Divisors

Introduction

- Weil divisors.
- Divisors on curves.
- Cartier Divisors.

→ Riemann-Roch
Jacobian Variety

Introduction:

Let C be a non singular projective curve in $\mathbb{P}^2_k$ (k algebraically closed)

For any line L in $\mathbb{P}^2$,

$L \cap C$ has exactly d points (d is the degree of the C)

(ex I.3.4)

$L \cap C \Rightarrow \sum n_i P_i$ where n_i is the multiplicity of P_i.

$\sum n_i P_i$ is called a divisor on C.

By varying L $\Rightarrow$ get a family of divisors on C, parametrized by the set of lines in $\mathbb{P}^2$.

- $\rightsquigarrow$ set of divisors is called a linear system of divisors on C.

Rk: knowing the linear system of divisors on C, one can recover the embedding of C in $\mathbb{P}^2_k$.

Given a point on C, say P.

consider the set of divisors containing P

$\Rightarrow$ set of lines passing through P

$\Rightarrow$ gives a unique characterisation of P in $\mathbb{P}^2_k$

Consider two lines L, L' in $\mathbb{P}^2_k$, $f=0$, $f'=0$

f/f' is a rational function on $\mathbb{P}^2_k$

which restricts to a rational function g on C.

Let $D = L \cap C$, $D' = L' \cap C$.

- g has zeros at points on D
  poles at points on D'

Group of divisors modulo linear equivalence. $\Rightarrow$ Picard group.

Invariant of the variety.

Weil Divisors:

Let X be a Noetherian, regular in Codimension one, Integral, Separated scheme.

<u>Def</u>: A scheme X is regular in codimension one if every local ring $\mathcal{O}_x$ of X of dimension one is regular.

$\Rightarrow$ $\mathcal{O}_x$ will be a discrete valuation ring.

Examples:

- Nonsingular Variety over a field
- Noetherian normal scheme.

Def: Let X be N.C.I.S.

A prime divisor on X is a

closed integral subscheme Y of codimension one.

A weil divisor is an element of the free abelian group $\mathrm{Div}\, X$ generated by the prime divisors.

We write $D = \sum n_i Y_i$

where Y_i are prime divisors on X

n_i are integers all

but finitely many are zero.

A divisor is <u>Effective</u> if $n_i \geq 0 \ \forall i$

If Y is a prime divisor on X, let
$\eta \in Y$ be its generic point.

The local ring $\mathcal{O}_{\eta,X}$ is a discrete
valuation ring.

with quotient field K

Call the corresponding valuation

v_Y

Let $f \in K^*$ be a nonzero rational function on X

If $v_Y(f)$ is positive say f has a zero along Y of order $v_Y(f)$

If $v_Y(f) < 0$ say f has a pole along Y of order $-v_Y(f)$.

<u>Lemma</u>: Let X be N.C.I.S., $f \in K^*$ then $v_Y(f) = 0$ for all but finitely many prime divisor Y of X.

Pf: let $U = \operatorname{spec} A$ be an open affine subset of X on which f is regular.

let $Z = X - U$. Z is a proper closed subset of X.

X is noetherian $\Rightarrow$ Z must contain finitely many prime divisors of X.

all the other prime divisors must meet $\mathcal{U}$.

$\Rightarrow$ Need to show that $\mathcal{U}$ contains finitely many divisors with $v_Y(f) \neq 0$

But f is regular on $\mathcal{U}$ $\Rightarrow v_Y(f) \geq 0$

If $v_Y(f) > 0 \Rightarrow Y$ is contained in the closed subset of U defined by the ideal $Af \subset A$.

Since $f \neq 0 \Rightarrow$ this is a proper closed subset.

$\Rightarrow$ it contains finitely many closed irreducible subsets of codimension one of U (which are the divisors) $\square$

Def: let X be NCIS, $f \in K^*$

Define the divisor of f

denoted $(f) = \sum_Y v_Y(f)\, Y$

where the sum is taken over all

prime divisors of X.

Any divisor in $\mathrm{DIV}\, X$ is called

<u>Principal</u> if it is the

divisor ~~of~~ of a function $f \in K^*$.

Pf: $f, g \in K^*$ then $(f/g) = (f) - (g)$

$$\varphi : f \mapsto (f)$$

is homomorphism from the multiplicative group of K^* to the additive gp $\mathrm{DIV}\, X$.

Def: Two divisors $D, D' \in \operatorname{Div} X$ are linearly equivalent

$$D \sim D'$$

if $D - D'$ is a principal divisor.

$\operatorname{Div} X / \operatorname{Im} \varphi$ = divisor class gp of X

$\operatorname{Cl} X$

Silverman The Arithmetic of E.C.
II.3.

DIVISORS ON CURVES

Def: let k be alg. closed

A curve over k is an integral separated scheme X of finite type over k.

If X is a nonsingular curve, then
X is N.C.I.S.

A prime divisor on X is a closed point

$$D = \sum_{P \subset X} n_i P_i \qquad n_i \in \mathbb{Z}$$

Def: The degree of D is

$$\deg D = \sum n_i$$

where $D = \sum_{P \subset X} n_i P_i$

Def: If $f : X \to Y$ is a finite morphism of non singular curves, we define

$$f^*: \mathrm{Div}\, Y \longrightarrow \mathrm{Div}\, X$$

a homomorphism as follows:

let $Q \in Y$ be given, $t \in \mathcal{O}_Q$

a local parameter at Q, $t \in K(Y)$

with $v_Q(t) = 1$

$$f^*Q = \sum_{f(P)=Q} v_P(t) \cdot P$$

Since f is a finite morphism we have finitely many $P \in X$ st $f(P) = Q$.

f^* preserves linear equivalence.

$\Rightarrow$ induces $f^*: Cl\,Y \to Cl\,X$

Rk: A principal divisor on a complete non singular curve has degree 0.

$\Rightarrow$ The degree of a divisor on X depends only on its linear equivalence

Prop: let $f: X \to Y$ be a finite morphism

$$\deg : \operatorname{Div} X \longrightarrow \mathbb{Z}$$

$$f^* D \longmapsto \deg f \cdot \deg D$$

The degree map is surjective

let $Cl^\circ X = \ker(\deg)$

There is a natural 1-1 correspondance between the set of closed points of X and $Cl\,\mathring{X}$.

<u>For E.C.</u>

Let $P_0 \in X$ $(P_0 = (0:1:0))$

tangent $z=0$ meets the curve in $3P_0$

any line passing through P, R, Q

$$P+Q+R \sim 3P_0$$

Now to any point $P \in X$

$$P \mapsto P - P_0 \in Cl^0 X$$

injective: $P - P_0 \sim Q - P_0$

$\Leftrightarrow P \sim Q$

$\Rightarrow$ ex. p. 139

$\leadsto$ rational ✓

Surjective!

Let $D \in Cl^0 X$

$$D = \sum n_i P_i \quad \text{with } \sum n_i = 0$$

$$\Rightarrow D = \sum n_i (P_i - P_0)$$

Now for any pt $R \in X$, $\exists T \in X$ st

$$P_0 + T + R \sim 3P_8$$

$R - P_0 \sim -(T - P_0)$

in $D = \sum n_i (P_i - P_0)$

if $n_i < 0$

complete Pf p 139 Hartshorne

$Cl^0 X \longleftrightarrow$ set of closed pt on X.

Rk: The divisor class gp of a variety has a discrete component ($\mathbb{Z}$), a continuous component ($Cl^\circ X$) which has itself the structure of an algebraic variety.

$Cl^0(X) \simeq$ gp of closed points
of an abelian variety
called the Jacobian
variety of X.

The dimension of $J(X)$ is the genus
of the curve.

For genus 2:

Cassels/Flynn:

Prolegomena to a Middlebrow Arithmetic of curves of genus 2.