Finding the Picard Rank of a Certain Family of K3 Surfaces

Florian Bouyer

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1 Introduction

This report aims to calculate the Picard number of a certain family of K3 surfaces.

In Section 2 we give a definition of K3 surfaces as well as useful properties of them, namely the Hodge Diamond. The theory behind he Hodge Diamond is both deep and complex, requiring a strong background in Differential Geometry and Complex Geometry. As the author works largely in the fields of Algebraic Geometry and Number Theory, it would be unreasonable to write it up in any detail in this report, and we would hope that the brief explanation behind the intuitions will be enough justification for the reader. The eager reader is encouraged to read the useful references given.

Although Section 2 is a very technical brief summary of the background that is needed Section is more detailed and assumes less of the reader, so it can be thought as the start of the report. In this section we show that for a K3 surface, its Picard Group is torsion free and hence determined by its Picard Rank. We then discuss the theory developed to bound the Picard rank: below by using algebraic geometry and divisors; above by using good reduction and point counting.

Section 4 is written more colloquially and explains how we put into practise the theory explained in Section 3 to a certain family of K3 surfaces. Along the way we note all the different directions or possible future work we could look into, summarising this in the last subsection.

As a note of interest the name K3 was coined by Weil in 1958, inspired by both the first three algebraic geometer to study them: Kummer, Kahler and Kodaira, and after the K2 mountain.

2 K3 Surfaces

We first define our object of interest.

Definition 2.1. A *K3 surface* is a nonsingular surface X with irregularity $q = \dim H^1(X, \mathcal{O}_X) = 0$ and trivial canonical sheaf $\omega_X \cong \mathcal{O}_X$. (Here $\mathcal{O}_X$ is as usual the sheaf of regular functions)

Here a surface is a projective integral separated scheme of finite type and dimension 2 over a field k. The reader is referred to [Har77] for an expansion on the terms in the previous sentence. By definition the canonical sheaf of a surface is $\omega_X = \wedge^2 \Omega_X$, where Ω_X is the sheaf of differentials of X.

Our next goal is to build the Hodge Diamond, which contains useful information for a surface X (and more generally for a compact Kahler manifold). Along the way we will be briefly giving some intuition to its usefulness, but first we need two definitions.

Definition 2.2. Let X be a surface. The *i-th Betti Number* is $b_i = \dim H^i(X, \mathbb{Z})$ for $i = 1, \ldots, 4$.

The *hodge numbers* are $h^{i,j} = \dim H^j(X, \Omega_X^i)$ for $i, j = 0, 1, 2$, where $\Omega_X^0 = \mathcal{O}_X, \Omega_X^1 = \Omega_X$ and $\Omega_X^2 = \wedge^2 \Omega_X$.

There are a fair number of relations between these numbers, but we will only give a few of these, pointing out the reasons for them. The reader is encouraged to read [Dem85, Sec 1] for more details. First by Serre's duality we have $h^{i,j} = h^{2-i,2-j}$, and we can work out that (since we are considering surfaces) $h^{0,0} = h^{2,2} = 1$, and if b_1 is even $b_1 = 2q$. Next by Poincaré's duality we have $b_1 = b_3$ and $b_0 = b_4 = 1$. Finally using Hodge decomposition, we have $b_1 = h^{1,0} + h^{0,1}$ and $b_2 = h^{0,2} + h^{1,1} + h^{2,0}$.

In the case X is a K3 surface, since $q = 0$ we have $b_1 = 0$ and since we have trivial canonical sheaf, we have $h^{2,0} = 1$. Hence to finish the Hodge Diamond, we need to find $h^{1,1}$; for this we use Noether Formula

$$\chi(X) = \frac{K^2 + c_2}{12},$$

where c_2 is the second Chern number, which is equal to the topological Euler characteristic, and K is the canonical divisor (see next section). Since for a K3 surface we have $K^2 = 0$ (see section below) and $\chi(X) = 2$ (as the arithmetic genus is the same as the arithmetic genus which is $h^{1,1}$ and $\chi(X) = g + 1$) we have that $c_2 = 24$. From this we can use the Hirzebruch-Riemann-Roch Theorem, which says that $\chi(X) = \sum(-1)^i b_i$, to find that $h^{1,1} = 24 - 4 = 20$. Hence for a K3 surface we get the Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The Adjunction Formula (special case). *Let X be a nonsingular subvariety of codimension 1 in a nonsingular variety Y over k. Let $\mathscr{L}$ be the invertible sheaf associated to X (when thought of as a divisor of Y, see next section why this can hold). Then $\omega_X \cong \omega_Y \otimes \mathscr{L} \otimes \mathcal{O}_X$.*

Proof. See [Har77, Sec II, Prop 8.20]. □

Theorem 2.3. *Let $F \in k[x, y, z, w]$ be homogeneous and quartic (of degree four). If $X = \{F(x, y, z, w) = 0\} \subset \mathbb{P}^3$ is nonsingular, then it is a* K3 *surface.*

Proof. First we note that Lefschetz Hyperplane Theorem (taking $X = X$ and $Y = \mathbb{P}^3$) tells us that $H^1(X, \mathbb{Z}) \cong H^1(\mathbb{P}^3, \mathbb{Z})$. But since $H^1(\mathbb{P}^3, \mathbb{Z})$ is trivial, we deduce that $\dim H^1(X, \mathbb{Z}) = b_1 = 0$. Hence by the previous work, we have that $\dim H^1(X, \mathcal{O}_X) = q = 0$.

To show that $\omega_X \cong \mathcal{O}_X$, we use the adjunction formula with $Y = \mathbb{P}^3$ and using the fact X has degree 4 to find $\omega_X \cong \omega_{\mathbb{P}^3} \otimes \mathcal{O}(4)|_X$. But we know that $\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-3-1)$, hence we have $\omega_{\mathbb{P}^3} \otimes \mathcal{O}(4)|_X \cong \mathcal{O}_X$ as required. □

3 Divisors and the Picard Group

In this section, we will define (Weil) divisors and the Picard group, also stating theorems that are useful.

Definition 3.1. Let X be a Noetherian integral separated scheme which is regular in co-dimension one. Then:

- A *prime divisor* on X is a closed integral subscheme Y of co-dimension one.
- A *(Weil) divisor* $D = \sum_i n_i Y_i$ is a finite formal sum of prime divisors, that is $n_i \in \mathbb{Z}$, Y_i is a prime divisor and all but finitely many n_i are zero.
- We say a divisor D is *effective* if all the $n_i \geq 0$.
- Let f be a rational function on X, then we define the divisor of f as $(f) = \sum \nu_Y(f)Y$, where ν_Y is the valuation of Y. A divisor which comes from such a function, i.e. $D = (f)$, is called a *principal divisor.*
- Two divisors D and D' are said to be *linearly equivalent,* written $D \sim D'$, if $D - D'$ is a principal divisor.
- The quotient group of all divisors (denoted $\mathrm{Div}X$) modulo the subgroup of principal divisors is called the *divisor class group* of X (and is denoted $\mathrm{Cl}X$)
- We define the *linear system* of a divisor D, denoted $L(D)$, to be the vector-space of rational functions on X such that $(f) + D$ is effective.
- A divisor D is said to be *very ample* if the rational map $\phi_D : X \dashrightarrow \mathbb{P}^n$ defined by $x \mapsto (s_0(x) : \cdots : s_n(x))$ is an embedding, where $\{s_i : i = 0, \ldots n\}$ is a basis for $L(D)$.

If we restrict ourself back to surfaces, then we see that a divisor is simply a formal sum of curves on the surface, where a curve is allowed to be singular and reducible. The rest of this section now assumes X is a surface. While the actual definition of the Picard Group of X, denoted $\mathrm{Pic}X$, is the group of isomorphism classes of invertible sheaves on X, under the right assumptions we have that $\mathrm{Pic}X \cong \mathrm{Cl}X$. (These are that$X$ is Noetherian, integral, separated and locally factorial, all of which hold for K3 surfaces). For this reason, after establishing some properties of $\mathrm{Pic}X$, we will think of it as the divisor class group; the next section will show it is useful to do so. Furthermore we can use this isomorphism and the fact that ω_X is an invertible sheaf to define the *canonical divisor* K, which is any divisor in the linear equivalence class of ω_X.

As well as linear equivalence, there is a second type of equivalence called *algebraical equivalence* - which we shall not define here for reasons that will become apparent later (see [Har77, V,Ex 1.7] for more details on algebraically equivalence) - that has the two following properties:

1. Any two divisors that are linearly equivalent are algebraically equivalent
2. The divisors algebraically equivalent to 0 form a subgroup of Div$X$, which we denote by $\text{Div}^0 X$.

Definition 3.2. Let $\text{Pic}^0 X$ denote the image of $\text{Div}^0 X$ in Pic$X$. Then the *Neron-Severi group, NS*(X), of X is the quotient $\text{Pic}X/\text{Pic}^0 X$.

Fact. *The Neron-Severi group is a finitely generated abelian group.*

This fact leads us to the next definition:

Definition 3.3. The *Picard number* of X, denoted $\rho(X)$, is the rank of NS(X).

Finally we remark that using the actual definition of Pic(X), we can show that $\text{Pic}X \cong H^1(X, \mathcal{O}_X^*)$ (see [Har77, III,Ex 4.5], where $\mathcal{O}_X^*$ is the sheaf whose section over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$ and the group operation is multiplication). With all these definitions in place, we can start trying to deduce some information about the Picard group of a K3 surface X. First consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

of sheaves. This leads to the following exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to \underset{\substack{\parallel \\ 0}}{H^1(X,\mathcal{O}_X)} \to \underset{\substack{\parallel \\ \text{Pic}X}}{H^1(X,\mathcal{O}_X^*)} \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to H^2(X,\mathcal{O}_X^*) \to H^3(X,\mathbb{Z}) \to 0$$

Hence we deduce straight away that $H^1(X, \mathbb{Z}) = 0$ and, by Poincaré duality, that $H^3(X, \mathbb{Z}) = 0$ (although we already knew this from the previous section). By [Har77, pg 447] we have that $\text{Pic}^0 X \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, hence for K3 surfaces $\text{Pic}^0 X$ is trivial and $\text{Pic}X \cong \text{NS}(X)$, meaning linear equivalence and algebraic equivalence are the same. Since a finitely generated abelian group is completely determined by its rank and torsion group, at this stage it would be useful to find the torsion group of Pic(X). For this we recall the following theorem:

Universal Coefficient Theorem for Cohomology. *For an abelian group A and topological space X we have the short exact sequence*

$$0 \to \text{Ext}(H_{i-1}(X,\mathbb{Z}), A) \to H^i(X, A) \to \text{Hom}(H_i(X,\mathbb{Z}), A) \to 0\ .$$

If we substitute $A = \mathbb{Z}$ and $i = 2$, we see from the short exact sequence that the torsion part of $H^2(X, \mathbb{Z})$ is isomorphic to $\text{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z})$ which is the same as the torsion part of $H_1(X, \mathbb{Z})$. Since it can be proven that $H_1(X, \mathbb{Z}) = 0$ (see [Mér85, prop 1.1]), we conclude that $H^2(X, \mathbb{Z})$ is torsion free. Since Pic$X$ injects into $H^2(X, \mathbb{Z})$, we have that Pic$X$ is torsion free, and hence we have the important result that $\text{Pic}X \cong \mathbb{Z}^{\rho(X)}$.

So to determine the Picard group, we just need to work out the Picard number, and we already have an upper bound for $\rho(X)$, namely the second Betti number, $b_2 = \dim H^2(X, \mathbb{Z})$. In particular a K3 surface has Picard number at most 22. Note that while this did use that we were working in characteristic 0, we can use étale cohomology to show that in positive characteristic, we still have an upper bound of 22. Furthermore in characteristic 0 we can use the Lefschetz Theorem on (1, 1)-classes which precisely tells us that $\text{NS}(X) = H^2(X, \mathbb{Z}) \cap H^1(X, \Omega_X^1)$ (see [Mér85, Thm 2.3]), so in particular we have that the Picard number is less than $\dim(H^1(X, \Omega_X^1)) = h^{1,1} = 20$.

Definition 3.4. Any K3 surface X with $\rho(X) = 20$ is called *singular* (not to be confused with the usual definition of a singular (non-smooth) surface).

This result does not hold in positive characteristic (in fact K3 surfaces which have $\rho(X) = 22$ are called *supersingular*, and the Fermat surface is one such example in characteristic $p \equiv 3 \mod 4$). While we now have a better picture of Pic$X$, it is still not easy to calculate. To find $\rho(X)$ one turns to the different methods of bounding it above and below, and hope the two bounds agree.

3.1 A lower bound using intersection numbers

As we already know a lot about the structure of Pic$X$, one is tempted to suggest we just need to find a way to enumerate a minimal set of generators. But this leads straight away to the question (which always comes up when talking about a quotient group), how easily can we tell if two divisors are linearly equivalent (if we can at all) ? Fortunately, this is where the theory of intersection numbers comes to help. Intuitively we want the intersection number between two curves to be the number of points in their intersection, but as on the plane, we want to count the points with multiplicities.

Definition 3.5. Let C and D be curves on X, and $P \in C \cap D$ be a point in the intersection. We say that C and D meet *transversally at* P if the local equations of C and D at P, say f and g respectively, generates the maximal ideal m_P of $\mathcal{O}_{P,X}$. If C and D meet transversally at all points in their intersection, then we say they meet *transversally.*

Meet transversally   Do not meet transversally

Example. Consider the curves C defined by the equation $x = 0$, and D defined by the equation $y^2z - x^3 = 0$ both in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$. They intersect at two points, $P_1 = [0 : 1 : 0]$ and $P_2 = [0 : 0 : 1]$. At the point P_1 we have that $\mathcal{O}_{P_1,\mathbb{P}^2} = \overline{\mathbb{Q}}[x, y, z]_{\langle x,z\rangle}$ and the local equations are the same as the global equations, hence we can check that $\langle x, y^2z - x^3\rangle = \langle x, z\rangle$ (since $z = \frac{(y^2z-x^3)+x^3}{y^2}$ and $\frac{1}{y^2} \in \mathcal{O}_{P_1,\mathbb{P}^2}$). Hence at P_1 we have that C and D intersect transversally. At P_2, we have $\mathcal{O}_{P_2,\mathbb{P}^2} = \overline{\mathbb{Q}}[x, y, z]_{\langle x,y\rangle}$ but $\langle x, y^2x - x^3\rangle = \langle x, y^2\rangle \neq \langle x, y\rangle$ hence they do not meet transversally at P_2.

With this definition we can use the following lemma to define the intersection number

Lemma 3.6. *There is a unique pairing (called the* intersection number*) Div$X \times$ Div$X \to \mathbb{Z}$, denoted $C \cdot D$ for any two divisors C, D such that*

1. *If C and D are curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$,*

2. *It is symmetric: $C \cdot D = D \cdot C$,*

3. *It is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$*

4. *It depends only on the linear equivalence class: if $C_1 \sim C_2$ then $C_1 \cdot D = C_2 \cdot D$.*

Proof. The proof is rather technical; so while we leave it to the reader to check up the details [Har77, V,Thm 1.1], we will give a brief sketch as it is a constructive proof.

If we assume properties 1 to 4, then there is only one way to define the intersection number, which will coincide with our construction.

The construction happens in two steps: first we define the intersection number for very ample divisors, then for all divisors. Given two very ample divisors C and D, first choose $C' \in L(C)$ which is non-singular and choose $D' \in L(D)$ which is non-singular and transversal to C'. This is possible by [Har77, V, Thm 1.2]. Then we define $C \cdot D = \#(C' \cap D')$. (One still needs to check this is well defined).

Given two arbitrary divisors C and D, then we pick four very ample divisors C', D', E', F', such that $C \sim C' - E'$ and $D \sim D' - F'$, again this is possible by [Har77, V, Thm 1.2]. Then we define $C \cdot D = C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F'$. Again, one should check this is well defined, but we can see that by construction it satisfies conditions 2. to 4. while condition 1. requires some more work. □

While the proof gives a construction to calculating the intersection number, it is still not a straightforward process, but the following definitions and theorems will help.

Definition 3.7. Let C and D be two divisors on a surface X defined over a field k, and let $P \in C \cap D$ be a point. We define the *intersection multiplicity* of C and D at P as $(C \cdot D)_P = \dim_k \mathcal{O}_{X,P}/(f, g)$, where f, g are the local equations of C and D at P respectively.

Example. Using our previous example with C being defined by $x = 0$ and D being defined by $y^2z - x^3 = 0$, we expect the intersection multiplicity at $P_1 = [0 : 1 : 0]$ to be 1, since they meet transversally at that point. Indeed we have $\mathcal{O}_{P_1,\mathbb{P}^2}/\left\langle x, y^2z - x^3\right\rangle \cong \overline{\mathbb{Q}}[x, y, z]_{\langle x,z\rangle}/\left\langle x, z\right\rangle \cong \overline{\mathbb{Q}}(y)$, which has dimension 1. At the point $P_2 = [0 : 0 : 1]$ we have $\mathcal{O}_{P_2,\mathbb{P}^3}/\left\langle x, y^2z - x^3\right\rangle \cong \overline{\mathbb{Q}}[x, y, z]_{\langle x,y\rangle}/\left\langle x, y^2\right\rangle \cong \overline{\mathbb{Q}}(z)[y]/\left\langle y^2\right\rangle$, and we can see that this has dimension 2.

Theorem 3.8. *If C and D are curves on X having no common irreducible component, then $C \cdot D = \sum_{P\in C\cap D}(C \cdot D)_P$.*

To prove this, one shows that $\sum(C \cdot D)_P$ only depends on the equivalence classes of C and D. Hence we can replace C and D by the difference of nonsingular curves all transversal to each other (as in the proof of Lemma 3.6) to see that it is the intersection number. Unfortunately this theorem comes with an assumption, and so does not allow us to calculate the self intersection number $D \cdot D = D^2$ of any divisor. For this we need to use the following theorem:

Theorem 3.9. *If C is a non-singular curve of genus g on the surface X then $2g - 2 = C \cdot (C + K)$, where K is the canonical divisor.*

Proof. By [Har77, V, Lem 1.3] we have $\deg_C(\omega_X \otimes \mathscr{L}(C) \otimes \mathcal{O}_C) = C \cdot (C + K)$, by the Adjunction Formula we have $\omega_C \cong \omega_X \otimes \mathscr{L}(C) \otimes \mathcal{O}_C$ and by Riemann-Roch we have $\deg_C(\omega_C) = 2g - 2$. □

In particular, if X is a K3 surface we have that $C^2 = 2g - 2$, since a K3 surface has trivial canonical divisor. To see why the intersection number is useful, we first give a recap on lattices.

Definition 3.10.

- A *integral lattice* is a free $\mathbb{Z}$-module L of finite rank, together with a symmetric, bilinear map $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$.
- A lattice is *even* if we have $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$.

- The *Gram Matrix* of L with respect to a set of elements $x_1, \dots, x_n \in L$ is the matrix

$$\begin{pmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_n\rangle \\ \langle x_1, x_2\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, x_n\rangle & \langle x_2, x_n\rangle & \cdots & \langle x_n, x_n\rangle \end{pmatrix}$$

Recall that the rank of the Gram Matrix will tell you how many of the elements in the set are linearly independent (where we see the elements as elements of the vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$). Note that by definition the intersection number is a symmetric, bilinear map which only depends on the linear equivalence class, hence it turns $\mathrm{Pic}(X)$ into a lattice. Furthermore, for K3 surfaces $\mathrm{Pic}(X)$ is an even lattice (due to $C^2 = 2(g-1)$). So we now have a strong tool with which we can tell if a divisor is a linear combination of other divisors that we already have. Hence to find a lower bound on $\rho(X)$, we list as many divisors as we can find and then calculate the Gram Matrix with respect to those divisors. Then the rank of the Gram Matrix will be a lower bound.

This method unfortunately can only give us a lower bound, as it does not give us any information or hint that we might have found all the "generating" divisors. Similarly this does not address the issue on how one goes about systematically finding some divisors.

3.2 An upper bound using good reduction

To find an upper bound to $\rho(X)$ we first use the following result taken from [Ful98, Ex 20.3.6]: Let R be a complete discrete valuation ring with quotient field K and residue field κ, then $\mathrm{rk}\,\mathrm{NS}(X \otimes_R \overline{K}) \leq \mathrm{rk}\,\mathrm{NS}(X \otimes_R \overline{\kappa})$. In our case, taking $R = \mathbb{Z}_p$ for a good prime p, we get $\rho(X_{\overline{\mathbb{Q}}}) = \rho(X_{\mathbb{Q}_p}) \leq \rho(X_{\overline{\mathbb{Q}_p}}) \leq \rho(X_{\overline{\mathbb{F}_p}})$, where $X_{\overline{\mathbb{F}_p}} = X \otimes \overline{\mathbb{F}_p}$ This is useful when used with the next lemma:

Lemma 3.11. *Let X be a* K3 *surface, $F_X : X \to X$ be the absolute Frobenius morphism of X, which acts by the identity on points and by $f \mapsto f^p$ on the structure sheaf. Let $\Phi_p(i)$ be the automorphism on $H^i_{\text{étale}}(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l)$ induced by $F_X \times 1$ acting on $X_{\overline{\mathbb{F}_p}}$. Then the rank of $NS(X_{\overline{\mathbb{F}_p}})$ is bounded above by the number of eigenvalues λ of $\Phi_p(2)$ for which λ/p is a root of unity (counted with multiplicity)*

For a more general setting, and a detailed proof see [vL05, Sec 2.6 & Thm 5.2.2]. So if we can find the characteristic polynomial of $\Phi_p(2)$, then counting the correct eigenvalues gives us an upper bound to $\rho(X_{\overline{\mathbb{F}_p}})$ and hence $\rho(X)$. To calculate the characteristic polynomial we use the Lefschetz formula:

$$\#X(\mathbb{F}_{p^n}) = \sum_{i=0}^{4} (-1)^i \,\mathrm{Tr}(\Phi_p^n(i)),$$

and the following lemma

Lemma 3.12. *Let V be a vector space of dimension n and T a linear operator on V. Let t_i denote the trace of T^i. The the characteristic polynomial of T is equal to*

$$f_T(x) = \det(x \cdot \mathrm{id} - T) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$$

where the c_i are given recursively by $c_1 = -t_1$ and

$$-kc_k = t_k + \sum_{i=1}^{k-1} c_i t_{k-i}$$

The Lefschetz formula helps us, as since X is a K3 surface, we know its Betti numbers with respect to étale cohomology, which as mentioned on Page 3 are the same as the Betti numbers calculated in Section 2. So we can deduce that $\mathrm{Tr}(\Phi_p^n(1)) = \mathrm{Tr}(\Phi_p^n(3)) = 0$, while for $i = 0$ and 4, we have that $\Phi_p^n(i)$ has only one eigenvalue, which by the Weil conjectures are 1 and p^{2n} respectively. Hence we have that $\mathrm{Tr}(\Phi_p^n(2)) = \#X(\mathbb{F}_{p^n}) - p^{2n} - 1$ and we can use the above lemma with $T = \Phi_p(2)$ to find the characteristic polynomial by just counting the number of points on X defined over $\mathbb{F}_{p^n}$ for $n = 1$ to 22 (since $b_2 = 22$).

The problem is that counting points on a surface is computationally expensive even over finite fields. So the first thing to note is that the Weil conjectures give a functional equation

$$p^{22} f_p(x) = \pm x^{22} f_p(p^2/x)$$

where f_p is the characteristic polynomial of $\Phi_p(2)$. Hence we can use the coefficient of x^i to calculate the coefficient of x^{22-i}, and so only need to count the points of X over $\mathbb{F}_{p^n}$ for $n \leq 11$. Furthermore we can slightly adapt a method taken from Elsenhans and Jahnel in [EJ08] which might allows us to stop before $n = 11$.

Suppose that we have found 1 or 2 divisors that are independent and defined over $\mathbb{F}_p$ (using intersection theory as described previously to see if they are independent) and that we have counted the points over $\mathbb{F}_{p^n}$ $n \leq 10$. Since we have at least one non-zero divisor defined over $\mathbb{F}_p$, we know that f_p has a zero at p. Then by the previous work we know all coefficients except c_{11} if we know the sign of the functional equation. If we suppose that the minus sign is present, then we deduce $c_{11} = 0$ and we know f_p. If on the other hand we assume the plus sign, then we can use $f_p(p) = 0$ to deduce c_{11} and hence determine f_p. So we end up with two different f_p. We can use Lemma 3.13 to see if we can rule out either one, otherwise we have to use the one that gives us a higher bound as our upper bound.

This whole argument can be extended to the case if we have found $2d - 1$ or $2d$ divisors (that are independent and defined over $\mathbb{F}_p$) and counted the points over $\mathbb{F}_{p^n}$ for $n \leq 11 - d$. Then we have that p is a $2d$-fold zero of f_p and using the fundamental equation we know all the coefficients except $c_{11-d+1}, c_{11-d+2}, \ldots, c_{11+d-1}$. Again we split into two cases, first assume the minus sign in the functional equation (and hence $c_{11} = 0$) and use $f_p(p) = 0, f_p^{(1)}(p) = 0, \ldots, f_p^{(d-1)}(p) = 0$ (where $f_p^{(i)}$ is the i-th derivation of f_p) to have a set of $2d$ linear equations which can be solved to find f_d. The same idea is used when assuming that the plus sign is present, and again we end with two different f_p. So we use the same idea as in the previous paragraph to find an upper bound.

Note that in the previous paragraph we had $2d$ linear equations for d unknowns in the second case ($d - 1$ unknowns in the first case). This was necessary due to the fact that the approach we use always yields an even number for an upper bound. Since we used

$$\rho(X_{\overline{\mathbb{F}_p}}) \leq \dim(H^2_{\text{étale}}(X_{\overline{\mathbb{F}_p}}, \overline{\mathbb{Q}_l})) - \#\{\text{zeros of } f_p \text{ not of the form } \zeta_n p\}$$

we have the relevant zeroes coming in pairs of complex conjugate numbers. This means that half of our linear equations are redundant as they are dependent. (In fact Tate's conjecture states that this bound is in fact an equality, hence the Picard number of a K3 surface over $\overline{\mathbb{F}_p}$ is always even)

The second thing to note, is that in the case we assume the minus sign and hence deduce $c_{11} = 0$, since we have only $d - 1$ unknowns, we can actually count points over $\mathbb{F}_{p^n}$ $n \leq 11 - d - 1$ to bring the number of unknowns to d. The reason one might want to do that is suppose we count points over $\mathbb{F}_{p^n}$ for $n \leq 11 - d - 1$, assume the minus sign and hence calculate f_p. If we find that f_p is a valid characteristic polynomial with 20 (or 22) roots of the required form, then there is no need for us to calculate the number of points over $\mathbb{F}_{p^n}$ for $n = 11 - d$, as whatever the second possible f_p we find, we have to take as an upper bound 20, which we already know is a natural upper bound.

While we have done so, we can also use the original idea of [EJ08]. Suppose we have found $2d$ divisors and counted points only over $\mathbb{F}_{p^n}$ for $n \leq 11 - d - 1$, i.e. we are missing one piece of information. We can then try to show that the Picard number is in fact $2d$ by finding some contradiction. If the Picard number was higher, then the characteristic polynomial would have a root of the form $\zeta_n p$, with ζ_n being a primitive nth root of unity. Furthermore, we know that the minimal polynomial is a factor of $f_p/(x-p)^{2d}$, hence ζ_n has order at most $22 - 2d$. As there are only a finite number of n such that $\phi(n) \leq 22 - 2d$ (the Euler phi function), for all such n we can try the following: test if the extra linear equation $f_p(\zeta_n p) = 0$ gives a contradiction, and if not we have determined f_p so test if it fails Lemma 3.13. If both tests fail for all such n then we have found all divisors, and hence we must have $\rho(X_{\overline{\mathbb{F}_p}}) = 2d$.

Lemma 3.13. *Fix a prime $p \in \mathbb{Z}$ and let $f_p \in \mathbb{Z}[x]$ be of degree 22. Calculate all its zeros as complex numbers to high precision. If any of them has an absolute value different from p then f_p is not the characteristic polynomial of the Frobenius of any* K3 *surface over $\mathbb{F}_p$.*

4 Our K3 Family

4.1 Preliminaries

Our main object of study is

$$\{A(x^4+y^4+z^4+w^4)+Bxyzw+C(x^2y^2+z^2w^2)+D(x^2z^2+y^2w^2)+E(x^2w^2+y^2z^2) = 0\} \subset \mathbb{P}^4_{[A:B:C:D:E]} \times \mathbb{P}^3_{[x:y:z:w]}$$

which can be considered as a family of surfaces in $\mathbb{P}^3_{[x:y:z:w]}$ parametrized by the variables $[A, B, C, D, E]$ (all our variables are defined over $\overline{\mathbb{Q}}$).

Notation. As often we will consider one surface in the family at a time, we will denote such a surface by $[A, B, C, D, E]$.

From now on, when talking about a curve, line or surface defined by equations $f_1, \ldots, f_n$, we mean the variety/zero set those equations define.

More particularly we are interested in the surfaces in our family that are K3 surfaces, and by 2.3 these are precisely the ones that are non-singular. The main aim is to work out the Picard number of the generic K3 surface, that is the surface that correspond to the generic point of $\mathbb{P}^4_{[A,B,C,D,E]}$ (which we know is a K3 surface as one can argue that being non-singular is true for general members of the family). To do this, we use the fact that the Picard group of the generic surfaces injects into the Picard group of any K3 surface of our family.

With this in mind the starting strategy to finding the generic Picard number would be to calculate the Picard number of a few K3 surfaces in our family. This is done using the methods detailed in the previous section. Then one would hope that along the way, a method to find general divisors would have been found to give a lower bound on the generic Picard number, while the Picard number of specific surfaces already gives an upper bound.

Most of the calculations were done in Magma [BCP97] and as such, we give a function which, given a list of divisors, returns the Gram Matrix with respect to those divisors.

```
GramMatrix := function(List)

    Mat := [];
    for S in List do
```

```
        Line := [];
        for D in List do
           if S eq D then
             Line := Append(Line, 2*ArithmeticGenus(S)-2);
           else

             Line := Append(Line,Degree(S meet D));
           end if;
           end for;
        Mat := Append(Mat,Line);
        end for;
     return Matrix(Mat);

  end function;
```

We first note that the Fermat Surface, defined by $x^4 + y^4 + z^4 + w^4$, is in our family.

Theorem 4.1. *The Fermat Surface ($[1, 0, 0, 0, 0]$) is a singular* K3 *surface.*

Proof. One can easily check that this surface is smooth as the Jacobian matrix

$$\begin{pmatrix} 4x^3 \\ 4y^3 \\ 4z^3 \\ 4w^3 \end{pmatrix} = 0$$

has no solutions in $\mathbb{P}^3$. To show that it has rank 20, note that we can group the terms in three different ways to give us pairs of quartic polynomials in two variables to solve. For example, we can group $x^4 + y^4 = 0$ and $z^4 + w^4 = 0$. Considering the first polynomial gives us four hyperplanes, namely $x = \zeta_8 y$ where ζ_8 is a primitive 8th root of unity. Hence the two polynomials give us 4^2 lines in $\mathbb{P}^3$ which also lie on the Fermat Surface. Considering all three different combination of groupings, we have found 48 different prime divisors on the Fermat Surface. One can then calculate the Gram Matrix of these 48 divisors and find it has rank 20. Hence we have $\rho([1, 0, 0, 0, 0]) \geq 20$, that is the Picard number is 20 (due to the natural upper bound of 20). □

We also note that we have the group $\Gamma = C_2 \times C_2 \times C_2 \times C_2$ as a subgroup of $\mathrm{Aut}(X)$ for any surface X in our family. Γ is generated by the elements $[x : y : z : w] \mapsto [y : x : w : z]$, $[x : y : z : w] \mapsto [z : w : x : y]$, $[x : y : z : w] \mapsto [x : y : -z : -w]$ and $[x : y : z : w] \mapsto [x : -y : z : -w]$. If we are considering the subfamily which has $B = 0$, then we have a bigger subgroup as we also have the automorphisms that negate each "coordinate" individually. Similarly if we have two of C, D and E the same, then we have at least the extra automorphism which swaps two (appropriate with respect to which pair of C, D, E are equal) coordinate without swapping the other two.

4.2 Attempts to an upper bound

The first thing that was attempted was to find various upper bounds by analyzing the various possible K3 surfaces that could occur over small finite fields. In each small finite field, we want to count how many K3 surfaces are in our family. Magma can use the following one line code to count:

```
#[pt: pt in Points(Q)| IsNonSingular(Scheme
    (P,pt[1]*eq1+pt[2]*eq2+pt[3]*eq3+pt[4]*eq4+pt[5]*eq5))];
```

where $P = \mathbb{P}^3_{[x:y:z:w],\mathbb{F}_p}$, $Q = \mathbb{P}^4_{[A:B:C:D:E],\mathbb{F}_p}$ and eq_1 to eq_5 are the 5 equations that goes with the 5 variables A to E. While this tells us how many surfaces in our family are non-singular (and which ones), we can remove some of them, as with appropriate relabeling of variables we can permute C, D and E. Hence, assuming that we have scaled $[A, B, C, D, E]$ from the left to the right, we can assume $C \leq D \leq E$ and remove from our list any which do not satisfy this condition.

Over the field $\mathbb{F}_2$, one finds that all the surfaces are singular. We do have that the 5 curves

$$\{[1,1,0,0,0],[1,1,0,0,1],[1,1,0,1,1],[1,1,1,1,1],[0,1,1,1,1]\}$$

have only points as singularities. This could be useful as we could solve the singularity at each point by blowing up and we know blowing up increases the Picard rank by 1 (See [Har77, II, Ex 8.5]). So this could be future work to do if other directions are found lacking.

Over the field $\mathbb{F}_3$, we find that the only non-singular surface is $[1, 0, 0, 0, 0]$. But since this is the Fermat surface, and by Theorem 4.1 it has rank 20 over $\mathbb{Q}$, hence over $\mathbb{F}_3$ it has rank at least 20 and will not give a useful bound. In fact it has been shown that the Fermat surface has rank 22 (and hence is supersingular) in $\mathbb{F}_p$ if and only if $p \equiv 3 \mod 4$. In view of possible future work, a note has been made of the 19 surfaces which have only points as singularities.

Over the field $\mathbb{F}_4$ we again find that there are no non-singular surfaces in our family.

Once we reach $\mathbb{F}_5$, we find 15 non-singular surfaces, and once we remove the ones which do not satisfy $C \leq D \leq E$ we are left with the 7 surfaces

$$\{[1,0,0,0,0],[1,0,0,0,1],[1,0,0,0,4],[1,1,1,1,1],[1,4,1,1,1],[1,1,1,4,4],[1,4,1,4,4]\}$$

At this point we can apply van Luijk's method to the surface of our choice, say $[1, 0, 0, 0, 1]$ (but we'll actually use $[1, 0, 1, 0, 0]$). While counting points over $\mathbb{F}_5, \mathbb{F}_{5^2}$ and $\mathbb{F}_{5^3}$ was relatively quick using the Magma's function

```
#Points(x,GF(5^2))
```

to find all points, things started to slow down after $\mathbb{F}_{5^4}$. Counting the points over $\mathbb{F}_{5^6}$ took a whole day, and the calculations had to be interrupted for $\mathbb{F}_{5^7}$ as it was starting to use too much memory. Using the data found so far, we can calculate that the first 6 coefficients are $26, 95, -3300, -34125, 91250$ and 2828125.

At this point, we try to find as many divisors as possible so we can use Elsenhans and Jahnel's method. To do this, recall that to find divisors on the Fermat Surface we "separated" the defining polynomial in two components. A good guess would be to look at the line L_1 defined by $(x^4 + x^2y^2 + y^4, w^4 + w^2z^2 + z^4)$. Using Magma we find that L_1 has 8 irreducible components, which in turn gives a rank 6 Gram Matrix. Next we try the line defined by $(x^4 + y^4 + z^4 + w^4, x^2y^2 + z^2w^2)$, which again gives us 8 distinct prime components which are all different from the 8 divisors already found. Unfortunately, the rank of the Gram Matrix of those 16 divisors is 7. We then try this technique with a few more lines we guess, but could not get a better rank than 7.

As having only 6 coefficients and finding only 7 divisors is not enough to find the characteristic polynomial of the Frobenius map, there are several options. Either we try to find the 7th coefficient, which probably involves developing better tools for point counting, or we keep trying to look for divisors knowing that we need to find at least 9. We could also look for divisors defined over different extensions of $\mathbb{F}_5$ and see if they give us more information on the characteristic polynomial which we can exploit. Finally we can decide to pick another surface and see if it is easier to work with. Having chosen the last option, we count points again only to find that the same numbers are appearing. Upon checking all other 5 surfaces we find that all 7 surfaces have the same number of points defined over $\mathbb{F}_5$ to $\mathbb{F}_{5^5}$ (they are $[0, 1112, 15360, 402072, 9753600]$).

The first thing we can conclude from that, since the characteristic polynomial only depends on the number of points, is that we end up with the same characteristic polynomial for all 7 surfaces. In particular, since we have the Fermat Surface as one of our surfaces defined over $\mathbb{F}_5$, we know that the characteristic polynomial will give us an upper bound of 20, which is not a useful upper bound. The second thing is that either it is just a coincidence, which is likely as $\mathbb{F}_5$ is small, or there something deeper going on. In the later case, as all K3 surfaces of rank 20 or higher are Kummer surfaces, one can guess that they are all the same Kummer surface. Currently more work is being done to investigate this possibility.

Over $\mathbb{F}_7$ we find 141 non-singular surfaces, of which we are interested in 51. Keeping in mind what happened with $\mathbb{F}_5$, we decided to group the different surfaces depending how many points were defined on the surface over small extensions of $\mathbb{F}_7$. The results are in the following table (note that we have $C \leq D \leq E$ except when any of them is 0, in which case we always put 0 to the right):

The surfaces	Number of points over			
	$\mathbb{F}_7$	$\mathbb{F}_{7^2}$	$\mathbb{F}_{7^3}$	$\mathbb{F}_{7^4}$
$[1,0,\pm3,0,0], [1,\pm2,3,3,6], [1,\pm2,4,4,6]$	48	3288	117936	5816856
$[1,\pm2,1,3,4]$	104	3288	120680	5816856
$[1,0,3,4,4], [1,0,4,6,6], [1,\pm1,3,0,0], [1,\pm2,6,6,0]$	16	3320	116944	5811864
$[1,0,1,1,1], [1,0,1,6,6], [1,\pm2,0,0,0]$	56	3320	116984	5811864
$[1,0,1,3,6], [1,\pm1,4,0,0], [1,\pm1,1,6,6], [1,\pm2,1,1,0], [1,\pm2,1,6,0]$	72	3320	119688	5811864
$[1,0,4,1,1], [1,0,3,3,3], [1,\pm1,1,1,1]$	128	3320	122432	5811864
$[1,0,1,3,3], [1,0,1,4,4], [1,\pm1,3,3,4], [1,\pm3,3,4,4]$	40	3384	118120	5808024
$[1,0,3,4,6], [1,\pm1,0,0,0], [1,\pm1,4,4,4], [1,\pm3,3,3,3,3]$	96	3384	120864	5808024
$[1,0,0,0,0], [1,0,6,0,0], [1,\pm3,6,6,6]$	64	3480	118336	5817624
$[1,0,1,0,0], [1,\pm3,1,1,6]$	120	3480	121080	5817624

As before we choose a surface, this time $[1, 0, 1, 3, 6]$, and count points on it. This time we run into computational problems from $\mathbb{F}_{7^6}$ onwards and as a result can only calculate the first 5 coefficients of the characteristic polynomial. Hence we try again to see how many divisors we can find, and using a similar approach as before we consider the prime components of the three curves defined by $(x^4 + 3x^2y^2 + y^4, f_1)$, $(x^4 + 4x^2z^2 + z^2, f_2)$ and $(x^4 + 6x^2w^2 + w^4, f_3)$. This gives 14 divisors, but if we apply every element of Γ to our list, and remove the repeated entries, we find a total of 20 divisors. Upon calculating the Gram Matrix, we see that these 20 divisors are generated from 7 of them. Again we do not have enough information to find the characteristic polynomial, so more work needs to be done here, which is currently being attempted.

Out of interest we also noted that there are no non-singular curves over $\mathbb{F}_8$, and we found that over $\mathbb{F}_9$ there are 227 surfaces that we are interested in. But upon counting points, we find that we only get 17 different values over $\mathbb{F}_9$ and 14 different values over $\mathbb{F}_{9^2}$, suggesting that the 227 surfaces can be grouped into 17 different sets.

4.3 Attempts to a lower bound

When looking for divisors over the finite field, a strategy that was often used was to "separate" the defining polynomial into two set of equations, and looking at the irreducible component of the curve defined by the two equations. As we are now looking at divisors over an algebraically closed field, finding the prime components corresponds to finding the factorisation of each polynomial. For simplicity let us first concentrate on the surfaces of the form $[A, 0, C, D, E]$ with A, C, D and E all non-zero. With surfaces of this form, we can break the defining polynomial of our surface into the two polynomials $A(x^4+y^4+z^4)+Cx^2y^2+Dx^2z^2+Ey^2z^2 =: F$ and $w^2\left(Aw^2+Cz^2+Dy^2+Ex^2\right) =: G$. Since A, C, D and E are all non-zero, we know that $Aw^2+Cz^2+Dy^2+Ex^2$ is an irreducible polynomial over $\overline{\mathbb{Q}}$. (One quick way to see this is, if it were reducible, then it has to factor as $(a_0x+a_1y+a_2z+a_3w)(b_0x+b_1y+b_2z+b_3w)$, but then we have $a_ib_i \neq 0$ for all i and $a_ib_j + a_jb_i = 0$ for all $i \neq j$. This leads to a contradiction). So the question is how does F factorise.

Lemma 4.2. *If X is a* K3 *surface defined by $[A, 0, C, D, E]$ with A, C, D and E all non-zero, then $A(x^4+y^4+z^4)+Cx^2y^2+Dx^2z^2+Ey^2z^2 =: F$ is irreducible over $\overline{\mathbb{Q}}$.*

Proof. We will show that if F factorises, then we have our surface X is singular and so by definition can not be a K3 surface.

First suppose that $F = Q_1Q_2$ where Q_1 and Q_2 are two quadratic polynomials with only x^2, y^2 and z^2 terms. In this case a relabelling of the variables $X := x^2, Y := y^2$ and $Z := z^2$ allows us to assume we are trying to factorise a quadratic. Hence if F can be factorised in that form, the corresponding conic in $\mathbb{P}^2$ is singular. That is, there is a non-trivial point $[X_1 : Y_1 : Z_1]$ which solves

$$\begin{pmatrix} 2AX+CY+DZ \\ CX+2AY+EZ \\ DX+EY+2AZ \end{pmatrix} = 0$$

On the other hand the Jacobian of F is

$$\begin{pmatrix} 4Ax^3+2Cxy^2+2CDxz^2+2Exw^2 \\ 4Ay^3+2Cyx^2+2Dyw^2+2Eyz^2 \\ 4Az^3+2Czw^2+2Dzx^2+2Ezy^2 \\ 4Aw^3+2Cwz^2+2Dwy^2+2Ewx^2 \end{pmatrix} = \begin{pmatrix} 2x(2AX+CY+DZ+Ew^2) \\ 2y(CX+2AY+EZ+Dw^2) \\ 2z(DX+EY+2AZ+Cw^2) \\ 2w(2Aw^2+Cz^2+Dy^2+Ex^2) \end{pmatrix}$$

which is zero at the point $[x_1 : y_1 : z_1 : 0]$ where $x_1^2 = X_1, y_1^2 = Y_1$ and $z_1^2 = Z_1$. Furthermore since the point $[x_1 : y_1 : z_1 : 0]$ solves F and the equation $w^2(Aw^2+Bz^2+Cy^2+Dx^2)$, we have that it is defined on our surface. Hence in that case we do not have a K3 surface.

Next suppose that $F = (a_0x^2+a_1y^2+a_2z^2+a_3xy+a_4xz+a_5yx)(b_0x^2+b_1y^2+b_2z^2+b_3xy+b_4xz+b_5yz)$ with at least one of a_3, a_4, a_5, b_3, b_4 or b_5 non-zero (as otherwise, we are in the above case), and without loss of generality assume $a_3 \neq 0$. Expanding the right hand side and comparing equations we get the following set of equations

$$\begin{aligned}
a_0b_0 = a_1b_1 = a_2b_2 &= A \neq 0 \\
a_0b_1 + a_1b_0 + a_3b_3 &= C \neq 0 \\
a_0b_2 + a_2b_0 + a_4b_4 &= D \neq 0 \\
a_1b_2 + a_2b_1 + a_5b_5 &= E \neq 0 \\
a_0b_3 + b_0a_3 = a_0b_4 + a_4b_0 = a_0b_5 + a_5b_0 &= 0 \\
a_1b_3 + b_1a_3 = a_1b_4 + a_4b_1 = a_1b_5 + a_5b_1 &= 0 \\
a_2b_3 + b_2a_3 = a_2b_4 + a_4b_2 = a_2b_5 + a_5b_2 &= 0 \\
a_3b_4 + a_4b_3 = a_3b_5 + a_5b_3 = a_4b_5 + a_5b_4 &= 0
\end{aligned}$$

The first set of equations implies that $a_0, b_0, a_1, b_1, a_2, b_2$ are all non-zero and hence we can express b_0, b_1 and b_2 in terms of A and a_0, a_1, a_2. Then using the set of equations of line 5, we can also express b_3, b_4 and b_5 in terms of a_i and A. Substituting all of this in the last set of equations we find that

$$a_3b_4 + a_4b_3 = \frac{-2Aa_3a_4}{a_0^2} = 0$$

Since A and a_3 are not 0, we have $a_4 = 0$, which in turn implies that b_4 is 0. Similarly we can deduce a_5 and b_5 are both zero. Next we calculate

$$a_1b_3 + b_1a_3 = -Aa_3\left(\frac{a_1}{a_0^2} - \frac{1}{a_1}\right) = 0$$

giving us that $\frac{a_1}{a_0} = \frac{a_0}{a_1}$. Similarly we can deduce $\frac{a_0}{a_2} = \frac{a_2}{a_0}$, and hence the three ratios $\frac{a_0}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_0}$ are all ± 1. Substituting all of this in the equations for C, D, E we find that

$$\begin{aligned}
C &= A\left(\frac{a_0}{a_1} + \frac{a_1}{a_0} - \frac{a_3^2}{a_0^2}\right) = A\left(\pm 2 - \left(\frac{a_3}{a_0}\right)^2\right) \\
D &= A\left(\frac{a_0}{a_2} + \frac{a_2}{a_0}\right) = \pm 2A \\
E &= A\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) = \pm 2A
\end{aligned}$$

In particular, we have D and E as explicit multiples of A. This means that F can be factorised into this form only if our surface belongs to one of the four families defined by $[1, 0, C', \pm 2, \pm 2]$. The next lemma shows that any surface in this family is in fact singular and hence not a K3 surface.

Lemma 4.3. *All the surfaces in the four families* $[1, 0, C', \pm 2, \pm 2]$ *are all singular and hence not* K3 *surfaces.*

Proof. Let X be the family of surface $[1, 0, 2, 2, E]$, where we permuted and relabelled the variables. Consider its Jacobian

$$\begin{pmatrix}
2x(2x^2 + 2y^2 + 2z^2 + Ew^2) \\
2y(2x^2 + 2y^2 + Ez^2 + 2w^2) \\
2z(2x^2 + Ey^2 + 2z^2 + 2w^2) \\
2w(Ex^2 + 2y^2 + 2z^2 + 2w^2)
\end{pmatrix}$$

One can see that setting $z = 0$ and $w = 0$, we just need to show that the points $[1 : \pm i : 0 : 0] \in X$ as $1^2 + (\pm i)^2 = 0$. The only term not involving z and w in the defining polynomial of X is $x^4 + 2x^2y^2 + y^4$, which $x = 1$, $y = \pm i$ clearly solves.

For the family $[1, 0, -2, -2, E]$ then $[1, \pm i, 0, 0]$ are still singular points, while for $[1, 0, \pm 2, \mp 2, E]$ we need to take the points $[1 : \pm 1 : 0 : 0]$. □

Finally we consider when F factors in a linear polynomial and a cubic. Again setting $F = (a_0x + a_1y + a_2z)(b_0x^3 + b_1y^3 + b_2z^3 + b_3x^2y + b_4xy^2 + b_5x^2z + b_6xz^2 + b_7z^2y + b_8zy^2 + b_9xyz)$, expanding and comparing coefficients, we get 15 equations. First we notice that since a_i are all non-zero, we must have all b_i are non-zero. Furthermore as before, we can use the equations to express everything in terms of A, a_0, a_1 and a_2. Doing so, we find equations of the form

$$2Aa_0 \left(\frac{a_1^2}{a_2^3} - \frac{a_2}{a_1^2} \right) = 0$$

allowing us to deduce that the three ratios $\left(\frac{a_0}{a_1}\right)^2, \left(\frac{a_1}{a_2}\right)^2, \left(\frac{a_2}{a_0}\right)^2$ are all ± 1, which in turn allows us to deduce

$$\begin{aligned} C &= A \left(\frac{a_0^2}{a_1^2} + \frac{a_1^2}{a_0^2} \right) = \pm 2A \\ D &= A \left(\frac{a_0^2}{a_2^2} + \frac{a_2^2}{a_0^2} \right) = \pm 2A \\ E &= A \left(\frac{a_1^2}{a_2^2} + \frac{a_2^2}{a_1^2} \right) = \pm 2A \end{aligned}$$

Hence, if F has a linear factor, then our surface is one of the eight surfaces $[1, 0, \pm 2, \pm 2, \pm 2]$ which are all part of the families $[1, 0, C', \pm 2, \pm 2]$. Hence reusing Lemma 4.3 we are done. □

Hence for surfaces of the form $[A, 0, C, D, E]$, we can start with two divisors: H defined by the equations F and w, L defined by the equations F and $Aw^2 + Cz^2 + Dy^2 + Ez^2$. As negating terms do not change the equations, applying the elements of Γ to H and L gives only a total of 8 divisors. Of those 8, we have 4 of them (H and the associated divisors under automorphism) belonging to the Hyperplane divisors class and so can only be counted once. Upon trying a few numerical examples, we keep finding that the rank of the Gram Matrix defined by these 5 divisors is 1.

To investigate why this happens let us have a closer look at what we are doing. We are taking a divisor L defined by two polynomials, say F and G. We have that, considering it as a subscheme of $\mathbb{P}^3$, L is by definition a complete intersection. To calculate the Gram Matrix defined by L and H (the Hyperplane divisor), we calculate the genus of L and H as well as the degree of the intersection of L and H. But the degree of the intersection of L and H is by definition the degree of L. Furthermore, as L is a complete intersection, we know that its degree is the product of the degree of F and G.

As for the calculation of the genus, using the Adjunction Formula twice, once on F and then to G, gives the canonical divisor of L is $(d_1 + d_2 - 4)H \cdot d_1H \cdot d_2H$, where d_1, d_2 are the degrees of the polynomials F and G respectively (c.f. [Har77, II, Ex 8.4]). In particular, by the Riemann-Roch Theorem

$$g_L = \frac{d_1d_2^2 + d_2d_1^2}{2} - 2d_1d_2 + 1.$$

Since $H^2 = 4$, (it has genus 3), we have that the Gram Matrix defined by L and H is

$$\begin{pmatrix} 4 & d_1d_2 \\ d_1d_2 & d_1d_2(d_1+d_2-4) \end{pmatrix}$$

This matrix has rank 1 if and only if there exists a such that $4a = d_1d_2$ and $d_1d_2a = d_1d_2(d_1+d_2-4)$. Rearranging, this gives the matrix has rank 1 if and only if d_1, d_2 are such that $d_1d_2 - 4(d_1+d_2) + 16 = 0$. This gives $d_1(d_2-4) = 4(d_2-4)$, so if $d_2 = 4$ then d_1 can be any integer. On the other hand if $d_2 \neq 4$, then $d_1 = 4$. So in either case, we find that the matrix has rank 1 if and only if at least one of d_1 or d_2 is of degree 4.

The immediate result of this to the work we have been doing is that our method is not going to work. So if we want to continue analysing the surfaces of the form $[A, 0, C, D, E]$ we need to split the defining polynomial into two smaller polynomials that are reducible, or we need to look at curves defined by higher degree polynomials (which are harder to systematically find without guessing). This result can also be applied to any K3 surface in our family: if we find a curve which is a complete intersection of two polynomials F and G such that at least one of them is irreducible of degree 4, then that curve can be ignored.

With this result in mind, we next tried to find subfamilies which can be split into two reducible polynomials. After finding various results on subfamilies of our family which are not K3 surfaces or polynomials which are irreducible over $\overline{\mathbb{Q}}$ we found the following lemma.

Lemma 4.4. *Let $C \neq \pm 2$, then every surface in the family $[1, 0, C, 0, 0]$ is a* K3 *surface with Picard rank at least* 10.

Proof. We first show that each surfaces are non-singular by noting that the Jacobian is

$$\begin{pmatrix} 2x(2x^2+Cy^2) \\ 2y(2y^2+Cx^2) \\ 2z(2z^2+Cw^2) \\ 2w(2w^2+Cz^2) \end{pmatrix}$$

If $C \neq \pm 2$, then for the Jacobian to be zero we need to solve either $(C^2-4)x^2 = 0$ or $(C^2-4)z^2 = 0$. Either one implies that the only point of singularity is $[0:0:0:0] \notin \mathbb{P}^3$.

Next we show that each surface has rank at least 10. Let $\Delta = \sqrt{C^2-4}$ and

$$\alpha_\pm = \sqrt{\frac{-C \pm \Delta}{2}}$$

then we have the factorisation $x^4 + Cx^2y^2 + y^4 = (x+\alpha_+y)(x+\alpha_-y)(x-\alpha_+y)(x-\alpha_-y)$ over $\overline{\mathbb{Q}}$. Hence we can define 16 curves on our surface, namely the 16 curves defined by $(x \pm \alpha_\pm y, z \pm \alpha_\pm w)$. We first remark that each of these lines are complete intersection, so using the work above, have self intersection -2. Furthermore we can see that they are all lines (we have the bijection $[x : y : z : w] \mapsto [x : z] \in \mathbb{P}^1$), and hence can intersected each other at most once (with multiplicity one). Furthermore as they all have the same form we know when they intersect, namely when they have one component in common. Hence putting our 16 lines in the order $(x+\alpha_+y, z+\alpha_+w), (x+\alpha_-y, z+\alpha_+w), (x-\alpha_+y, z+\alpha_+w), (x-\alpha_-y, z+\alpha_+w), \ldots, (x-\alpha_-y, z-\alpha_-w)$ we get the Gram Matrix:

$$\begin{pmatrix} M & I_4 & I_4 & I_4 \\ I_4 & M & I_4 & I_4 \\ I_4 & I_4 & M & I_4 \\ I_4 & I_4 & I_4 & M \end{pmatrix}$$

where I_4 is the four by four identity matrix and

$$M = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$$

Note that this Gram Matrix does not depend on the value of C. We then get a computer telling us that the rank of that matrix is 10, concluding that each surface in that family has rank at least 10. □

4.4 Further work

There are several areas that are worth exploring to help us find the rank of our family. We should try to find a better way to count points defined over a surface over a finite field. When counting points we never exploited the fact we had a group of order 16 acting on our surface. One way to do so would be to look up the algorithms that the Magma function uses to find points, and see if we can adapt it to include the information we know using the group Γ. As mentioned before, we could have a look to see if we can find different divisors defined over extensions of our original field. Then if we can work out which root of the characteristic polynomial they correspond to, we can use this to work out the characteristic polynomial.

Another thing to keep in mind is that while our work over finite fields is aimed at giving an upper bound on the rank of the surface $X_{\overline{\mathbb{F}_p}}$, what we really want is the actual rank of $X_{\overline{\mathbb{F}_p}}$ so as to have an upper bound on X. Recall that X is an elliptic surface if it has a fibration $f : X_{\overline{\mathbb{F}_p}} \to C$, where C is a projective curve, such that all but finitely many of the fibers are elliptic. If we know that $X_{\overline{\mathbb{F}_p}}$ is an elliptic surface with E being the generic fiber of f, then we can use the Shioda-Tate formula:

$$\rho(X) = 2 + r + \sum_{v \in C} (m_v - 1)$$

where $r = \text{rk}(E)$ and m_v is the number of irreducible components of $f^{-1}(v)$. Hence only the singular fiber contributes, and the possible singular fibers that can occur have been classified; see [vL05, Sec 2.4] for more details including a proof of the Shioda-Tate formula. If we use this in conjunction with the result that states "Any K3 surface with Picard number $\rho \geq 5$ admits an elliptic fibration" [Sch12, Thm 6], then we know that all the surfaces we have looked at so far have been elliptic surfaces and in theory we can use Shioda-Tate. The difficulty comes in finding the elliptic fibration in question, so we need to look into how one can find the elliptic fibration of an elliptic surface.

Recall that when looking at our family in $\mathbb{F}_5$, we found that all possible surfaces defined had the same number of points. From this we concluded that they all had Picard number at least 20. This implies that they are all Kummer surfaces and hence we can ask if they are all the same K3 surface. A K3 Kummer surface is the minimal resolution of a quartic in $\mathbb{P}^3$ with 16 singular points. All such surfaces, including the quartic in $\mathbb{P}^3$ with the 16 singular points also called a Kummer surface, come from an Abelian surface quotiented by the involution $x \mapsto -x$. Finally since an Abelian surface comes from the Jacobian of a hyperelliptic surface of genus 2, we have that a Kummer surface is defined by a hyperelliptic surface C. So to answer our question, for each Kummer surface we need to find the hyperelliptic curve that defines it $C_X : y^2 = f_X(x)$. Then as a hyperelliptic curve is a double cover of $\mathbb{P}^1$, we can use the automorphisms of $\mathbb{P}^1$ to check if $f_{X_1}(x)$ and $f_{X_2}(x)$ are isomorphic.

While we did not find any non-singular surfaces in $\mathbb{F}_2$, we can use the surfaces which only had singular points. For such a surface we can use blow-up to resolve the singularities at each points, and as blowing up increases the Picard number by 1, we can calculate the Picard number of the new non-singular surface to deduce the Picard number of our original surface.

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