More on intersections

· Relations with conormal sheaves and blowups

X fibered surface

$D \subseteq X$ Cartier divisor

$D = V(I) \quad I = \mathcal{O}_X(-D) \quad i : D \hookrightarrow X$

$$1 \longrightarrow I \longrightarrow \mathcal{O}_X \longrightarrow i_*(\mathcal{O}_D) \longrightarrow 1$$

$$\begin{aligned} C_{D|X} &= i^*(I/I^2) \\ &= I \otimes_{\mathcal{O}_X} \mathcal{O}_X/I \\ &= \mathcal{O}_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_D = \mathcal{O}_X(-D)|_D \end{aligned}$$

$$\omega_{D|X} = C_{D|X}^\vee = \mathcal{O}_X(D)|_D$$

For a blowup:

$Y \hookrightarrow X$ both regular, $Y = V(I)$

$\rightsquigarrow \underset{\text{inv im of } Y}{Y'} \hookrightarrow X'$ blowup in Y

$Y' = V(\underbrace{I\mathcal{O}_{X'}}_{J})$ and $Y' \cong \mathbb{P}^{r-1}_Y$

By construction

$X' \overset{i}{\hookrightarrow} \mathbb{P}^{r-1}_X$ and $J = i^*(\mathcal{O}_{\mathbb{P}^{r-1}_X}(1))$

$J/J^2 = i^*(\mathcal{O}_{\mathbb{P}^{r-1}_Y}(1))$ and

$$\omega_{Y'|X'} = (J/J^2)^\vee = \mathcal{O}_{Y'}(-1)$$

Specializing to $Y = pt$, one sees that Y' has self-intersection -1 by taking the degree.

Adjunction:

Let X be a fibered surface.

$E < X_s$ component of special fiber.

$$\omega_{E|k(s)} = \left(\mathcal{O}_X(E) \otimes \omega_{X|S}\right)\Big|_E$$

Pf One has

$$\begin{tikzcd} & X \arrow[dr] & \\ E \arrow[ur, hook] \arrow[rr] \arrow[dr] & & S \\ & \mathrm{Spec}\, k(s) \arrow[ur] & \end{tikzcd}$$

$$\omega_{E|S} = \underbrace{\omega_{E|X}}_{\omega_{E|X}|_E = \mathcal{O}_X(E)|_E} \otimes\, \omega_{X|S}\big|_E \rightsquigarrow \text{RHS}$$

$$\| $$

$$\omega_{E|k(s)} \otimes \overbrace{\omega_{\mathrm{Spec}\, k(s)|S}}^{\text{trivial.}}$$

$\square$

By taking degrees and using the relation between $\deg(\omega)$ and p_a, one gets

$$p_a(E) = 1 + \frac{1}{2}(E^2 + E \cdot K_{X|S})$$

where $\omega_{X|S} = \mathcal{O}_X(K_{X|S})$.

Cohomology:

$\mathcal{F}$ sheaf on X scheme

$\rightsquigarrow$

$H^0(\mathcal{F}), H^1(\mathcal{F}), H^2(\mathcal{F}), \dots$

sequence of groups st

- $H^0(\mathcal{F}) = \mathcal{F}(X)$
- If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H})$$
$$\to H^1(\mathcal{F}) \to H^1(\mathcal{G}) \to H^1(\mathcal{H})$$
$$\to \dots$$

Key properties for projective space:

$X = \mathbb{P}^d_A = \mathrm{Proj}\, B$ where $B = A[T_0, \dots, T_d]$.

Then

$H^0(X, \mathcal{O}_X(n)) = B_n$ n-th graded parts.

$H^i(X, \mathcal{O}_X(n)) = 0$ if $i \neq 0, d$

$H^d(X, \mathcal{O}_X(n)) = H^0(X, \mathcal{O}_X(-n-d-1))^\vee$
$= 0$ $n \gg 0$.

Serre:

Let $\mathcal{F}$ be a general sheaf on $X = \mathbb{P}^d_A$. Then

$H^i(X, \underset{\mathcal{F} \otimes \mathcal{O}_X(n)}{\underset{\shortparallel}{\mathcal{F}(n)}}) = 0$ for $i > 0$

if $n \gg 0$.

- Subschemes: if $Z \overset{i}{\hookrightarrow} X$ closed immersion then

$H^i(Z, \mathcal{F}) \cong H^i(X, i_* \mathcal{F})$

sheaf on Z

X arithmetic surface,

$E < X_s$ irr st

$\begin{cases} E \cong \mathbb{P}^1_{k'} & \text{for } k'/k \text{ finite} \\ E^2 < 0 \end{cases}$

H effective divisor on X st

$H^1(X, \mathcal{O}_X(H)) = 0$.

$r = \dfrac{-H \cdot E}{E \cdot E} \in \mathbb{Q}$

Then (i) $H^1(X, \mathcal{O}_X(H + iE)) = 0$ for $i \leq r$

(ii) If $r \in \mathbb{Z}$, $\mathcal{O}_X(H)$ generated by global sections.

$\begin{cases} \mathcal{O}_X(H + rE)|_E \cong \mathcal{O}_E \\ \mathcal{O}_X(H + rE) \text{ gener.} \end{cases}$

Pf induction on i:

For the given $i \leq r$ we have

$$(H+(i+1)E)\cdot E \geq 0.$$

So $\mathcal{O}_X(H+(i+1)E)|_E$ is of positive degree, hence isomorphic to $\mathcal{O}_E(a)$ for some $a \geq 0$.

(i) Use

$$0 \to \mathcal{O}_X(H+iE) \to \mathcal{O}_X(H+(i+1)E) \to i_* \mathcal{O}_X(H+(i+1)E)|_E \to 0$$

Use cohomology:

- $H^1(X, \mathcal{O}_X(H+iE)) = 0$ by ind. hyp
- $H^1(X, i_* \mathcal{O}_X(H+(i+1)E)|_E) = 0$

$= H^1(E, \mathcal{O}_X(H+(i+1)E)|_E)$

$= H^1(E, \mathcal{O}_E(a)) = 0$ by description for $\mathbb{P}^1$.

Hence $H^1(X, \mathcal{O}_X(H+(i+1)E)) = 0$ because of the exact sequence.

(ii) $\mathcal{O}_X(H+rE)|_E \cong \mathcal{O}_E$

because its degree is zero and $E \cong \mathbb{P}^1_k$...

GGS:

- We only have to check this at points of E, since

$$H^0(H+rE) \supseteq H^0(X, H)$$

and outside E these sheaves coincide

Now

$$H^0(X, \mathcal{O}_X(H+rE)) \quad \text{global sections of } H+rE$$

$$\twoheadrightarrow H^0(E, \mathcal{O}_X(H+rE)|_E) \quad \text{generated by its global sections as a sheaf on } E$$

$$\to H^1(X, \mathcal{O}_X(H+(r-1)E))$$

$$\| $$
$$0$$

Thm X arithmetic surface

$E \subseteq X_s$ irr comp of special fiber

st $\begin{cases} E \cong \mathbb{P}^1_{k'} \\ E^2 < 0 \end{cases}$

Then a contraction

$f : X \longrightarrow Y$

of E exists

Pf Let $\mathcal{L}$ be an ample sheaf on X.

$\mathcal{L} \longrightarrow \mathcal{L}^{\otimes n}$

$\mathcal{L}$ very ample

$\mathcal{L} \longmapsto \mathcal{L}^{\otimes n}$, Serre

$\mathcal{L}$ very ample and $H^1(X, \mathcal{L}) = 0$

$\mathcal{L} = \mathcal{O}_X(H_0)$ H_0 effective

Let Γ be a component of a special fiber. Then $\mathcal{O}_X(H_0)|_\Gamma$ is still ample, so

$H_0 \cdot \Gamma > 0$ $\left(\text{ample} \Leftrightarrow \deg > 0 \text{ on } \mathbb{P}^1\right)$

Let $m = -E^2 > 0$

$r = H_0 \cdot E > 0$.

Construct $D = mH_0 + rE$.

By our previous result D is generated by global sections and so defines a morphism $f = f_D$

(i) E gets contracted because

$D \cdot E = 0$ by construction, so

$\deg(\mathcal{O}_X(D)|_E) = 0$ so $\mathcal{O}_X(D)|_E = \mathcal{O}_E$

$(E \cong \mathbb{P}^1_{k'})$

(ii) Other Γ do not get contracted:

$\deg(\mathcal{O}_X(D)|_\Gamma) = D \cdot \Gamma$

$= m \underbrace{H_0 \cdot \Gamma}_{\substack{>0 \\ \text{because } H_0 \text{ ample}}} + r \underbrace{E \cdot \Gamma}_{\substack{\geq 0 \text{ since no} \\ \text{common} \\ \text{component}}} > 0$

so $\mathcal{O}_X(D)|_\Gamma$ is not trivial. $\square$

Using the theorem on formal functions, one shows:

if $d = -E^2 / [k' : k(s)]$

$= \deg_{k'}(\mathcal{O}_X(-E)|_E)$

$= \deg_{k'}(N_{E/X})$

$E \mapsto y$ contraction:

$\dim_{k(y)} T_{Y,y} = d+1$

So the contraction is regular

$\Longleftrightarrow$

$$\begin{cases} E \cong \mathbb{P}^1_{k'} \\ E^2 = -[k' : k(s)] \end{cases}$$

In such a case E is called <u>exceptional</u>.

The arithmetic obtained after successively contracting all exep. div is the <u>relatively minimal model</u> of X.

Criteria for a divisor to be exceptional:

(i) E is exceptional $\Longleftrightarrow$

$$E^2 < 0 \text{ and } K_{X/S} \cdot E < 0.$$

(ii) $p_a(X_\eta) \geq 1$: E excep $\Longleftrightarrow K_{X/S} \cdot E < 0$.

<u>Pf</u> (i) Use adjunction:

$$K_{X/S} \cdot E + E^2 = -2\chi_{k(s)}(\mathcal{O}_E)$$
$$= -2 + 2\dim_k (H^1(E, \mathcal{O}_E))$$

This shows $H^1 = 0$, which means that E is a conic, and in fact

$$E \cong \mathbb{P}^1_{k'}.$$

In fact then $K_{X/S} \cdot E = E^2$.

(ii) $H^0(X, \omega_{X/S}) \otimes K(S) \neq 0$

because of hyp.

Therefore $\omega_{X/S}$ is effective:

$$\omega_{X/S} = \mathcal{O}(K_{X/S}) \text{ for } K_{X/S} > 0$$

We have $K_{X/S} = aE + D$ where

D has no common component with E.

Because of the intersection $\neq$, we see $a \geq 1$. Then

$$aE^2 = \underbrace{K_{X/S} \cdot E}_{<0} - \underbrace{D \cdot E}_{\geq 0} < 0$$

Recall: Y arithmetic is <u>minimal</u> if for all other X arithmetic we have that a birational map

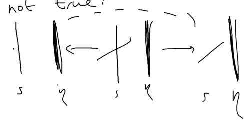
$$X \dashrightarrow Y$$

is in fact a morphism.

We want: rel. min $\Longrightarrow$ min.

This is <u>true</u> if $p_a(X_\eta) \geq 1$

For $p_a(X_\eta) = 0$ the statement is not true:



<u>Lemma</u> X_1, X_2 arithmetic birational without morphisms between them:

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \arrow[ll, dashed] \end{tikzcd}$$

There exist a Z common birational cover and $\underbrace{E_1, E_2}_{\text{exceptional}}$ on Z st $p_1(E_1)=pt$

either $\begin{cases} p_2(E_1) = \text{still exceptional} \\ \\ \text{or } (E_1+\mu E_2)^2 \geq 0 \text{ for some } \mu \geq 0 \end{cases}$

<u>Thm</u> Rel minimal implies minimal if $p_a(X_\eta) \geq 1$.

<u>Pf</u> $2p_a(X_\eta)-2 = 2p_a(Z_\eta)-2$

$= K_{Z/S} \cdot Z_s$

Now consider $D = E_1 + \mu E_2$.
These are contained in the same fiber,
so $D = r Z_s$ for some r because
of negative-definiteness.

$$= (\underbrace{K_{Z/S} \cdot E_1}_{<0} + \mu \underbrace{E_2 \cdot K_{Z/S}}_{<0})/r < 0.$$

□

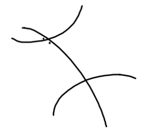
Other models

(1) E/K elliptic curve where $K = K(S)$ for affine Dedekind scheme.

Then there exists a normal model of E over S so a minimal regular model $\mathcal{E}$ as well.

Let N = smooth locus of $\mathcal{E} \to S$

$$\begin{array}{c} N \\ \cap \\ \mathcal{E} \end{array} \text{ open immersion}$$



We have

$$\mathcal{E}(S) = E(K)$$

$\mathcal{E}(S) = \mathrm{Hom}(S, \mathcal{E})$ because $\mathcal{E}$ is proper.

$E(K) = \mathrm{Hom}(\mathrm{Spec}\, K, \mathcal{E})$

$N(S)$

because rational points intersect special fibers once so not in singular points

$\overline{E_N}$ =

N is called the Néron model of E.

It is the unique smooth model of E st for X/S smooth there is a bijection

$$\mathrm{Hom}_S(X, \mathcal{E}) = \mathrm{Hom}_K(X_K, E)$$

This gives a filtration of $E(K)$:

$$E^1(K) \hookrightarrow E^0(K) \hookrightarrow E(K)$$

$N_S^0(k)$

group of conn. components of special fiber

$E^1(K) \cong \underline{\mathfrak{m}}$ for the max. ideal of a discrete valuation ring R with f.o.f. K.

$S = \mathrm{Spec}(R)$ $\quad k = R/\mathfrak{m}$

conn comp cont 0 of special fiber of N over S