac{n}{2}}}.$$

By direct computations, we have

$$\begin{aligned}
\partial_t v &= 4\beta(1-4\beta t)^{-\frac{n}{2}-1}\left(\frac{n}{2} + \frac{\beta|x|^2}{1-4\beta t}\right)e^{\frac{\beta|x|^2}{1-4\beta t}},\\
\partial_{x_i} v &= (1-4\beta t)^{-\frac{n}{2}-1}2\beta x_i e^{\frac{\beta|x|^2}{1-4\beta t}},\\
\partial^2_{x_ix_i} v &= 4\beta(1-4\beta t)^{-\frac{n}{2}-1}\left(\frac{1}{2} + \frac{\beta x_i^2}{1-4\beta t}\right)e^{\frac{\beta|x|^2}{1-4\beta t}},\\
\Delta v &= \sum_{i=1}^n \partial^2_{x_ix_i} v = 4\beta(1-4\beta t)^{-\frac{n}{2}-1}\left(\frac{n}{2} + \frac{\beta|x|^2}{1-4\beta t}\right)e^{\frac{\beta|x|^2}{1-4\beta t}},
\end{aligned}$$

which implies that $v(x,t)$ satisfies the heat equation in S_{T_1}. In addition, since $(1-4\beta t)^{-1} > 1$ in S_{T_1}, we have that

$$v(x,t) \geq e^{\beta|x|^2}.$$

Let $\varepsilon > 0$ and set $w := u + \varepsilon v - I$. To prove $u \geq I$ in S_{T_1}, it suffices to prove that $w \geq 0$ in S_{T_1} for any $\varepsilon > 0$. The function w satisfies that $Lw = Lu + \varepsilon Lv \geq 0$ in S_{T_1} and $w(x,0) = u(x,0) + \varepsilon v(x,0) - I \geq 0$ for all $x \in \mathbb{R}^n$. By the assumption on u we have that

$$w = u + \varepsilon v - I \geq \varepsilon e^{\beta|x|^2} - Ce^{\alpha|x|^2} - I,$$

from where we deduce that

$$\inf_{\partial B_r(0)\times[0,T_1]} \geq \varepsilon e^{\beta r^2} - Ce^{\alpha r^2} - I.$$

Since $\beta > \alpha$ the term $\varepsilon e^{\beta r^2}$ dominates for r large enough, i.e., there exists $R \geq 0$ such that $w \geq 0$ on $\partial B_r(0) \times [0, T_1]$ for all $r > R$.

On the parabolic boundary of the cylinder $B_r(0) \times (0, T_1)$, we have $w \geq 0$. Therefore, applying Theorem 20 to this cylinder, we obtain that

$$\inf_{B_r(0)\times(0,T_1)} w = \inf_{\partial_p(B_r(0)\times(0,T_1))} w \geq 0,$$

so that $w \geq 0$ in $B_r(0) \times (0, T_1)$ for any $r > R$. Thus $w \geq 0$ in S_{T_1} for any $\varepsilon > 0$ and so $u \geq I$. Now dividing the interval $[0, T]$ into finitely many intervals of lengths less than T_1 and successively applying the result on each interval, we obtain

$$u(x,t) \geq \inf_{y\in\mathbb{R}^n} u(y,0)$$

for all $(x,t) \in S_T$. Taking the infimum over S_T we get that

$$\inf_{S_T} u \geq \inf_{x\in\mathbb{R}^n} u(x,0).$$

On the other hand, it is obvious that

$$\inf_{S_T} u \leq \inf_{x\in\mathbb{R}^n} u(x,0).$$

Therefore, we have $\inf_{S_T} u = \inf_{x\in\mathbb{R}^n} u(x,0)$ as desired. □

Corollary 7. *Suppose that $u \in C^{2,1}(S_T) \cap C^0(\overline{S}_T)$ and there exist constants $C, \alpha > 0$ such that*

$$|u(x,t)| \leq Ce^{\alpha|x|^2}, \quad \forall x \in \mathbb{R}^n, 0 \leq t \leq T.$$

Suppose further that u solves the heat equation

$$\partial_t u(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^n, t > 0,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n,$$

where $u_0 \in C^0(\mathbb{R}^n)$ is bounded. Then u is unique.

Proof. Suppose that there are two functions u_1 and u_2 satisfying the hypotheses of the theorem. Define $w := u_1 - u_2$. Then

$$|w(x,t)| \leq |u_1(x,t)| + |u_2(x,t)| \leq Ce^{\alpha|x|^2} + C'e^{\alpha'|x|^2} \leq C''e^{\alpha''|x|^2},$$

where $C'' = 2\max\{C, C'\}$ and $\alpha'' = \max\{\alpha, \alpha'\}$. In addition, w solves

$$\begin{cases} w_t - \Delta w = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T), \\ w = 0 \quad \text{on} \quad \mathbb{R}^n \times \{0\}. \end{cases}$$

Applying Theorem 21 and noting that $Lu = 0$, we conclude that $w = 0$ and hence $u_1 = u_2$ as desired. $\square$

Note that if the growth assumption on u, $|u(x,t)| \leq Ce^{\alpha|x|^2}$ is removed then the uniqueness statement above does not hold. A counter example for this was constructed by Tychonoff in 1935.

Problem Sheet 5

Exercise 20. Suppose that $n \geq 3$ and G is given by

$$G(x) = \frac{1}{\sigma_{n-1}(2-n)}|x|^{2-n}.$$

Let $\rho \in C_c^2(\mathbb{R}^n)$ be supported in $B_R(0)$ and define u by

$$u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} G(x-y)\rho(y)\, dy.$$

Show that for $|\alpha| \leq 2$,

$$\sup_{\mathbb{R}^n} |D^\alpha u| \leq \frac{R^2}{2(n-2)} \sup_{\mathbb{R}^n} |D^\alpha \rho|.$$

Deduce that the problem of finding $u \in C^2(\mathbb{R}^n)$ such that $u \to 0$ as $|x| \to \infty$ satisfying

$$\Delta u = \rho \quad \text{in} \quad \mathbb{R}^n$$

is well-posed.

Exercise 21 (The inhomogeneous heat equation)**.** (a) Let $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$. Show that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} \Gamma(x,t)\phi(x,t)\, dx = \phi(0,0),$$

where Γ denotes the fundamental solution of the heat equation.

Deduce that the function

$$t \mapsto \int_{\mathbb{R}^n} \Gamma(x,t)\phi(x,t)\, dx,$$

is uniformly continuous on $(0,\infty)$ and vanishes for large values of t.

(b) Define the distribution $T \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ by

$$T\phi = \int_0^\infty \int_{\mathbb{R}^n} \Gamma(x,t)\phi(x,t)\, dx\, dt, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}).$$

Let $\delta > 0$. Show that, for any $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$, we have

$$D_t T(\phi) = -\int_0^\delta \int_{\mathbb{R}^n} \Gamma(x,t)\phi_t(x,t)\, dx\, dt + \int_{\mathbb{R}^n} \Gamma(x,\delta)\phi(x,\delta) dx + \int_\delta^\infty \Gamma_t(x,t)\phi(x,t)\, dx\, dt,$$

and

$$D_{x_i x_j} T(\phi) = \int_0^\delta \int_{\mathbb{R}^n} \Gamma(x,t)\phi_{x_i x_j}(x,t)\,dx\,dt + \int_\delta^\infty \Gamma_{x_i x_j}(x,t)\phi(x,t)\,dx\,dt.$$

(c) Deduce that for any $\delta > 0$, we have

$$(D_t T - \Delta T)(\phi) = -\int_0^\delta \int_{\mathbb{R}^n} \Gamma(x,t)(\phi_t + \Delta\phi)(x,t)\,dx\,dt + \int_{\mathbb{R}^n} \Gamma(x,\delta)\phi(x,\delta)\,dx.$$

By bounding the first integral and making use of part (a), show that

$$(D_t T - \Delta T)(\phi) = \phi(0,0).$$

(d) Find a distributional solution to the inhomogeneous heat equation

$$u_t - \Delta u = f \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R},$$

where $f \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$.

Exercise 22. Let $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ be the half-space. Consider the boundary value problem

$$\begin{cases} u_t = \Delta u \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u = u_0 \quad \text{on} \quad \Omega \times \{0\}. \end{cases}$$

Suppose that $u_0 \in C_c^0(\Omega)$. By considering a reflection in the plane $x_n = 0$, find a solution u.

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[1]which was in turn based on notes of Andrea Malchiodi, who lectured this course from 2011-2013

Appendix

We review relevant knowledge about differentiation of functions of several variables.

Suppose Ω is an open subset of $\mathbb{R}^n$. $f : \Omega \to \mathbb{R}^m$ is said to be differentiable at a point $x \in \Omega$ if there exists a linear map, $Df(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that if $h \in \mathbb{R}^n$, we have

$$f(x+h) = f(x) + Df(x) \cdot h + o(h), \quad \text{as} \quad h \to 0. \tag{5.9}$$

We relate this definition to the usual partial derivatives. If $f = (f^i)_{i=1}^m$, i.e., f^i is the i-th component of $f(x)$ with respect t the canonical basic of $\mathbb{R}^m$, we define the partial derivative $\frac{\partial f^i}{\partial x_j}(x)$ to be the limit (if exists)

$$\frac{\partial f^i}{\partial x_j}(x) = \lim_{\varepsilon \to 0} \frac{f(x_1, \ldots, x_j + \varepsilon, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_n)}{\varepsilon}.$$

In other words, the partial derivative of f^i in the x_j direction is simply the derivative of f^i when considering it as a function of x_j alone. By setting $h = \varepsilon e_j$ in (5.9), we see that if f is differentiable at x then $\frac{\partial f^i}{\partial x_j}(x)$ exists for all $i = 1, \ldots, m; j = 1, \ldots, n$. To obtain the conserve, we need a bit stronger condition.

Lemma 9. *Suppose that the partial derivative $\frac{\partial f^i}{\partial x_j}(x)$ exist and are continuous on an open neighbourhood of x, then f is differentiable at x.*

Since $Df(x)$ is a linear map, we can represent it with respect to the canonical bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ as a matrix. Then the partial derivatives of f are the components of this matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \ldots & \frac{\partial f_2}{\partial x_n}(x) \\ \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

or in a compact way

$$[Df(x)]^i_j = \frac{\partial f^i}{\partial x_j}.$$

If f is differentiable at each point in Ω, we can think of Df as a map from Ω to the space of $m \times n$ matrices which is identified to $\mathbb{R}^{m\times n}$

$$Df : \Omega \to \mathbb{R}^{m\times n}$$
$$x \mapsto Df(x).$$

If this map is continuous, the we say that $f \in C^1(\Omega)$. Equivalently $f \in C^1(\Omega)$ if all its partial derivatives exist and are continuous in Ω.

Similarly, we say that $f \in C^2(\Omega)$ if $Df \in C^1(\Omega)$ and so on inductively. We can think of $D^2 f$ as a map from Ω to $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$, then its components are

$$[D^2 f(x)]^i_{j_1 j_2} = \frac{\partial^2 f^i}{\partial x_{j_1} \partial x_{j_2}}(x).$$

Lemma 10. *If $f \in C^k(\Omega)$ if and only if all the partial derivatives*

$$\frac{\partial^k f^i}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_k}} : U \to \mathbb{R}$$

exist and are continuous in Ω.

We now introduce multi-indices notations. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{Z}_{\geq} 0)^n$, we define $|\alpha| := \sum_{i=1}^n \alpha_i$ and

$$\frac{\partial^{|\alpha| f}}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^\alpha_1 \cdots \left(\frac{\partial}{\partial x_n}\right)^\alpha_n,$$

i.e., we differentiate α_1 times with respect to x_1, α_2 times with respect to x_2 and so on. We also use more compact notation

$$D_i := \frac{\partial}{\partial x_i}, \quad D^\alpha := \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Similarly for $h = (h_1, \dots, h_n)$, we denote

$$h^\alpha = h_1^{\alpha_1} \dots h_n^{\alpha_n}.$$

Finally, we introduce the multi-index factorial $\alpha! := \alpha_1! \dots \alpha_n!$.

Theorem 22 (Multivariate Taylor's theorem)**.** *Suppose that Ω is an open subset of $\mathbb{R}^n$, $x \in \Omega$ and $f : U \to \mathbb{R}$ belongs to $C^k(\Omega)$. Then writing $h = (h_j)_{j=1}^n$ and using multi-indices notation, we have*

$$f(x+h) = f(x) + \sum_{|\alpha| \leq k} \frac{h^\alpha}{\alpha!} D^\alpha f(x) + R_k(x,h),$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ *with* $|\alpha| \leq k$ *and the remainder term satisfies*

$$\frac{R_k(x,h)}{|h|^k} \to 0 \quad as \quad h \to 0.$$

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