Continued Fractions

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Introduction

The origin of continued fractions as we know them today is a source of debate among historians. Many claim that continued fractions were developed in the last decades of the 16^{th} century by the Italian mathematicians **Raphael Bombelli** and **Pietro Cataldi**, while others believe that the theory of continued fractions began one century later with **William Brouncker** and **John Wallis**. Some historians even claim that the Euclidean algorithm marks the birth of continued fraction theory and date this theory back to the 3^{rd} century B.C. with **Euclid** [3].

Regardless of its origins, continued fractions have played a central role in the development of many mathematical theories and, even today, are still a very active line of research. In the present dissertation, I shall define continued fractions, deduce their main properties, and explain their contribution to two fields of mathematics: **complex analysis** and **number theory**.

In chapter 1 of this dissertation, I focus on the **computation** of continued fractions and the main **transformations** that allow us to work with them.

In chapter 2, I work with continued fractions with coefficients in the **complex numbers**. I start by setting the foundations of the theory of **convergence** and then apply these results to define **meromorphic functions** as continued fractions. In order to do so, I link the theory of continued fractions with the theory of **formal power series** and a relation is established between **Padé approximants** and a particular kind of continued fractions known as **regular C-fractions**. The chapter finishes with the analysis of the associated C-fractions of a useful family of functions known as **hypergeometric functions**.

In chapter 3, the applications of **simple continued fractions** (a special case of continued fractions with integer coefficients) to many areas of **number theory** are shown. An **equivalence** is defined between real numbers and simple continued fractions and, based on this equivalence, one can study how well irrational numbers can be **approximated** by rational numbers. Afterwards, I study the correspondence between **periodic simple continued fractions** and **quadratic irrationalities** and lastly, I apply the theory of simple continued fractions to find the solutions of **Pell's equation**. Based on this, I discuss the computation of **fundamental units in real quadratic fields.**

I hope that this dissertation helps the reader understand some of the most important advances of continued fractions throughout history. However, I would like to remark that there are many areas in which continued fractions had an impact that I have read about and, due to the limitation on the number of pages, I did not have the chance to include.

For example, although not limited to, **dynamical systems** [12], the **geometry of numbers** [21] or **hyperbolic geometry** [37].

Unlike many other mathematical concepts that are abstract and hard to grasp, the nature of continued fractions is **algorithmic**, which makes them easier to understand and apply to concrete examples. This makes them **approachable**, in the sense that a strong background in mathematics is not needed to understand and follow many of the proofs in this dissertation. Nevertheless, I would not like the reader to mistakenly think that the theory of continued fractions is "simple" or basic. That is why I have added to this dissertation sections about **open problems and lines of research** that I hope can help the reader to understand that most of the problems that involve continued fractions are hard despite their easy formulation.

Another aspect that I hope this dissertation shows is the huge impact that computer science has in the study of continued fractions nowadays. In that regard, some programming languages like Mathematica have implemented functions and routines to work with continued fractions both numerically and symbolically [44]. However, while I was working with Mathematica, I found that the few functions it provided me with were insufficient to study all necessary computations with continued fractions. That is why I have developed a **fully functional Mathematica package** for continued fractions with more than 30 functions that allow the reader to reproduce most of the computations that I present in this dissertation.

The appendix A provides an organised **sample of this package** that can be reviewed while reading the dissertation. In this package, two ways of working with continued fractions have been coded: the usual standard, in which continued fractions are defined by the **lists** of their coefficients, and another one in which the coefficients are given to the functions as **pure functions**. The first ones are specially designed for **numerical computation**, while the second ones (which can be identified because they have a `K` at the end of their names) are designed for **symbolic computations**.

Appendices B and C follow the trend of most books on continued fractions of displaying and collecting information through **figures** and **tables**. These could have never been elaborated (quickly) if it were not for the functions that I coded in the package.

As a last comment, I would like to highlight the fact that this dissertation has specifically been written to be read in **electronic format**. The text in red provides hyperlinks that connect the different parts of the dissertation to allow the reader to go back and forth to check references, equations, code, figures, tables...

Notation

In terms of notation, we will use $\mathcal{R}$ to represent a (commutative) integral domain, $\mathcal{Q}$ to represent its quotient field, and $\mathbb{F}$ to represent a field. We will use $\mathbb{N}$ to represent the natural numbers (including zero) and $\mathbb{N}^+$ to represent all positive integers.

1 Properties and computation methods for continued fractions

1.1 Introduction

1.1.1 Continued fractions

Let $\mathcal{R}$ be an integral domain and $\mathcal{Q}$ its quotient field.

Definition 1.1. *A **linear fractional transformation** or a **Möbius transformation** on $\mathcal{R}$ is a mapping $\tau : \mathcal{Q} \to \mathcal{Q}$ of the form:*

$$\tau(w) = \frac{aw + b}{cw + d} \tag{1.1}$$

with $a, b, c, d \in \mathcal{R}$ and $ad - bc$ a unit. The representation of τ is not unique, as we can always multiply the numerator and the denominator by any unit of $\mathcal{R}$ without changing the transformation. Nevertheless, we will consider all these equivalent transformations the same. We will denote the set of all linear fractional transformations with coefficients in $\mathcal{R}$ by $\mathcal{M}_{\mathcal{R}}$.

Definition 1.2. *A **continued fraction** is a triple $\left\{\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{S_n\}_{n=0}^{\infty}\right\}$ where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of elements of $\mathcal{R}$ and $\{S_n\}_{n=0}^{\infty}$ is the sequence in $\mathcal{M}_{\mathcal{R}}$ defined in the following way:*

$$S_n = s_0 \circ s_1 \circ s_2 \circ \dots s_n$$
$$s_0(w) = w + b_0 \qquad s_k(w) = \frac{a_k}{w + b_k} \quad \text{for } k \in \mathbb{N}^+$$

An intuitive idea of the concept of a continued fraction is described by this notation:

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

We will use the following notation to represent abstract continued fractions:

$$b_0 + \operatorname*{K}_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)$$

Inside the text, these will be typed as $\mathbf{K}(\frac{a_n}{b_n})$. The letter **K** comes from *Kettenbruch*, which is *continued fraction* in German. This was the notation used in the classical text of Perron [32].

For concrete examples, we will use the following notation, developed by Pringsheim:

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots$$

I have chosen these conventions, but I also acknowledge that many other different ones have been used throughout history to represent continued fractions [4, Section 1.4.].

The concept of a finite or terminating continued fraction arises from the following analogous definition:

Definition 1.3. *A **terminating continued fraction** is a triple of **finite** sequences* $\left\{\{a_n\}_{n=1}^N, \{b_n\}_{n=0}^N, \{S_n\}_{n=0}^N\right\}$ *as described before.*

I will represent these continued fractions with any of these two notations:

$$b_0 + \mathop{\mathbf{K}}_{n=1}^{N}\left(\frac{a_n}{b_n}\right) \qquad\qquad b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_N|}{|b_N}$$

To a continued fraction, one may associate the following main elements:

Definition 1.4. *The **approximants** or **convergents** of a continued fraction are the elements* $w_n = S_n(0)$.

The linear fractional transformations are identified with the projective linear group $\mathrm{PGL}(2, \mathcal{R})$ via the identification:

$$\tau(w) = \frac{aw+b}{cw+d} \quad \Leftrightarrow \quad M_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where composition of transformations corresponds to matrix multiplication.

Definition 1.5. *We will call the elements p_n, q_n of the second column of the matrix:*

$$S_n = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix} := \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$$

*respectively the **n-th partial numerator** p_n and the **n-th partial denominator** q_n of the continued fraction $\mathbf{K}(\frac{a_n}{b_n})$. By convention, $p_{-1} = 1$, $p_0 = b_0$, $q_{-1} = 0$ and $q_0 = 1$.*

With this definition, it is easy to see that we have $w_n = \frac{p_n}{q_n}$ for $n \in \mathbb{N}$.

1.1.2 Basic properties

Proposition 1.1. A continued fraction $\mathbf{K}(\frac{a_n}{b_n})$ satisfies the following properties:

1. The **recurrence relations**:

$$\begin{cases} p_n = b_n p_{n-1} + a_n p_{n-2} \\ q_n = b_n q_{n-1} + a_n q_{n-2} \end{cases} \tag{1.2}$$

2. The **determinant formulas**:

$$\Delta_n = p_{n-1}q_n - p_n q_{n-1} = \prod_{k=1}^{n}(-a_k) \tag{1.3}$$

$$b_n\Delta_{n-1} = p_{n-2}q_n - p_n q_{n-2} = b_n \prod_{k=1}^{n-1}(-a_k) \tag{1.4}$$

3. The **Euler-Minding formula**:

$$w_n = b_0 - \sum_{k=1}^{n} \frac{\Delta_k}{q_k q_{k-1}} \tag{1.5}$$

Proof.

- (1.2) is deduced from the definition of matrix multiplication.
- (1.3) is deduced from the fact that $\Delta_n = \det(S_n)$ and that the determinant of a product is the product of the determinants. By definition, $\Delta_0 = 1$.
- (1.4) follows from

$$p_{n-2}q_n - p_n q_{n-2} = \begin{vmatrix} p_{n-2} & p_n \\ q_{n-2} & q_n \end{vmatrix} \overset{(1.2)}{=} \begin{vmatrix} p_{n-2} & b_n p_{n-1} + a_n p_{n-2} \\ q_{n-2} & b_n q_{n-1} + a_n q_{n-2} \end{vmatrix} = b_n \begin{vmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{vmatrix} = b_n \Delta_{n-1}.$$

- (1.5) can be proven by induction, since $w_0 = b_0$ and

$$w_n - w_{n-1} = \frac{p_n q_{n-1} - p_{n-1}q_n}{q_n q_{n-1}} \overset{(1.3)}{=} -\frac{\Delta_n}{q_n q_{n-1}}.$$

□

Theorem 1.1. *Two sequences of $\mathcal{R}$ $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ are the partial numerators and denominators of some continued fraction $\mathbf{K}(\frac{a_n}{b_n})$ if and only if $\Delta_n \neq 0$ for $n \in \mathbb{N}^+$, $\Delta_{n-1}|\Delta_n$, $\Delta_{n-1}|(p_{n-2}q_n - q_{n-2}p_n)$ for $n \geq 2$ and $q_0 = 1$. Then $\mathbf{K}(\frac{a_n}{b_n})$ is uniquely determined by*

$$\begin{cases} a_1 = -\Delta_1 \\ a_n = -\dfrac{\Delta_n}{\Delta_{n-1}} \quad \text{for } n \geq 2 \end{cases} \qquad \begin{cases} b_0 = p_0, \quad b_1 = q_1 \\ b_n = \dfrac{p_{n-2}q_n - q_{n-2}p_n}{\Delta_{n-1}} \quad \text{for } n \geq 2 \end{cases} \tag{1.6}$$

Proof. Let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be given. Then the elements a_n and b_n must be solutions of the system of linear equations defined by (1.2). This system has a solution in $\mathcal{Q}$ if and only if $\Delta_n \neq 0$ for $n \geq 1$ and $q_0 = 1$ and this solution is unique. Solving the equations, we get that the $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ must be as in (1.6) and our hypothesis guarantees that all the a_n and b_n belong to $\mathcal{R}$.
The procedure to compute the $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ has been coded in (A.1.1). □

1.2 Transformations of continued fractions

1.2.1 Equivalence relation on continued fractions

Definition 1.6. *Two continued fractions* $\mathbf{K}(\frac{a_n}{b_n})$ *and* $\mathbf{K}(\frac{c_n}{d_n})$ *are said to be equivalent if they have the same sequences of approximants* $\{w_n\}_{n=0}^{\infty}$ *and* $\{\tilde{w}_n\}_{n=0}^{\infty}$ *or, equivalently, if for all* $n \in \mathbb{N}$, $p_n\tilde{q}_n - \tilde{p}_n q_n = 0$.

This idea was introduced by Seidel, who also proved the following result:

Theorem 1.2. $\mathbf{K}(\frac{a_n}{b_n})$ *is equivalent to* $\mathbf{K}(\frac{c_n}{d_n})$ *(both with coefficients in* $\mathcal{R}$*) if and only if there is a sequence of elements of* $\mathcal{Q}$, $\{z_n\}_{n=0}^{\infty}$ *with* $z_0 = 1$, $z_n \neq 0$ *for all* $n \in \mathbb{N}$ *such that*

$$c_n = z_{n-1}z_n a_n \qquad d_n = z_n b_n$$

Proof. $\mathbf{K}(\frac{a_n}{b_n}) \sim \mathbf{K}(\frac{c_n}{d_n})$ if and only if there exist numbers $z_k \neq 0$ such that

$$\tilde{p}_n = p_n \prod_{k=0}^{n} z_k \qquad \tilde{q}_n = q_n \prod_{k=0}^{n} z_k$$

Since $q_0 = \tilde{q}_0 = 1$, it is clear that $z_0 = 1$. The expressions for c_n and d_n are obtained from theorem 1.1. □

This result is fundamental for proving the equivalence of certain kinds of complex continued fractions. We will analyse the following examples:

- Let $\mathbf{K}(\frac{a_n}{b_n})$ such that $b_1|a_1$ and $(b_{n-1}b_n)|a_n$ for all $n \in \mathbb{N}^+$ and let $\{z_n\}_{n=0}^{\infty}$ be the sequence:

$$z_0 = 1 \qquad z_n = \frac{1}{b_n}$$

  Then, $\mathbf{K}(\frac{a_n}{b_n}) \sim \mathbf{K}(\frac{c_n}{1})$, a continued fraction with partial denominators all equal to 1, other than d_0, which would be equal to b_0 (A.1.2). The expression for c_n would be:

$$c_1 = \frac{a_1}{b_1} \qquad c_n = \frac{a_n}{b_{n-1}b_n} \tag{1.7}$$

- Let $\mathbf{K}(\frac{a_n}{b_n})$ such that $(\prod_{k=1}^{n} a_{2k})|b_{2n}$ and $(\prod_{k=1}^{n} a_{2k-1})|b_{2n-1}$ for $n \in \mathbb{N}^+$ and let $\{z_n\}_{n=0}^{\infty}$ be the following sequence:

$$z_0 = 1 \qquad z_1 = \frac{1}{a_1}$$

$$z_{2m} = \frac{a_1a_3\ldots a_{2m-1}}{a_2a_4\ldots a_{2m}} \qquad z_{2m+1} = \frac{a_2a_4\ldots a_{2m}}{a_1a_3\ldots a_{2m+1}}$$

  Then, $\mathbf{K}(\frac{a_n}{b_n}) \sim \mathbf{K}(\frac{1}{d_n})$, a continued fraction with partial numerators all equal to 1 (A.1.3) and the d_n can be obtained by the expression:

$$d_0 = b_0 \qquad d_n = b_n \prod_{k=1}^{n} a_k^{(-1)^{n+1-k}} \tag{1.8}$$

1.2.2 Euler's identity

Let $\{c_n\}_{n=n=0}^{\infty}$ be a sequence of elements of $\mathcal{R}$ with $c_{n-1}|c_n$ for all $n \in \mathbb{N}^+$ and let f_n be

$$f_n = \sum_{k=0}^{n} c_k$$

If we set $p_n = f_n$ and $q_n = 1$ for all $n \in \mathbb{N}$, by theorem 1.1, we know that there exist sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ such that the approximants of $\mathbf{K}(\frac{a_n}{b_n})$ are f_n. Applying that theorem, we get that they are given by the formula:

$$\begin{cases} a_1 = c_1 \\ a_n = -\dfrac{c_n}{c_{n-1}} \quad \text{for } n \geq 2 \end{cases} \qquad \begin{cases} b_0 = c_0, \quad b_1 = 1 \\ b_n = 1 + \dfrac{c_n}{c_{n-1}} \quad \text{for } n \geq 2 \end{cases}$$

Now, defining the following sequence $\{\rho_n\}_{n=0}^{\infty}$:

$$\rho_0 = c_0 \qquad \rho_1 = c_1 \qquad \rho_n = \frac{c_n}{c_{n-1}}$$

we have

$$c_0 = \rho_0 \qquad c_n = \prod_{j=1}^{n} \rho_j$$

and we get the following identity:

Proposition 1.2 (Euler's identity). (A.1.4) For any sequence $\{\rho_n\}_{n=0}^{N}$ with $\rho_n \neq 0$ for all $n \in \{1, \dots, N\}$, we have

$$\rho_0 + \sum_{k=1}^{N} \Big(\prod_{j=1}^{k} \rho_j\Big) = \rho_0 + \frac{\rho_1|}{|1} + \frac{-\rho_2|}{|1+\rho_2} + \dots + \frac{-\rho_N|}{|1+\rho_N} \tag{1.9}$$

Wherever this makes sense (for example, in topological fields), this identity remains true if we let $N \to \infty$.

One of the main uses of Euler's identity is that it allows us to understand power series as continued fractions. For example, let us consider in $\mathbb{C}$, the exponential series $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. Then, letting $c_k = \frac{z^k}{k!}$, we get that $\rho_n = \frac{z}{n}$, so we can express e^z as a continued fraction:

$$e^z = 1 + \frac{z|}{|1} + \frac{-\frac{z}{2}|}{|1+\frac{z}{2}} + \frac{-\frac{z}{3}|}{|1+\frac{z}{3}} + \dots = 1 + \frac{z|}{|1} + \frac{-z|}{|1+z} + \frac{-z|}{|2+z} + \frac{-2z|}{|3+z} + \dots$$

This is a small preview of chapter 2, where we will explore methods to represent complex functions as continued fractions in ways that ensure us a better convergence than the power series we initially start from.

1.2.3 Contractions and extensions of continued fractions

Definition 1.7. *We say that* $\mathbf{K}(\frac{c_n}{d_n})$ *is a* ***contraction*** *of* $\mathbf{K}(\frac{a_n}{b_n})$ *if its approximants* $\tilde{w}_n$ *are a subsequence of the approximants* w_n *of* $\mathbf{K}(\frac{a_n}{b_n})$*. In this case, we say that* $\mathbf{K}(\frac{a_n}{b_n})$ *is an* ***extension*** *of* $\mathbf{K}(\frac{c_n}{d_n})$*.*

We will focus on the following two cases of contraction:

Definition 1.8. *An* ***even part*** *of* $\mathbf{K}(\frac{a_n}{b_n})$ *is a contraction whose approximants are* $\tilde{w}_n = w_{2n}$*. It is the* ***canonical even part*** (A.1.5) if $\tilde{p}_n = p_{2n}$ and $\tilde{q}_n = q_{2n}$ for all $n \in \mathbb{N}$.

Theorem 1.3. $\mathbf{K}(\frac{a_n}{b_n})$ *has an even part if and only if* $b_{2n} \neq 0$ *for all* $n \in \mathbb{N}^+$*. If* $b_{2n-2}|(a_{2n-1}b_{2n})$ *for* $n \geq 2$*, it has a canonical even part that is given by* $\mathbf{K}(\frac{c_n}{d_n})$ *where*

$$\begin{cases} c_1 = a_1 b_2 \\ c_n = -\dfrac{a_{2n-2}a_{2n-1}b_{2n}}{b_{2n-2}} & \text{for } n \geq 2 \end{cases} \qquad \begin{cases} d_0 = b_0, \quad d_1 = a_2 + b_1 b_2 \\ d_n = a_{2n} + b_{2n-1}b_{2n} + \dfrac{a_{2n-1}b_{2n}}{b_{2n-2}} & \text{for } n \geq 2 \end{cases}$$

Proof. It is a standard application of theorem 1.1, as we can see in the book of Lorentzen and Waadeland [28, Theorem 2.19.]. □

Definition 1.9. *An* ***odd part*** *of* $\mathbf{K}(\frac{a_n}{b_n})$ *is a contraction with approximants* $\tilde{w}_n = w_{2n+1}$*. An odd part is the* ***canonical odd part*** (A.1.6) if $\tilde{p}_n = p_{2n+1}$ and $\tilde{q}_n = q_{2n+1}$ for all $n \in \mathbb{N}^+$.

Theorem 1.4. $\mathbf{K}(\frac{a_n}{b_n})$ *has an odd part if and only if* $b_{2n+1} \neq 0$ *for all* $n \in \mathbb{N}^+$*. If* $b_1|a_1$ *and* $b_{2n-1}|(a_{2n}b_{2n+1})$ *for all* $n \geq 2$*, it has a canonical odd part that is then given by* $\mathbf{K}(\frac{c_n}{d_n})$ *where*

$$\begin{cases} c_2 = -\dfrac{a_3 a_4 b_1 b_5}{b_3} \\ c_n = -\dfrac{a_{2n-1}a_{2n}b_{2n+1}}{b_{2n-1}} & \text{for } n \neq 2 \end{cases} \qquad \begin{cases} d_0 = b_0 + \dfrac{a_1}{b_1}, \quad d_1 = a_3 b_1 + b_1 b_2 b_3 + a_2 b_3 \\ d_n = a_{2n+1} + b_{2n}b_{2n+1} + \dfrac{a_{2n}b_{2n+1}}{b_{2n-1}} & \text{for } n \geq 2 \end{cases}$$

Proof. Again, it is a standard application of theorem 1.1. □

If instead of contractions, we want to study extensions, the following theorem is useful:

Theorem 1.5. *Let* $\mathbf{K}(\frac{a_n}{b_n})$ *be a continued fraction, let* $k \in \mathbb{N}^+$ *and let* $r \in \mathcal{Q}$ *such that* $rq_k, rq_{k+1} \in \mathcal{R}$*,* $(p_{k-1} - q_{k-1}r)|(p_k - q_k r)$ *and* $w_{k-1} \neq r \neq w_k$*. Then, there exists a continued fraction* $\mathbf{K}(\frac{c_n}{d_n})$ *whose approximants* $\tilde{w}_n$ *are*

$$\tilde{w}_n = \begin{cases} w_n & \text{for } n < k \\ r & \text{for } n = k \\ w_{n-1} & \text{for } n > k \end{cases}$$

Proof. Once again, it can be proved with theorem 1.1. The expressions of the partial numerators and denominators can be checked in the implementation of this function (A.1.7). □

This theorem shows how a single element of $\mathcal{Q}$ can be inserted in the sequence of approximants of a continued fraction and, by doing this repeatedly, it is possible to insert any denumerable set of elements in the sequence.

1.3 Computation of continued fractions

1.3.1 Algorithms for the computation of continued fractions

There are three ways of computing the approximants of continued fractions, each of which has different advantages and disadvantages.

1. The **forward recurrence algorithm** (A.2.1) consists of using the recurrence relation (1.2) to compute p_n and q_n.

2. **Euler-Minding summation** (A.2.2) consists of computing q_n by the recurrence relation and then finding w_n by means of the following expression deduced from the Euler-Minding formula:
$$w_n = w_{n-1} - \frac{\Delta_n}{q_n q_{n-1}}$$
3. The **backward recurrence algorithm** (A.2.3) consists of initialising the variable l_n to zero and then computing the l_k going backwards by setting
$$l_{k-1} := \frac{a_k}{b_k + l_k}$$
At the end of the process, the n^{th} approximant would be $w_n = b_0 + l_0$.

The first two algorithms have complexity $\mathcal{O}(n)$, whereas the third has complexity $\mathcal{O}(n^2)$. However, the backward recurrence algorithm is used in most cases over the first two algorithms because it is more stable numerically [28, Subsection 1.1.3.].

1.3.2 The Bauer-Muir transformation

Once we work in a field $\mathbb{F}$ and we introduce a notion of convergence, we can identify a continued fraction with the value it converges to. Hence, it is useful to develop transformations that change the value of the approximants without changing the limit. The following transformation is an example of this:

Definition 1.10. *The **Bauer-Muir transformation** (A.1.8) of a continued fraction* $\mathbf{K}(\frac{a_n}{b_n})$ *with respect to a sequence* $\{g_n\}_{n=0}^{\infty}$ *from* $\mathbb{F}$ *is the continued fraction* $\mathbf{K}(\frac{c_n}{d_n})$ *whose partial numerators* $\tilde{p}_n$ *and denominators* $\tilde{q}_n$ *are given by*

$$\begin{aligned} \tilde{p}_0 &= g_0 + b_0 & \tilde{q}_0 &= 1 & \\ \tilde{p}_n &= p_{n-1} g_n + p_n & \tilde{q}_n &= q_{n-1} g_n + q_n & \text{for } n \in \mathbb{N}^+ \end{aligned}$$

Theorem 1.6. *The Bauer-Muir transformation of* $\mathbf{K}(\frac{a_n}{b_n})$ *with respect to a sequence* $\{g_n\}_{n=0}^{\infty}$ *exists if and only if*
$$\lambda_n := a_n - g_{n-1}(b_n + g_n) \neq 0 \qquad \text{for } n \in \mathbb{N}^+.$$
If it exists, it is given by $d_0 + \mathbf{K}(\frac{c_n}{d_n})$ *where*
$$\begin{cases} c_1 = \lambda_1 \\ c_n = a_{n-1} \dfrac{\lambda_n}{\lambda_{n-1}} & \text{for } n \geq 2 \end{cases} \qquad \begin{cases} d_0 = g_0 + b_0, \quad d_1 = g_1 + b_1 \\ d_n = g_n + b_n - g_{n-2} \dfrac{\lambda_n}{\lambda_{n-1}} & \text{for } n \geq 2 \end{cases}$$

Proof. This is an application of theorem 1.1, where in this case:

$$\begin{aligned}\tilde{\Delta}_n &= \tilde{p}_{n-1}\tilde{q}_n - \tilde{p}_n\tilde{q}_{n-1} \\ &= (p_{n-2}g_n + p_{n-1})(q_{n-1}g_n + q_n) - (p_{n-1}g_n + p_n)(q_{n-2}g_n + q_{n-1}) \\ &= -(p_{n-2}q_{n-1} - p_{n-1}q_{n-2})(a_n - g_{n-1}b_n - g_{n-1}g_n).\end{aligned}$$

The first factor is non zero by the determinant formula (1.3), and the second one is equal to λ_n. Thus, $\tilde{\Delta}_n \neq 0$ and as we are in a field, all the conditions of theorem 1.1 are satisfied and so, the transformation exist and it is given by those c_n and d_n. □

The Bauer-Muir transformation is particularly useful if $\mathbf{K}(\frac{a_n}{b_n})$ and its transformation converge to the same value. This is the case for continued fractions with real positive coefficients and positive g_n, as shown by Perron [33, p. 27].

In the book of Lorentzen and Waadeland two applications of this transformation are described: for carrying out stable computations in $\mathbb{C}$ [28, Example 17.] and for expressing continued fractions as solutions of functional equations [28, Example 18.]. A survey of this transformation has also been written by Jacobsen [19].

1.4 Generalisations of continued fractions

The multiple uses of continued fractions in all fields of mathematics have encouraged the development of generalisations of continued fractions [9] where the coefficients are in more "exotic" rings than the ones we are going to work with in the next two chapters. These generalisations of continued fractions are out of the scope of this dissertation, but to mention some of them, there exist generalisations where the partial numerators and denominators are:

- Continued fractions themselves and multivariate expressions [10].
- Vectors in $\mathbb{C}_n$ [7].
- Square matrices in a field [16].
- Elements of a Banach space [11].

All these generalisations arise from the same motivation: since continued fraction have useful properties in rings such as $\mathbb{C}$ or $\mathbb{Z}$, by working with generalisations we would expect that some of these properties will still hold in those more exotic rings.

The main drawback with most of these multidimensional and multivariate generalisations is that they are very complex due to the fact that the lack of commutativity causes problems.

These generalisations have allowed mathematicians to tackle problems such as the **multivariate Padé approximation** [8] or **simultaneous Diophantine approximation**, whose simpler versions we will address in the following chapters.

2 Continued fractions in complex analysis

In this section, we will assume that $\mathcal{R} = \mathbb{C}$ and so, $\mathcal{Q}$ is also $\mathbb{C}$.

2.1 Convergence

We give $\mathbb{C}$ the usual Euclidean topology and study various notions of convergence of continued fractions with complex coefficients.

Definition 2.1. *A continued fraction* $\mathbf{K}(\frac{a_n}{b_n})$ *with coefficients in* $\mathbb{C}$ ***converges in the classical sense*** *to a value* $z \in \hat{\mathbb{C}}$ *if* $\lim_{n\to\infty} w_n = z$.

2.1.1 Examples of continued fractions

- $1 + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{1}{1}\right) = 1 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \dots$

  Its approximants are $w_n = \frac{F_{n+1}}{F_n}$ where F_n is the n^{th} Fibonacci number defined by the recurrence relation given by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

  It is well known that the continued fraction converges to $\lim_{n\to\infty} w_n = \frac{1+\sqrt{5}}{2}$, the **golden ratio** ϕ.

- $2 + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{-1}{2}\right) = 2 + \frac{-1|}{|2} + \frac{-1|}{|2} + \frac{-1|}{|2} + \dots$

  Its approximants are $w_n = \frac{n+2}{n+1}$, therefore, its continued fraction converges to $\mathbf{1}$.

- $\frac{2|}{|1} + \frac{1|}{|1} + \frac{-1|}{|1} + \frac{2|}{|1} + \frac{1|}{|1} + \frac{-1|}{|1} + \dots$

  Its approximants are $w_n = 0$ if $n \equiv 0 \mod 3$, $w_n = \frac{2^n}{2^{n+1}-3}$ if $n \equiv 1 \mod 3$ and $w_n = \frac{2^n}{2^{n+1}-2}$ if $n \equiv 1 \mod 2$. Therefore, $\lim_{n\to\infty} w_n$ does not exist and the continued fraction **does not converge** in the classical sense.

2.1.2 General convergence

It is worth noting that there is an additional notion of convergence for continued fractions called **general convergence** [18, 28]. It arises from the fact that when we work with Möbius transformations in the complex plane, there is another naturally arising metric which has some advantages over the Euclidean metric:

Definition 2.2. *We define the **chordal metric** (A.3.1) in the extended complex plane $\hat{\mathbb{C}}$ from the Euclidean metric in the following way:*

$$\mathfrak{m}(z_1, z_2) := \begin{cases} \dfrac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} & \textit{for } z_1, z_2 \in \mathbb{C} \\ \dfrac{2}{\sqrt{1+|z_1|^2}} & \textit{for } z_1 \in \mathbb{C}, z_2 = \infty \\ 0 & \textit{for } z_1 = z_2 = \infty \end{cases} \tag{2.1}$$

This metric has several advantages over the Euclidean metric such as:

- It is a **bounded metric**, since $\mathfrak{m}(z_1, z_2) < 2$ for all $z_1, z_2 \in \hat{\mathbb{C}}$. This is because this metric is, in essence, the Euclidean distance between the mapped points in the stereographic projection.
- $\hat{\mathbb{C}}$ is **compact** under this metric.
- The chordal metric and the Euclidean metric are **equivalent** on $\mathbb{C}$, which means that whenever we are working on $\mathbb{C}$ both notions of convergence of sequences are the same, that is, $z_n \to \hat{z} \in \hat{\mathbb{C}}$ if and only if $\mathfrak{m}(z_n, \hat{z}) \to 0$.
- If $\tau_n \to \tau \in \mathcal{M}_{\mathbb{C}}$, then the convergence is **uniform** with respect to the chordal metric. Furthermore, the chordal metric defines a **metric** in $\mathcal{M}_{\mathbb{C}}$ in the following way:

$$\sigma(\tau_1, \tau_2) := \sup_{z \in \hat{\mathbb{C}}} \mathfrak{m}(\tau_1(z), \tau_2(z))$$

Definition 2.3. *A continued fraction **converges generally** to a value $z \in \hat{\mathbb{C}}$ if and only if there exist two sequences $\{v_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ such that*

$$\liminf_{n\to\infty} \mathfrak{m}(v_n, c_n) > 0 \qquad \textit{and} \qquad \lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(c_n) = z \tag{2.2}$$

The value z can be shown to be unique and hence this is well-defined. Classical convergence implies general convergence. This can be seen by taking the sequences $\{v_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$, with $v_n = 0$ and $c_n = \infty$ for all $n \in \mathbb{N}$, as

$$\liminf_{n\to\infty} \mathfrak{m}(0, \infty) = 2 \qquad \text{and} \qquad z = \lim_{n\to\infty} S_n(0) = \lim_{n\to\infty} S_{n-1}(0) = \lim_{n\to\infty} S_n(\infty)$$

Nevertheless, general convergence does not imply classical convergence. The last continued fraction in the previous section is an example of this. We saw that it does not converge in the classical sense, but it can be seen that it converges in the general sense, as shown by Cuyt et al. [9, Example 1.2.1.]. This example shows how for every sequence $\{v_n\}_{n=0}^{\infty}$ bounded away from $-1, 0$ and ∞, $S_n(v_n)$ converges to $\frac{1}{2}$ and so, it makes sense to assign this value to that continued fraction.

2.1.3 Convergence criteria

As it often happens in discussing convergence, there is no infallible trick which allows us to analyse the convergence of every single continued fraction. However, there are certain criteria which are useful to work with, as they allow us to transform the problem of convergence of continued fractions into the problem of convergence of infinite series, which is more familiar.

Definition 2.4. *We say that a* ***sequence*** $\{c_n\}_{n=0}^{\infty}$ ***converges absolutely*** *if*

$$\sum_{n=1}^{\infty} |c_n - c_{n-1}| < \infty$$

I advise the reader to be careful with this definition. This notion of absolute convergence is a stronger notion of convergence for sequences that is well suited for continued fractions and is of course **not** the concept of absolute convergence of the **series** $\sum c_n$, rather of its differences.

Absolute convergence implies convergence to a finite value, since the convergence of the series implies that the telescopic sum $\sum_{n=1}^{\infty}(c_n - c_{n-1})$ converges and so,

$$\lim_{n\to\infty} c_n = c_0 + \sum_{n=1}^{\infty}(c_n - c_{n-1}).$$

Definition 2.5. *We say that a continued fraction* ***converges absolutely*** *if its sequence of approximants converges absolutely.*

Theorem 2.1 (Stern-Stolz criterion). *If* $\sum_{n=1}^{\infty} |b_n|$ *converges, then the continued fraction* $\mathbf{K}(\frac{1}{b_n})$ *diverges, for* $m = \{0, 1\}$ *the sequences* $\{p_{2n+m}\}_{n=0}^{\infty}$ *and* $\{q_{2n+m}\}_{n=0}^{\infty}$ *converge absolutely to finite values* $\mathcal{P}_m$ *and* $\mathcal{Q}_m$ *respectively, and*

$$\mathcal{P}_1\mathcal{Q}_0 - \mathcal{P}_0\mathcal{Q}_1 = 1 \tag{2.3}$$

Proof. We shall give a classical proof of this theorem. Let $\sum |b_n| < \infty$. Now, $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ are solutions of the recurrence relation:

$$x_n = b_n x_{n-1} + x_{n-2}.$$

We want to prove that for all n, there exists a constant λ such that:

$$|x_n| \leq \lambda \prod_{k=1}^{n}(1 + |b_k|).$$

To prove this, we apply induction. For the base case, it suffices to take $\lambda > \max\{x_{-1}, x_0\}$.

Let the previous statement be true for every $m < n$. For every solution $\{x_n\}_{n=0}^{\infty}$ of this relation,

$$\begin{aligned} |x_n| &\overset{(1.2)}{\leq} |b_n||x_{n-1}| + |x_{n-2}| \\ &\leq |b_n||x_{n-1}| + (1+|b_{n-1}|)|x_{n-2}| \\ &\leq \lambda |b_n| \prod_{k=1}^{n-1}(1+|b_k|) + \lambda(1+|b_{n-1}|)\prod_{k=1}^{n-2}(1+|b_k|) \\ &\leq \lambda \prod_{k=1}^{n}(1+|b_k|) \end{aligned}$$

Since $\prod_{k=1}^{n}(1+|b_k|)$ converges if and only if $\sum_{k=1}^{n}|b_n|$ converges, this means that $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ are bounded under our conditions. This bound implies that the two series $\sum|b_n p_{n-1}|$ and $\sum|b_n q_{n-1}|$ converge. Since solutions $\{x_n\}_{n=0}^{\infty}$ satisfy $x_n = b_n x_{n-1} + x_{n-2}$, this means that $\sum|p_n - p_{n-2}| < \infty$ and $\sum|q_n - q_{n-2}| < \infty$.

In other words, $\{p_{2n}\}_{n=1}^{\infty}$, $\{p_{2n+1}\}_{n=1}^{\infty}$, $\{q_{2n}\}_{n=1}^{\infty}$ and $\{q_{2n+1}\}_{n=1}^{\infty}$ converge absolutely to finite values $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{Q}_0$ and $\mathcal{Q}_1$. The identity (2.3) follows from taking the limit in the formula:

$$p_{2n-1}q_{2n} - p_{2n}q_{2n-1} \overset{(1.2)}{=} (-1)^{2n} = 1.$$

Finally, the divergence of $\mathbf{K}(\frac{1}{b_n})$, is deduced from the fact that

$$\lim_{n\to\infty} w_{2n} = \lim_{n\to\infty}\frac{p_{2n}}{q_{2n}} = \frac{\mathcal{P}_0}{\mathcal{Q}_0} \qquad \lim_{n\to\infty} w_{2n+1} = \lim_{n\to\infty}\frac{p_{2n+1}}{q_{2n+1}} = \frac{\mathcal{P}_1}{\mathcal{Q}_1}$$

and $\frac{\mathcal{P}_1}{\mathcal{Q}_1} - \frac{\mathcal{P}_0}{\mathcal{Q}_0} = \frac{1}{\mathcal{Q}_0\mathcal{Q}_1} \neq 0$ as $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are bounded. □

As classical convergence only depends on the value of the approximants, for any continued fraction $\mathbf{K}(\frac{a_n}{b_n})$, it is always possible to apply the transformation (1.7) such that $\mathbf{K}(\frac{a_n}{b_n}) \sim \mathbf{K}(\frac{1}{d_n})$. Thus, the Stern-Stolz criterion can be conditioned to the divergence of the following series:

Definition 2.6. *We define the* ***Stern-Stolz series*** *of* $\mathbf{K}(\frac{a_n}{b_n})$ *as the series:*

$$\mathcal{S} := \sum_{n=1}^{\infty}|d_n| = \sum_{n=1}^{\infty}|b_n\prod_{k=1}^{n} a_k^{(-1)^{n+1-k}}| \tag{2.4}$$

The study of the divergence of this series (A.3.2) is, in general, difficult. Nevertheless, the following proposition gives us some equivalent criteria:

Proposition 2.1. The Stern-Stolz series of $\mathbf{K}(\frac{a_n}{b_n})$ diverges if any of the following two conditions hold:

$$(i) \quad \sum_{n=2}^{\infty}\sqrt{\left|\frac{b_{n-1}b_n}{a_n}\right|} = \infty \tag{2.5}$$

$$(ii) \quad \liminf_{n\to\infty}\left|\frac{a_n}{b_{n-1}b_n}\right| < \infty \tag{2.6}$$

Proof. (i) It is clear that under the transformation (1.8), this criteria is equivalent to proving that

$$\sum_{n=2}^{\infty} \sqrt{|d_{n-1}d_n|} = \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} |d_n|$$

This is true because of the arithmetic-geometric mean inequality, as

$$\sum_{n=1} |d_n| \geq \sum_{n=2} \tfrac{1}{2}(|d_{n-1}| + |d_n|) \geq \sum_{n=2}^{\infty} \sqrt{|d_{n-1}d_n|}.$$

(ii) If the inferior limit of the sequence $\left\{\left|\frac{a_n}{b_{n-1}b_n}\right|\right\}_{n=2}^{\infty}$ is a finite number r, there is an infinite subsequence of finite numbers which are bounded below by r. Hence, there is a subsequence of finite numbers all greater than $\frac{1}{r}$ in $\left\{\left|\frac{b_{n-1}b_n}{a_n}\right|\right\}$ and so, (2.5) diverges. □

The Stern-Stolz criterion gives us conditions for the divergence of classical approximants. It would be interesting to see if there are conditions for the convergence of $\mathbf{K}(\frac{a_n}{b_n})$. The following theorem, which is due to Lane and Wall [25], gives a very practical answer:

Theorem 2.2 (Lane-Wall characterization). *Let $\mathbf{K}(\frac{a_n}{b_n})$ a continued fractions with approximants $\{w_n\}_{n=0}^{\infty}$. If both the even and odd part of $\mathbf{K}(\frac{a_n}{b_n})$ are absolutely convergent, then $\mathbf{K}(\frac{a_n}{b_n})$ converges if and only if its Stern-Stolz series diverges*

$$\sum_{n=1}^{\infty} |b_n \prod_{k=1}^{n} a_k^{(-1)^{n+1-k}}| = \infty \tag{2.7}$$

Proof. The proof of this statement is not specially complicated but it has been done in a very general setting which would be too long to include in this dissertation [25]. A simpler, less rigorous adaptation of this proof is also available [28, Theorem 3.3.]. □

Remark 2.1. The condition of absolute convergence of the odd and even part is necessary, regular convergence is not enough. Wall proved this and gave an example of it [43].

2.1.4 The parabola theorem

Probably, the most important theorem regarding convergence of continued fractions is the **parabola theorem**. Many proofs have been given of this theorem, some more analytical, others more geometrical. I will give a modern one [28, Theorem 3.43.], for which we will first need to explain the following concept:

Definition 2.7. *A sequence $\{V_n\}_{n=0}^{\infty}$ of sets $V_n \subset \hat{\mathbb{C}}$ is a sequence of **value sets** for $\mathbf{K}(\frac{a_n}{b_n})$ if and only if*

$$s_n(V_n) = \frac{a_n}{b_n + V_n} \subseteq V_{n-1} \quad \textit{for } n \in \mathbb{N}^+ \tag{2.8}$$

The reason why value sets are important is because if we consider for $n \in \mathbb{N}^+$ the sets $K_n := S_n(V_n)$, then $K_n = S_{n-1}(s_n(V_n)) \subseteq S_{n-1}(V_{n-1}) = K_{n-1}$, which implies that

$$S_n(w_n) \in K_n \subseteq \cdots \subseteq V_0 \qquad \text{for } w_n \in V_n.$$

In the case where the V_n are non-empty, closed sets; the K_n are non-empty and closed as well, which implies that the limit set:

$$K := \lim_{n\to\infty} K_n = \bigcap_{n=1}^{\infty} K_n$$

exists (and it is non-empty and closed). If we define $\mathrm{diam}(K) := \sup\{|v-w| : v, w \in K\}$ then it is easy to see that if $\mathrm{diam}(K) = 0$ then, K consists of a single point $z \in \mathbb{C}$ and if $0 \in V_n$ for all $n \in \mathbb{N}$, then, $\mathbf{K}(\frac{a_n}{b_n})$ converges to z in the classical sense.

We also need the following lemma:

Lemma 2.1. Let $\mathcal{T}_n := \tau_1 \circ \tau_2 \circ \ldots \tau_n$ for all $n \in \mathbb{N}^+$ where all $\tau_n \in \mathcal{M}_{\mathbb{C}}$ map the unit disk $\overline{\mathbb{D}}$ into a subset of itself, and assume there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset \hat{\mathbb{C}}$ such that

$$\liminf_{n\to\infty} \big||v_n| - 1\big| > 0 \quad \text{and} \quad \liminf_{n\to\infty} \big||\tau_n(v_n)| - 1\big| > 0.$$

If $\mathrm{diam}(\bigcap_{n=1}^{\infty} \mathcal{T}_n(\overline{\mathbb{D}})) \neq 0$, then for every $v \in \overline{\mathbb{D}}$, $\{\mathcal{T}_n(v)\}_{n=1}^{\infty}$ converges absolutely to the same constant $\gamma \in \overline{\mathbb{D}}$.

Proof. A slightly more general version of this lemma and its proof can be found in the book of Lorentzen and Waadeland [28, Lemma 3.8]. □

Now we can proceed with the theorem.

Theorem 2.3 (Parabola Theorem). *Let $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and let P_α be the parabolic region given by*

$$P_\alpha := \{z \in \mathbb{C} : |z| - \mathrm{Re}(ze^{-2\alpha i}) \leq \tfrac{1}{2}\cos^2(\alpha)\}$$

Let $\mathbf{K}(\frac{a_n}{b_n}) \sim \mathbf{K}(\frac{c_n}{1})$ be a continued fraction such that $\{c_n\}_{n=0}^{\infty} \subseteq P_\alpha$. Then the even and odd parts of the continued fraction $\mathbf{K}(\frac{a_n}{b_n})$ converge absolutely. Hence, by theorem 2.2, $\mathbf{K}(\frac{a_n}{b_n})$ converges if and only if its Stern-Stolz series (2.6) diverges.

The approximants $\{w_n\}_{n=0}^{\infty}$ of $\mathbf{K}(\frac{a_n}{b_n})$ are in the half plane

$$V_\alpha := \{w \in \mathbb{C} : \mathrm{Re}(we^{-\alpha i}) \geq -1/2\cos(\alpha)\} \cup \{\infty\}.$$

and, if it converges, it does to a value in the closure of V_α.

The regions P_α and V_α for some values of α are shown in appendix (B.1).

The boundary of P_α is a parabola with its focus at the origin, its axis of symmetry along the line $r_\alpha \equiv \{z \in \mathbb{C} : z = te^{2\alpha i}, t \in \mathbb{R}\}$, its vertex at $z_\alpha = -\frac{1}{4}e^{2\alpha i}\cos^2(\alpha)$ and its directrix at $d_\alpha \equiv \{z \in \mathbb{C} : z = -\frac{1}{2}e^{2\alpha i}\cos^2(\alpha) + te^{(2\alpha+\pi/2)i}, t \in \mathbb{R}\}$.

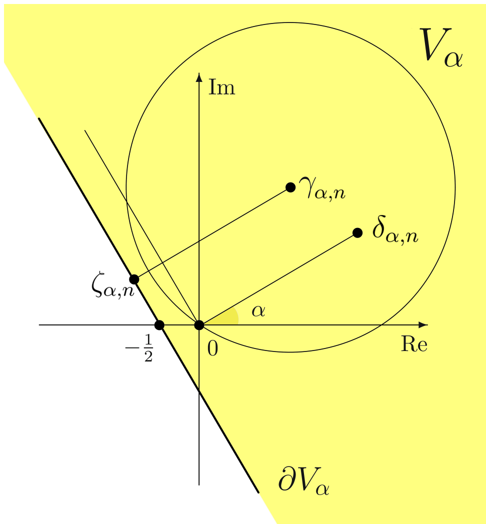
The boundary of V_α is the line $\partial V_\alpha \equiv \{z \in \mathbb{C} : z = -\frac{1}{2} + te^{(\alpha+\pi/2)i}, t \in \mathbb{R}\}$.

Proof. In order to prove this theorem, we will first have to prove that $\{V_\alpha\}_{n=0}^{\infty}$ is a sequence of value sets if and only if $c_n \in P_\alpha$ for every $n \in \mathbb{N}^+$. In order to do this, let us see what is the necessary condition for $s_n(V_\alpha) = \frac{c_n}{1+V_\alpha} \subseteq V_\alpha$ to hold.

As s_n is a Möbius transformation, we know that it maps lines to either circumferences or lines. It is easy to see that s_n maps ∂V_α to a circumference, as s_n satisfies $s_n^{-1}(\infty) = -1$ and $-1 \notin \partial V_\alpha$ for any $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\infty \notin s_n(\partial V_\alpha)$. Furthermore, either by performing some computations or by applying the result from Lorentzen and Waadeland [28, Theorem 3.6], it can be seen that $s_n(V_\alpha)$ is a closed disk with centre $\gamma_{\alpha,n}$ and radius $\rho_{\alpha,n}$, where

$$\gamma_{\alpha,n} = \frac{c_n e^{-\alpha i}}{\cos(\alpha)}, \qquad \rho_{\alpha,n} = \frac{|c_n|}{\cos(\alpha)}.$$

The disk $s_n(V_\alpha)$ is contained in V_α if and only if $\gamma_{\alpha,n}$ is in $s_n(V_\alpha)$ and the distance from $\gamma_{\alpha,n}$ to ∂V_α is less or equal than $\rho_{\alpha,n}$.



Let $\zeta_{\alpha,n}$ be the closest point of ∂V_α from $\gamma_{\alpha,n}$. Then,

$$\zeta_{\alpha,n} = \gamma_{\alpha,n} - (\tfrac{1}{2}\cos(\alpha) + \mathrm{Re}(\gamma_{\alpha,n} e^{-\alpha i}))e^{\alpha i}.$$

Let $\delta_{\alpha,n} = \gamma_{\alpha,n} - \zeta_{\alpha,n}$. From the diagram in the right we see that $\gamma_{\alpha,n} \in s_n(V_\alpha)$ if and only if $e^{-\alpha i}\delta_{\alpha,n} \geq 0$. The distance from $\gamma_{\alpha,n}$ to ∂V_α is

$$|\delta_{\alpha,n}| = \tfrac{1}{2}\cos(\alpha) + \mathrm{Re}(\gamma_{\alpha,n} e^{-\alpha i}).$$

Thus, a sufficient (and, in fact, necessary) condition for $s_n(V_\alpha) \subseteq V_\alpha$ is that

$$|\delta_{\alpha,n}| \geq \rho_{\alpha,n} \qquad \Leftrightarrow \qquad \tfrac{1}{2}\cos(\alpha) + \mathrm{Re}\Big(\frac{c_n e^{-2\alpha i}}{\cos(\alpha)}\Big) \geq \frac{|c_n|}{\cos(\alpha)}$$

Multiplying by $\cos(\alpha)$, which is always positive, we see that this is equivalent to $c_n \in P_\alpha$. Let $\mathbf{K}(\frac{c_n}{1})$ be a continued fraction with $c_n \in P_\alpha$. As $\infty \notin S_1(V_\alpha)$, the sets $K_n := S_n(V_\alpha)$ are bounded disks and $K_n \subseteq K_{n-1}$ for all $n \in \mathbb{N}^+$. Let $K = \bigcap_{n=1}^{\infty} K_n$. If $\mathrm{diam}(K) = 0$, as $0 \in V_\alpha$, from what we have explained before, we deduce that $\mathbf{K}(\frac{a_n}{b_n})$ converges and by theorem 2.2, the Stern-Stolz series must diverge, so the parabola theorem holds.

For the case $\mathrm{diam}(K) > 0$, we will use lemma 2.1 to prove the absolute convergence of the odd and even parts of $\mathbf{K}(\frac{a_n}{b_n})$. To do so, we first need to find a series of transformations that map the unit disk into a subset of itself. As we already know that $s_n(V_\alpha) \subseteq V_\alpha$ we can use this information by considering the Möbius transformation

$$\varphi(w) := \frac{-1 + e^{i\alpha}\cos(\alpha) - w}{1 + w}$$

which maps the closed unit disk $\overline{\mathbb{D}}$ onto V_α. It satisfies $\varphi(\infty) = -1$ and $\varphi(-1) = \infty$.

Let us now consider the Möbius transformation:

$$\tau_n := \varphi^{-1} \circ s_{2n-1} \circ s_{2n} \circ \varphi \qquad \text{for } n \in \mathbb{N}^+.$$

Then, $\tau_n(\overline{\mathbb{D}}) = \varphi^{-1} \circ s_{2n-1} \circ s_{2n}(V_\alpha) \subseteq \varphi^{-1} \circ s_{2n-1}(V_\alpha) \subseteq \varphi^{-1}(V_\alpha) = \overline{\mathbb{D}}$. If we consider the sequence $\{v_n\}_{n=1}^{\infty}$ with $v_n = \infty$ for all $n \in \mathbb{N}^+$ it can easily be seen that for $v_n = \infty$, $\tau_n(\infty) = \varphi^{-1} \circ s_{2n-1} \circ s_n(-1) = \varphi^{-1} \circ s_{n-1}(\infty) = \varphi^{-1}(0) = -1 + e^{\alpha i}\cos(\alpha)$, hence, $|\tau_n(\infty)| = |(-1+\cos^2(\alpha)) + \sin(\alpha)\cos(\alpha)i| = |\sin(\alpha)||-\sin(\alpha)+\cos(\alpha)i| = |\sin\alpha| < 1$.

Let $\mathcal{T}_n = \tau_1 \circ \cdots \circ \tau_n$. As $\mathcal{T}_n = \varphi^{-1} \circ S_n \circ \varphi$, $\operatorname{diam}(\bigcap_{n=1}^{\infty} \mathcal{T}_n(\overline{\mathbb{D}})) = \operatorname{diam}(\varphi^{-1}(K)) > 0$ and from lemma 2.1, it follows that for every $v \in \overline{\mathbb{D}}$, $\{\mathcal{T}_n(v)\}_{n=1}^{\infty}$ converges absolutely to the same constant $\gamma \in \overline{\mathbb{D}}$ so, in particular, if we set $\lambda = \tau_n(\infty)$, as $|\lambda| < 1$ we have that

$$\sum_{n=2}^{\infty} |\mathcal{T}_n(\lambda) - \mathcal{T}_{n-1}(\lambda)| < \infty.$$

Moreover, $\mathcal{T}_n(\lambda) = \varphi^{-1} \circ S_{2n} \circ \varphi(\lambda) = \varphi^{-1}(w_{2n})$ where we recall that φ^{-1} is a fixed Möbius transformation with pole at $z = -1$ and since $w_n \in K_n \subset V_\alpha$ for all $n \in \mathbb{N}^+$, $\{w_n\}_{n=1}^{\infty}$ is bounded away from -1 and so, the even part of $\mathbf{K}(\frac{a_n}{b_n})$ converges absolutely to $\varphi(\lambda)$.

Similarly, by considering $\tau_n = \varphi^{-1} \circ s_{2n} \circ s_{2n+1} \circ \varphi$ and repeating the same reasoning, we deduce that the odd part of $\mathbf{K}(\frac{a_n}{b_n})$ also converges absolutely. □

There are several aspects that make the parabola theorem one of the best among the convergence theorems. For instance, it has been proven by Lorentzen [27, Theorem 2.] that if you enlarge the set P_α you immediately lose the property that the continued fraction $\mathbf{K}(\frac{c_n}{1})$ with coefficients in that set converges if and only if the Stern-Stolz series diverges.

2.1.5 Other convergence theorems

There are two theorems of convergence that must be mentioned because of their historical importance that can easily be deduced from the parabola theorem in the case where $\alpha = 0$:

Corollary 2.1 (Seidel-Stern theorem). Let $\mathbf{K}(\frac{a_n}{b_n}) \sim \mathbf{K}(\frac{1}{d_n})$. If all coefficients d_n are real, strictly positive numbers, the even and odd parts are absolutely convergent and so, the convergence on $\mathbf{K}(\frac{a_n}{b_n})$ is conditioned to the divergence of the Stern-Stolz series [20, Theorem 4.4.1].

Proof. If all $d_n > 0$, $\mathbf{K}(\frac{1}{d_n}) \sim \mathbf{K}(\frac{c_n}{1})$ with c_n being the inverse of a positive real number, hence a positive number (this is by equation (1.7)). As all positive numbers are in P_0, we can apply the parabola theorem and finish. □

Corollary 2.2 (Worpitzsky's theorem). Let $|c_n| \leq \frac{1}{4}$ for all $n \in \mathbb{N}$. Then $\mathbf{K}(\frac{c_n}{1})$ converges [20, Corollary 4.36.B].

Proof. It is easy to see that all the c_n are in P_0 and also, $\liminf_{n\to\infty}|c_n| < \infty$ so, by proposition 2.1, the Stern-Stolz series diverges and the result follows from the parabola theorem. □

2.2 Functions defined as continued fractions in $\mathbb{C}$

As we have defined most of the concepts in the continued fraction theory for rings, there is no problem to work abstractly with continued fractions whose coefficients are in the ring of meromorphic functions. We will have to be careful though when making sense of these fractions as meromorphic functions, as convergence is not guaranteed.

We will represent these fractions with the notation $\mathbf{K}(\frac{a_n(z)}{b_n(z)})$. Its n^{th} partial numerator will be represented as $p_n(z)$, the n^{th} partial denominator as $q_n(z)$ and the n^{th} approximant as $w_n(z)$.

The most common examples of these fractions in the literature are the following:

Definition 2.8. *A **C-fraction** (where the C stands for **corresponding**) is a continued fraction of the form:*

$$a_0 + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n z^{\alpha_n}}{1}\right) = a_0 + \frac{a_1 z^{\alpha_1}|}{|1} + \frac{a_2 z^{\alpha_2}|}{|1} + \frac{a_3 z^{\alpha_3}|}{|1} + \dots \tag{2.9}$$

*where the $a_k \in \mathbb{C}$ are all different from zero except possibly for a_0, and α_k are positive integers. If all $\alpha_k = 1$, we say that it is a **regular C-fraction**.*

A special case of C-fractions is the following:

Definition 2.9. *A **S-fraction** (or a Stieltjes continued fraction) is a regular C-fraction such that $a_k > 0$ for all $k \in \mathbb{N}^+$.*

We will see that these kinds of functions are incredibly useful, as many functions can be expressed by expansions of this form. Furthermore, they are also closely related to **Stieltjes moment theory** [18, Section 12.9.], which is relevant in measure theory.

Now that we have showcased some of the possible continued fractions that can be used to represent meromorphic functions, we can now proceed to explain the theory behind these representations.

2.2.1 Formal power series

The representation of the function is closely linked to its expansion as a formal power series.

Definition 2.10. *A series $L(z)$ is a **formal power series** at $z = a \in \mathbb{C}$ if and only if it is of the form*

$$L(z) = \sum_{k=m}^{\infty} c_k (z-a)^k, \qquad \text{where } c_k \in \mathbb{C} \text{ and } c_m \neq 0 \text{ for } k, m \in \mathbb{Z}, \tag{2.10}$$

or $L(z) = 0$.

Under the operations of addition and multiplication, the set $\mathbb{L}_a$ of all formal power series at $z = a$ is a field over $\mathbb{C}$.

Definition 2.11. *For every $L(z) \in \mathbb{L}_a$, its* ***order*** *$\lambda(L)$ is defined as:*

$$\lambda(L) = \begin{cases} m & \text{if } L(z) \text{ is as in equation (2.10)} \\ \infty & \text{if } L(z) = 0 \end{cases} \tag{2.11}$$

If $\lambda(L) \geq 0$, we say that L is a **formal Taylor series**.
The order function satisfies that, for every pair of formal power series $L_1(z)$ and $L_2(z)$ in $\mathbb{L}_a$,

$$\lambda(L_1 \pm L_2) \geq \min\{\lambda(L_1), \lambda(L_2)\}, \tag{2.12}$$
$$\lambda(L_1 \pm L_2) = \min\{\lambda(L_1), \lambda(L_2)\} \qquad \text{if } \lambda(L_1) \neq \lambda(L_2), \tag{2.13}$$
$$\lambda(L_1 L_2) = \lambda(L_1) + \lambda(L_2), \tag{2.14}$$
$$\lambda(L_1/L_2) = \lambda(L_1) - \lambda(L_2) \qquad \text{if } L_2 \neq 0 \tag{2.15}$$

Remark 2.2. With these properties it is easy to prove that if we consider the norm defined as $\|L\| = 2^{-\lambda(L)}$, $\mathbb{L}_a$ is what is called a **normed field**. This notion is very important, as it defines a notion of convergence in more abstract rings such as the field of formal power series over a finite field or the p-adic numbers [26].

Let $f(z)$ be a function meromorphic at $z = a$. We will denote by $\Lambda_a(f)(z)$ the Laurent expansion in a deleted neighbourhood of a.

Definition 2.12. *Let $\{f_n(z)\}_{n=0}^{\infty}$ be a sequence of meromorphic functions at $z = a$, $L(z)$ a formal power series at $z = a$, and $v_n := \lambda(L - \Lambda_a(f_n))$. Then, we say that the sequence $\{f_n(z)\}_{n=0}^{\infty}$* ***corresponds to*** *$L(z)$ if and only if $\lim_{n\to\infty} v_n = \infty$.*

The integer v_n is known as the **order of correspondence** of $f_n(z)$ to $L(z)$ and the condition $v_n := \lambda(L - \Lambda_a(f_n))$ is sometimes written as

$$L(z) - \Lambda_0(f_n)(z) = O((z-a)^{v_n}). \tag{2.16}$$

Remark 2.3. Without any loss of generality this last definition can be generalised to the case $z = \infty$ by defining the formal power series as

$$L(z) = \sum_{k=m}^{\infty} c_{-k}(z-a)^{-k}, \qquad \text{where } c_{-k} \in \mathbb{C} \text{ and } c_{-m} \neq 0 \text{ for } k, m \in \mathbb{Z} \tag{2.17}$$

and for all $L \in \mathbb{L}_\infty$, its order as

$$\lambda(L) = \begin{cases} m & \text{if } L(z) \text{ is as in equation (2.17).} \\ \infty & \text{if } L(z) = 0, \end{cases} \tag{2.18}$$

Definition 2.13. *A continued fraction $\mathbf{K}(\frac{a_n(z)}{b_n(z)})$* ***corresponds*** *to a formal power series $L(z)$ at $z = a$ if all of its approximants $w_n(z)$ are meromorphic at $z = a$ and if the sequence $\{w_n(z)\}_{n=0}^{\infty}$ corresponds to $L(z)$.*

Theorem 2.4. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of functions meromorphic at $z = a$. Then:*

1. *There exists a formal power series at $z = a$ such that $\{f_n\}_{n=0}^{\infty}$ corresponds at $z = a$ if and only if for $k_n := \lambda(\Lambda_a(f_{n+1} - f_n))$, we have that*

$$\lim_{n\to\infty} k_n = \infty. \tag{2.19}$$

2. *If equation (2.19) holds, then the formal power series $L(z)$ to which $\{f_n\}_{n=0}^{\infty}$ corresponds is uniquely determined.*
3. *If $\{k_n\}_{n=0}^{\infty}$ tends monotonically to ∞, then $k_n = v_n$ for all $n \in \mathbb{N}^+$.*

Proof. This theorem can be proven by working with the previously defined properties of λ and the norm defined in the remark 2.2. The complete proof can be found in the book of Jones and Thron [20, Theorem 5.1.]. □

The reason why I started this section defining different kinds of continued fractions will now become apparent, as we will be able to set a correspondence between formal Taylor series at $z = 0$ and C-continued fractions [9, Theorem 2.4.1.].

Theorem 2.5. *There is the following one-to-one correspondence between the set of all C-fractions (2.9), including terminating C-fractions, and the set of formal Taylor series at $z = 0$.*

1. *Every C-fraction corresponds to a unique formal Taylor series $L(z)$ at $z = 0$ and the order of correspondence of the n^{th} approximant $w_n(z)$ is*

$$v_n = \sum_{k=1}^{n+1} \alpha_k. \tag{2.20}$$

2. *Let $L(z)$ be a given formal Taylor series at $z = 0$ with $L(0) = c_0$. Then, either there exists a C-fraction corresponding to $L(z)$ at $z = 0$, or for some $n \in \mathbb{N}$, there exists a terminating C-fraction:*

$$w_n(z) = c_0 + \mathop{\mathrm{K}}_{m=1}^{n} \left(\frac{a_m z^{\alpha_m}}{1}\right) \tag{2.21}$$

such that $L(z) = \Lambda_0(w_n)(z)$.

3. *If $f(z)$ is a rational function holomorphic at $z = 0$ and if $L(z) = \Lambda_0(w_n)(z)$ is the Taylor series expansion of $f(z)$ about $z = 0$, then there exists a terminating C-fraction $w_n(z)$ of the form (2.21) such that $L(z) = \Lambda_0(w_n)(z)$.*

Proof. The proof simply involves matching the expressions of both Taylor series with the help of theorem 2.4. For more details, I again refer to Jones and Thron [20, Corollary 5.3.]. □

The reason why this concept of correspondence of series and continued fractions is useful is because the correspondence gives us a good insight on the convergence of continued fractions, as we will now see.

2.2.2 Uniform convergence

Let $D \subseteq \mathbb{C}$ be a domain, that is, an open and connected subset of $\mathbb{C}$.

Definition 2.14. *A sequence $\{f_n\}_{n=0}^{\infty}$ of meromorphic functions in a domain D is said to* ***converge uniformly*** *on a compact subset $K \subset D$ if and only if:*

- *There exists a $N_K \in \mathbb{N}$ such that for all $n \geq N_K$, $f_n(z)$ is holomorphic in some domain containing K.*
- *For every $\epsilon > 0$, there exists a $N_\epsilon > N_K$ such that*

$$\sup_{z \in K} |f_{n+m}(z) - f_n(z)| < \epsilon \qquad \text{for all } n \geq N_\epsilon \text{ and all } m \in \mathbb{N} \tag{2.22}$$

A continued fraction is said to ***converge uniformly*** *on a compact subset $K \subseteq D$ if and only if $\{w_n(z)\}_{n=0}^{\infty}$ satisfies the conditions from above.*

Definition 2.15. *A sequence $\{f_n\}_{n=0}^{\infty}$ of functions meromorphic in a domain D is said to be* ***uniformly bounded*** *on a compact set K of D if and only if there exist N_K and a bound B_K such that*

$$\sup_{z \in K} |f_n(z)| \leq B_K \qquad n \geq N_K \tag{2.23}$$

Theorem 2.6. *Let $\mathbf{K}(\frac{a_n(z)}{b_n(z)})$ correspond to a formal Taylor series $L(z)$ at $z = 0$ and let D be a domain which also contains $z = 0$. Then, the continued fraction converges uniformly to a holomorphic function $f(z)$ on any compact subset of D if and only if the sequence of approximants of $\mathbf{K}(\frac{a_n(z)}{b_n(z)})$ is uniformly bounded on every compact subset K of D. The series $L(z)$ is then the formal Taylor series at $z = 0$ of $f(z)$.*

Proof. This theorem is deduced by working with the formal Taylor series in the domain D and applying a classical result in the theory of complex analysis known as the **Stieltjes-Vitali theorem**. The full proof is in [20, Theorems 5.11-5.13]. □

This theorem shows how correspondence (between holomorphic functions and continued fractions) alone does not imply convergence. Nevertheless, whenever there is an additional boundedness property, both lead to convergence. This is the reason why this theorem has been used repeatedly to find convergence criteria for many families of continued fractions.

Furthermore, theorem 2.6 shows that if D contains the disk where $L(z)$ converges, the continued fraction provides an analytic continuation of $f(z)$ in D.

The next section will give an insight on how to find the continued fraction representation of certain kinds of functions.

2.3 Padé approximants

Let $f(z)$ be a formal Taylor series at $z = 0$. For simplicity, from now on, we will denote by $f(z)$ both the formal Taylor series $\Lambda_0(f)$ and its limit function f when it exists. Let $p_{m,n}(z)$ and $q_{m,n}(z)$ be polynomials of degree at most m and n respectively with $q_{m,n}(z)$ non-zero.

Definition 2.16. *The* ***(m,n) Padé approximant*** *at $z = 0$ of $f(z)$ is the quotient*

$$r_{m,n}(z) = \frac{p_{m,n}(z)}{q_{m,n}(z)}$$

satisfying

$$f(z)q_{m,n}(z) - p_{m,n}(z) = O(z^{m+n+1}). \tag{2.24}$$

When $m = 0$, for example, $r_{0,n}(z) = p_{0,n}(z)$ is the sum of the first n terms of the Taylor expansion of $f(z)$.

Proposition 2.2. The Padé approximant of $f(z)$ always exists and it is uniquely determined (that is, for a fixed $f(z)$ there is a unique Padé approximant).

Proof. **Existence:**

Let $f(z) = \sum_{k=0}^{n+m} \lambda_k z^k + O(z^{n+m+1})$, $p_{m,n}(z) = \sum_{k=0}^{m} \mu_k z^k$ and $q_{m,n}(z) = \sum_{k=0}^{n} \nu_k z^k$ with $\lambda_k, \mu_k, \nu_k \in \mathbb{C}$ for all possible k. By the definition of the multiplication of polynomials, the solutions of $f(z)q_{m,n}(z) - p_{m,n}(z) = O(z^{m+n+1})$ arise from the solution of the following linear systems in the unknowns μ_k and ν_k:

$$\begin{array}{r|r} \lambda_0\nu_0 = \mu_0 & \lambda_{m+1}\nu_0 + \lambda_m\nu_1 + \cdots + \lambda_{m-n+1}\nu_n = 0 \\ \lambda_1\nu_0 + \lambda_0\nu_1 = \mu_1 & \lambda_{m+2}\nu_0 + \lambda_{m+1}\nu_1 + \cdots + \lambda_{m-n+2}\nu_n = 0 \\ \vdots & \vdots \\ \lambda_m\nu_0 + \lambda_{m-1}\nu_1 + \cdots + \lambda_0\nu_m = \mu_m & \lambda_{m+n}\nu_0 + \lambda_{m+n-1}\nu_1 + \cdots + \lambda_m\nu_n = 0 \end{array} \tag{2.25}$$

The linear system on the right has n equations but $n+1$ unknowns, so it must have a least one solution $(\nu_0, \nu_1, \ldots, \nu_m)$ besides the trivial one. By solving the system on the left we obtain the values of the $(\mu_0, \mu_1, \ldots, \mu_m)$ and we have found two polynomials $p_{m,n}(z)$ and $q_{m,n}(z)$, so $r_{m,n}(z)$ is a solution.

Uniqueness:

Let $\overline{p}_{m,n}(z)$ and $\overline{q}_{m,n}(z)$ be another solution of (2.25), so $f(z)\overline{q}_{m,n}(z) - \overline{p}_{m,n}(z) = O(z^{m+n+1})$. Then,

$$\begin{cases} f(z)q_{m,n}(z)\overline{q}_{m,n}(z) - p_{m,n}(z)\overline{q}_{m,n}(z) = O(z^{m+n+1}) \\ f(z)\overline{q}_{m,n}(z)q_{m,n}(z) - \overline{p}_{m,n}(z)q_{m,n}(z) = O(z^{m+n+1}) \end{cases}$$

So, by subtracting the right hand sides of both equalities, we get that

$$s(z) := \overline{p}_{m,n}(z)q_{m,n}(z) - p_{m,n}(z)\overline{q}_{m,n}(z) = O(z^{m+n+1})$$

and, as $s(z)$ is a polynomial of degree at most $m+n$, we deduce that $s(z) = 0$. Thus, we get that $\frac{\overline{p}_{m,n}(z)}{\overline{q}_{m,n}(z)} = \frac{p_{m,n}(z)}{q_{m,n}(z)} = r_{m,n}(z)$, as we wanted to see. □

Historically the Padé approximants of a function $f(z)$ are represented in a table known as the **Padé table**:

$$\begin{matrix} r_{0,0}(z) & r_{0,1}(z) & r_{0,1}(z) & r_{0,3}(z) & \cdots \\ r_{1,0}(z) & r_{1,1}(z) & r_{1,2}(z) & r_{1,3}(z) & \cdots \\ r_{2,0}(z) & r_{2,1}(z) & r_{2,2}(z) & r_{2,3}(z) & \cdots \\ r_{3,0}(z) & r_{3,1}(z) & r_{3,2}(z) & r_{3,3}(z) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

In section C.1, I have computed the Padé table for the exponential function.

The standard way of representing this table is writing the $r_{m,n}(z)$ as irreducible fractions where the denominators $q_{m,n}(0) = 1$. This is always possible, as the $r_{m,n}(z)$ are holomorphic at $z = 0$, so they cannot have any poles at $z = 0$, so $q_{m,n}(0) \neq 0$.

From now, on, I will denote by $p_{m,n}(z)$ and $q_{m,n}(z)$ the relatively prime polynomials such that $r_{m,n}(z) = \frac{p_{m,n}(z)}{q_{m,n}(z)}$ and $q_{m,n}(0) = 1$. It is worth noting, though, that these $p_{m,n}(z)$ and $q_{m,n}(z)$ do not necessarily have to verify equation (2.24), but there must always be a polynomial such that when multiplied by $p_{m,n}(z)$ and $q_{m,n}(z)$, those verify (2.24).

The Padé tables have an interesting structure that will now be stated.

Proposition 2.3. Suppose that a rational function $r(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are relatively prime polynomials of degrees m and n, occurs at some place in the Padé table of $f(z)$. Then, the set of all places in the Padé table where $r(z)$ occurs is a square block and if

$$\lambda(qf - p) = m + n + r + 1,$$

then $r \geq 0$ and the square block consists of the $(r+1)^2$ places with coordinates $(m+i, m+j)$ where $i, j \in \{0, \dots, r\}$. The case $r = \infty$ is also possible and in this case $q(z)f(z) - p(z)$ is the zero polynomial, hence, $f(z) = r(z)$.

Proof. The proof is available in Gragg's survey article [15, Theorem 3.2.]. □

To illustrate these square blocks in a concrete example, they have been marked for the Padé table of the function $f(z) = \frac{x^2-1}{x^2+1}$ in C.2.

Definition 2.17. *A (m, n) Padé approximant is said to be* ***normal*** *if the degrees of $p_{m,n}(z)$ and $q_{m,n}(z)$ are exactly m and n respectively and $\lambda(q_{m,n}f - p_{m,n}) = m + n + 1$.*

The formal power series $f(z)$ and its Padé table are also said to be ***normal*** *if every Padé approximant is normal.*

There are many reasons why normal Padé approximants are important. One of them is that if $r_{m,n}(z)$ is normal, then $r_{m,n}(z)$ is a solution of the **Hermite interpolation problem**, that is, $\lambda(f - \Lambda_0(r_{m,n})) = m + n + 1$ (this is trivial by the definition of normal and the fact that $q_{m,n}(0) \neq 1$). This is generally not true for regular Padé approximants of $f(z)$.

As another interesting fact, it can also be seen that a rational function is a normal Padé approximant of $f(z)$ if and only if it occurs in exactly one place in the Padé table of $f(z)$.

Let $f(z) = \sum_{k=0}^{\infty} \lambda_k z^k$. I will denote by T_n^m the determinant of the following Toeplitz matrix[1]:

$$T_n^m = \begin{vmatrix} \lambda_m & \lambda_{m-1} & \cdots & \lambda_{m-n+1} \\ \lambda_{m+1} & \lambda_m & \cdots & \lambda_{m-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m+n-1} & \lambda_{m+n-2} & \cdots & \lambda_m \end{vmatrix} \tag{2.26}$$

The complete characterisation of normal approximants is given by the following theorem:

Proposition 2.4. The (m, n) Padé approximant of $f(z)$ is normal if and only if the determinants T_n^m, T_{n-1}^m, T_n^{m+1} and T_{n+1}^{m+1} are all non-zero. Therefore $f(z)$ and its Padé table are normal if and only if $T_n^m \neq 0$ for all $m, n \in \mathbb{N}$. In particular, each $T_1^m = \lambda_m$ must be non-zero.

Proof. The proof of this can also be found in Gragg's article [15, Corollary 2.]. □

The reason why we introduced this notion of normal Padé approximants is because there is a connection between the C-fractions and the elements of the "staircase" sequence of the Padé table:

$$\begin{matrix} r_{0,0}(z) & & & \\ r_{1,0}(z) & r_{1,1}(z) & & \\ & r_{2,1}(z) & r_{2,2}(z) & \\ & & r_{3,2}(z) & \ddots \\ & & & \ddots \end{matrix}$$

Theorem 2.7. *Let $f(z) = \sum_{k=0}^{\infty} \lambda_k z^k$ be a Taylor series such that the Padé approximants of the staircase sequence $\{r_{0,0}(z), r_{1,0}(z), r_{1,1}(z), \ldots\}$ are all normal. Then, there exists a regular C-fraction*

$$a_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{a_n z}{1} \right)$$

whose approximants satisfy $w_{2n} = r_{n,n}$ and $w_{2n+1} = r_{n+1,n}$.

Proof. For the first part, let us define the sequences $\{p_n(z)\}_{n=0}^{\infty}$ and $\{q_n(z)\}_{n=0}^{\infty}$ by

$$p_{2n}(z) := p_{n,n}(z) \qquad p_{2n+1}(z) := p_{n+1,n}(z)$$
$$q_{2n}(z) := q_{n,n}(z) \qquad q_{2n+1}(z) := q_{n+1,n}(z)$$

[1]This is, a matrix in which each descending diagonal from left to right is constant

Let

$$\Delta_{2n}(z) \overset{(1.3)}{=} p_{n,n-1}(z)q_{n,n}(z) - p_{n,n}(z)q_{n,n-1}(z) \quad \text{for all } n \in \mathbb{N}^+$$
$$\Delta_{2n+1}(z) \overset{(1.3)}{=} p_{n,n}(z)q_{n+1,n}(z) - p_{n+1,n}(z)q_{n,n}(z) \quad \text{for all } n \in \mathbb{N}.$$

As all the approximants of the staircase are normal, we know that $r_{n,n}(z) \neq r_{n+1,n}(z)$ and also $r_{n,n}(z) \neq r_{n+1,n}(z)$, so $\Delta_k \neq 0$ for all $k \in \mathbb{N}^+$ and by theorem 1.1, there exists $\{a_n(z)\}_{n=1}^{\infty}$ and $\{b_n(z)\}_{n=1}^{\infty}$ in the field of rational functions with coefficients in $\mathbb{C}$ such that $\mathbf{K}(\frac{a_n(z)}{b_n(z)})$ has as partial numerators and denominators $\{p_n(z)\}_{n=0}^{\infty}$ and $\{q_n(z)\}_{n=0}^{\infty}$.

These $\{a_n(z)\}_{n=1}^{\infty}$ and $\{b_n(z)\}_{n=1}^{\infty}$ are given by $a_1(z) = \lambda_1 z$, $b_0(z) = \lambda_0$, $b_1(z) = 1$ and

$$a_{2n}(z) = -\frac{\Delta_{2n}(z)}{\Delta_{2n-1}(z)} \qquad a_{2n+1}(z) = -\frac{\Delta_{2n+1}(z)}{\Delta_{2n}(z)}$$
$$b_{2n}(z) = \frac{p_{n-1,n-1}(z)q_{n,n}(z) - p_{n,n}(z)q_{n-1,n-1}(z)}{\Delta_{2n-1}(z)} \qquad b_{2n+1}(z) = \frac{p_{n,n-1}(z)q_{n+1,n} - p_{n+1,n}(z)q_{n,n-1}}{\Delta_{2n}(z)}$$

for $n \in \mathbb{N}^+$.

Let us now see what the Δ_{2n} are like. Since all the approximants of the staircase are normal, we know that $\lambda(f - \Lambda_0(r_{n,n-1})) = 2n$ and $\lambda(f - \Lambda_0(r_{n,n})) = 2n+1$, so

$$\begin{aligned}\lambda(\Delta_{2n}) &= \lambda((r_{n,n-1} - r_{n,n})q_{n,n-1}q_{n,n}) \\ &\overset{(2.14)}{=} \lambda(r_{n,n-1} - r_{n,n}) + \lambda(q_{n,n-1}) + \lambda(q_{n,n-1}) \\ &\overset{(2.13)}{=} \min\{2n, 2n+1\} + 0 + 0 = 2n.\end{aligned}$$

where the orders of $q_{n,n}$ and $q_{n,n-1}$ are both zero as $q_{n,n}(0) \neq 0$ and $q_{n,n-1}(0) \neq 0$.

Therefore, since $\Delta_{2n}(z)$ is a polynomial of degree $2n$ and it also has order $2n$, we infer that $\Delta_{2n}(z) = \mu_{2n} z^{2n}$ with $\mu_{2n} \in \mathbb{C}$. Repeating this same reasoning without barely any changes, it can be proven that

$$\begin{aligned}\Delta_{2n+1}(z) &= \mu_{2n+1} z^{2n+1}, &&\text{with } \mu_{2n+1} \in \mathbb{C}/\{0\} \\ p_{n-1,n-1}(z)q_{n,n}(z) - p_{n,n}(z)q_{n-1,n-1}(z) &= \nu_{2n-1} z^{2n-1}, &&\text{with } \nu_{2n-1} \in \mathbb{C}/\{0\} \\ p_{n,n-1}(z)q_{n+1,n} - p_{n+1,n}(z)q_{n,n-1} &= \nu_{2n} z^{2n}, &&\text{with } \nu_{2n} \in \mathbb{C}/\{0\}\end{aligned}$$

Furthermore, by setting $\gamma_k = -\frac{\mu_k}{\mu_{k-1}}$ and $\delta_k = \frac{\nu_{k-1}}{\mu_{k-1}}$ for all $k \geq 2$, it is easy to see that $a_n(z) = \gamma_n z$, $b_n(z) = \delta_n$ for $n \geq 2$. Finally, if we consider $a_0 = \lambda_0$, $a_1 = \lambda_1$, $a_2 = \frac{\gamma_2}{\delta_2}$ and $a_n = \frac{\gamma_n}{\delta_{n-1}\delta_n}$, by (1.7), we get that the continued fraction

$$a_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n z}{1}\right)$$

satisfies $w_{2n} = r_{n,n}(z)$ and $w_{2n+1} = r_{n+1,n}$. □

Remark 2.4. Theorem 2.7 is constructive, so it gives us a way to compute the coefficients of the regular C-fraction of a normal function (A.3.3). However, it is worth noting that this is a very inefficient way to perform this computation and that there is an algorithm called the **qd-algorithm** (where *qd* stands for *quotient-difference*) that allows us to compute them much faster. This and other algorithms are described in Henrici's book [18, Chapter 12].

This close relationship between Padé tables and continued fractions make it possible to apply the convergence theory of one of them to obtain results for the other. The most complete results about the convergence of regular C-fractions is the following:

Theorem 2.8. *Let $a_0 + \boldsymbol{K}(\frac{a_n z}{1})$ be a regular C-fraction such that $\lim_{n\to\infty} a_n = a \neq \infty$ and let $R_0 = \mathbb{C}$ and $R_a = \{z \in \mathbb{C} : -\pi < \arg(az + \frac{1}{4}) < \pi\}$ if $a \neq 0$. Then, for all $a \in \mathbb{C}$:*

(i) $a_0 + \boldsymbol{K}(\frac{a_n z}{1})$ converges to a meromorphic function $f(z)$ in R_a.

(ii) The convergence is uniform on every compact subset $K \subseteq R_a$ which contains no poles of $f(z)$.

(iii) $f(z)$ is holomorphic in $z = 0$ and $f(0) = a_0$.

Proof. This proof of this theorem is a direct application of theorem 2.6 where $D = R_\alpha$. It was found by the mathematician Van Vleck in 1904 [20, Theorems 5.14-5.15]. □

This theorem illustrates in a concrete way how useful continued fractions are for representing meromorphic functions, as I commented in theorem 2.6 - they provide analytic continuations outside their disks of convergence.

2.3.1 Hypergeometric functions

We have seen that normal Taylor series can be represented as regular C-fractions, but in most cases, the procedure is computationally complex and it does not give an expression for the general term of the coefficients.

In that regard, the family of hypergeometric functions is a family of power series that, in some cases, have very simple continued fraction representations, and this can be used to find beautiful formulas for some of the most famous constants in mathematics.

Besides this, they appear naturally as solutions of a certain differential equation called the **hypergeometric differential equation** that it is very important in the study of differential equations of second order with three singular points (that is, three points where the functions that are the coefficients of the equation diverge) [18, Section 9.9.].

Definition 2.18. *For $n \in \mathbb{N}$, we define the **Pochhammer symbol** $(a)_n$ as:*

$$(a)_0 := 1, \quad (a)_n := \prod_{k=0}^{n-1}(a+k) \tag{2.27}$$

In particular, for $m \in \mathbb{N}^+$, $(m)_n = \frac{(m+n-1)!}{(m-1)!}$.

Definition 2.19. *Let $a, b, c \in \mathbb{C}$ with c non-zero and not a negative integer. The* ***hypergeometric function*** *or* ***Gauss hypergeometric function*** *${}_2F_1(a, b; c; z)$ is the analytic function whose power series expansion is*

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \tag{2.28}$$

If a and b are either zero or a negative integer, it is easy to see that from a certain term on, the series is identically zero, so the hypergeometric function ${}_2F_1(a, b; c; z)$ is a polynomial, converging for all $z \in \mathbb{C}$. Otherwise, it can be seen through the ratio test that the radius of convergence of ${}_2F_1(a, b; c; z)$ is 1.

Remark 2.5. There is a broader family of series called the **generalised hypergeometric functions**

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!} \tag{2.29}$$

Whenever the b_n are non-zero and not negative integers, and $p \leq q + 1$, these series define complex analytic functions. The Gauss hypergeometric functions would be the case where $p = 2$ and $q = 1$.

There are many identities concerning hypergeometric series, but there are two in particular that we can use to construct continued fractions. These identities can be established from the equation (2.28) if we compare the power series of both sides term by term.

$$\begin{aligned} {}_2F_1(a, b; c; z) &= {}_2F_1(a, b+1; c+1; z) - \tfrac{a(c-b)}{c(c+1)} z \, {}_2F_1(a+1, b+1; c+2; z) \\ {}_2F_1(a, b+1; c+1; z) &= {}_2F_1(a+1, b+1; c+2; z) - \tfrac{(b+1)(c-a+1)}{(c+1)(c+2)} z \, {}_2F_1(a+1, b+2; c+3; z) \end{aligned}$$

Thus, if there are no zeros in the denominators, we have:

$$\frac{{}_2F_1(a, b; c; z)}{{}_2F_1(a, b+1; c+1; z)} = 1 + \frac{-\frac{a(c-b)}{c(c+1)} z}{\frac{{}_2F_1(a, b+1; c+1; z)}{{}_2F_1(a+1, b+1; c+2; z)}} \tag{2.30}$$

$$\frac{{}_2F_1(a, b+1; c+1; z)}{{}_2F_1(a+1, b+1; c+2; z)} = 1 + \frac{-\frac{(b+1)(c-a+1)}{(c+1)(c+2)} z}{\frac{{}_2F_1(a+1, b+1; c+2; z)}{{}_2F_1(a+1, b+2; c+3; z)}} \tag{2.31}$$

The denominator on the right hand side of the equation (2.30) equals the left hand side of the equation (2.31). In addition, the denominator of the right hand side of (2.31) can be interpreted as the left hand side of (2.30) by replacing a by $a + 1$, b by $b + 1$ and c by $c + 2$.

Therefore, we have that,

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} = 1 + \cfrac{a_1 z}{\cfrac{{}_2F_1(a,b+1;c+1;z)}{{}_2F_1(a+1,b+1;c+2;z)}}$$

$$= 1 + \cfrac{a_1 z}{1 + \cfrac{a_2 z}{\cfrac{{}_2F_1(a+1,b+1;c+2;z)}{{}_2F_1(a+1,b+2;c+3;z)}}}$$

$$= 1 + \frac{a_1 z|}{|1} + \frac{a_2 z|}{|1} + \frac{a_3 z|}{|1} + \cdots$$

with

$$a_{2n-1} = -\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)} \text{ for } n \in \mathbb{N} \quad \text{and} \quad a_{2n} = -\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)} \text{ for } n \in \mathbb{N}^+ \tag{2.32}$$

This continued fraction is called **Gauss's continued fraction**. The following theorem describes this kind of continued fractions [20, Theorem 6.1.].

Theorem 2.9. *Let $\{a_n\}_{n=0}^{\infty}$ be defined as in (2.32). Then:*

1. *The regular C-fraction $1+\mathbf{K}(\frac{a_n z}{1})$ converges to a function $f(z)$ meromorphic in the domain $D = \{z \in \mathbb{C} : 0 < \arg(z-1) < 2\pi\}$, which is $\mathbb{C}$ cut along the real axis from 1 to $+\infty$.*

2. *The convergence is uniform on every compact subset of D which contains no poles of $f(z)$.*

3. *For all z such that $|z| < 1$, $f(z) = \frac{F(a,b;c;z)}{F(a,b+1;c+1;z)}$ and hence $f(z)$ provides the analytic continuation of this quotient of hypergeometric functions in D.*

Proof. It is easy to check that $\lim_{n\to\infty} a_n = -\frac{1}{4}$. Thus, this theorem can be deduced from theorem 2.8, where the D in this theorem corresponds to $R_{\text{-}1/4}$ as

$$\begin{aligned} R_{\text{-}1/4} &= \{z \in \mathbb{C} : -\pi < \arg(-\tfrac{1}{4}z + \tfrac{1}{4}) < \pi\} \\ &= \{z \in \mathbb{C} : -\pi < \arg(-z+1) < \pi\} \\ &= \{z \in \mathbb{C} : -\pi < \arg(z-1) - \pi < \pi\} = D \end{aligned}$$

□

Gauss's continued fraction is particularly used for the case $b = 0$, as ${}_2F_1(a,0;c;z) = 1$, so we have that

$${}_2F_1(a,1;c+1;z) = \frac{1|}{|1} + \frac{a_1 z|}{|1} + \frac{a_2 z|}{|1} + \frac{a_3 z|}{|1} + \cdots$$

where

$$a_{2n-1} = -\frac{(a+n)(c+n)}{(c+2n)(c+2n+1)} \text{ for } n \in \mathbb{N} \quad \text{and} \quad a_{2n} = -\frac{n(c-a+n)}{(c+2n-1)(c+2n)} \text{ for } n \in \mathbb{N}^+ \tag{2.33}$$

Let us now see some examples of functions of this kind:

- **The general binomial**

$$(z+1)^\alpha = {}_2F_1(-\alpha,1;1;-z) = \frac{1|}{|1} + \frac{(-\alpha)z|}{|1} + \frac{1(1+\alpha)z|}{|2} + \frac{1(1-\alpha)z|}{|3} + \frac{2(2+\alpha)z|}{|4} + \frac{2(2-\alpha)z|}{|5} + \cdots$$

where $\alpha \in \mathbb{C}$ and $\alpha \notin \mathbb{Z}$.

The last equality corresponds to performing a transformation as in theorem 1.2 with $z_1 = 1$ and $z_n = n-1$ for $n \geq 2$. This continued fraction represents a single-valued branch of the analytic function $(z+1)^\alpha$ in the cut complex plane along the real axis from -1 to $-\infty$. This is significantly better than the Taylor series of $(z+1)^\alpha$, whose radius of convergence is 1.

- **The natural logarithm**

$$\log(z+1) = z\,{}_2F_1(1,1;2;-z) = \frac{z|}{|1} + \frac{1^2z|}{|2} + \frac{1^2z|}{|3} + \frac{2^2z|}{|4} + \frac{2^2z|}{|5} + \frac{3^2z|}{|6} + \cdots$$

where the last equality corresponds to performing a transformation as in theorem 1.2 with $z_n = n$.

This regular C-fraction represents the principal value of the logarithm[2] of $z+1$ and converges in the cut complex plane along the real axis from -1 to $-\infty$.

- **The arctangent function**

$$\arctan(z) = z\,{}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};-z^2) = \frac{z|}{|1} + \frac{1^2z^2|}{|3} + \frac{2^2z^2|}{|5} + \frac{3^2z^2|}{|7} + \frac{4^2z^2|}{|9} + \frac{5^2z^2|}{|11} + \cdots$$

where the last equality corresponds to performing a transformation as in theorem 1.2 with $z_n = 2n-1$.

This continued fraction converges and represents a single-valued branch of the analytic function $\arctan$ in the cut z-plane that has cuts along the imaginary axis from i to $+i\infty$ and from $-i$ to $-i\infty$ (this set is the preimage of D by the function $-z^2$).

This contrasts with the Taylor series expansion of $\arctan(z)$ at $z=0$, which only converges for $|z| \leq 1$ except for $z = \pm i$.

[2]The **principal value of the logarithm** is the logarithm whose imaginary part lies in $(-\pi,\pi]$. It is often written as $\mathrm{Log}(z)$.

The convergence of the C-fractions is faster than the Taylor series expansion. For example, in order to get a smaller error than the approximation of $\frac{\pi}{4} = \arctan(1)$ that we get with 8 terms of the continued fraction, we would need 261199 terms of the Taylor series expansion.

There are many other examples of functions that can be represented with Gauss's continued fractions such as

$$\frac{\arcsin(z)}{\sqrt{1-z^2}} = \frac{z\ {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)}{{}_2F_1(\frac{1}{2},-\frac{1}{2};\frac{3}{2};z^2)} \qquad \text{or} \qquad \int_0^z \frac{t^p dt}{t^q+1} = \frac{z^{p+1}}{q}\ {}_2F_1(\tfrac{p+1}{q},1;\tfrac{p+q+1}{q};-z^q)$$

with $p, q \in \mathbb{N}$.

This last integral converges in the domain $D = \{z \in \mathbb{C} : -\pi < \arg(z^q+1) < \pi\}$, and it makes sense in that domain, as $f(t) = \frac{t^p}{t^q+1}$ is continuous and has an antiderivative on D, therefore, by the **path-independence theorem**, all contour integrals $\int_\Gamma f(t)\,dt$ are independent of the path Γ, and so, they only depend on the endpoints. This integral is a general case that includes $\log(z+1)$ and $\arctan(z)$ as the cases where $p = 0$, $q = 1, 2$ respectively.

2.4 Open problems and lines of research

In this dissertation, I have focused on the family of C-fractions, but there are many other families of continued fractions that have different uses:

- **Thiele type continued fractions** are a very important tool in the theory of interpolation with complex numbers [4, p. 65-74].

- **P-fractions** (where the P stands for **principal part**) play a similar role to C-fractions in representing formal Laurent series $f(z)$, and their approximants are also elements of the Padé table of $f(z)$ [4, Corollary 2.8.].

- **T-fractions**, **M-fractions**, **g-fractions**, **PC-fractions**... all of them are involved in the representation and acceleration of convergence of some analytic functions [4, 9, 20].

The complex continued fractions also play a very important role in the study of **general hypergeometric functions** ${}_pF_q$, as analogous expressions to Gauss's continued fraction are known for many other values of p and q in addition to $p = 2, q = 1$. This gives us a way to efficiently compute functions such as the **gamma function** or the **complementary error function** $\operatorname{erfc}(z)$ that naturally appear in many fields such as physics, statistics and differential equations. These expressions can be found alongside many others in [28, Appendix A.]

Furthermore, if we want an even more general setting, there are ***q*-hypergeometric series** that also generate interesting continued fractions. These have allowed mathematicians to prove some of the results that Ramanujan wrote in his famous notebooks and that involve modular functions represented as continued fractions [34].

As an example of a different area where complex continued fractions naturally appear, I would like to highlight that there is a relation between complex continued fractions and **para-odd rational functions**, a sort of rational function that is involved in the study of **stable polynomials**, which are those whose zeros all have a negative real part. This theory is explained in [20, Section 7.4.] and the results derived from them have multiple applications in the study of mechanical and electrical systems.

2.5 A bridge between analysis and number theory

In this dissertation we study the applications of continued fractions to both complex analysis and number theory. Despite the fact that it seems that these are two completely different fields that do not have anything in common, there is a link that we will now present.

We said that general hypergeometric functions had analogous expressions to Gauss's continued fraction. Let us analyze the particular case ${}_0F_1(;c;z)$ with $c \in \mathbb{C}$ not a negative integer. These are sometimes known as **confluent hypergeometric functions** ψ. They satisfy

$$ {}_0F_1(;c;z) = {}_0F_1(;c+1;z) + \tfrac{1}{c(c+1)} z \, {}_0F_1(;c+2;z) $$

so reasoning as in the case ${}_2F_1$, it can be proven that the equality

$$ \frac{{}_0F_1(;c;z)}{{}_0F_1(;c+1;z)} = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{a_n z}{1} \right) \qquad \text{with } a_n = \frac{1}{(c+n-1)(c+n)} $$

holds for every $z \in \mathbb{C}$ by theorem 2.8, as the sequence of a_n tends to zero. Therefore, in particular,

$$ \begin{aligned} z\coth(z) = \frac{{}_0F_1(;\frac{1}{2};\frac{z^2}{4})}{{}_0F_1(;\frac{3}{2};\frac{z^2}{4})} &= 1 + \frac{z^2|}{|3} + \frac{z^2|}{|5} + \frac{z^2|}{|7} + \cdots \\ &= 1 + \frac{2z^2|}{|6} + \frac{4z^2|}{|10} + \frac{4z^2|}{|14} + \cdots \end{aligned} $$

Letting $z = 1$ in the first line and $z = \frac{1}{2}$ in the second line, we end up with the equalities:

$$ \frac{e^2+1}{e^2-1} = \coth(1) = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{1}{2n+1} \right) \qquad \frac{e+1}{e-1} = \coth(\tfrac{1}{2}) = 2 + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{1}{4n+2} \right) $$

In the next chapter we will see that these continued fraction representations of $\frac{e^2+1}{e^2-1}$ and $\frac{e+1}{e-1}$ are more special than other representations in the sense that they give us additional information of the nature of these numbers as irrational numbers.

3 Simple continued fractions in number theory

3.1 Simple continued fractions

Definition 3.1. *A **simple continued fraction** is a continued fraction with $a_i = 1$, $b_0 \in \mathbb{Z}$ and $b_i \in \mathbb{N}^+$ for all $i \in \mathbb{N}^+$.*

This is the standard type of continued fraction used in number theory. We will represent the simple continued fractions with the notation $[b_0, b_1, b_2, \ldots]$. If we work with terminating continued fractions, we will represent them by $[b_0, b_1, \ldots, b_N]$.

3.1.1 Examples of simple continued fractions

1. $[1, 2, 3]$ is $\frac{10}{7}$.

2. As we had previously seen, $[1, 1, 1, 1, \ldots]$ converges to ϕ.

3. $[1, 3, 5, 7, 9, \ldots]$ converges to $\frac{e^2+1}{e^2-1}$ and $[2, 6, 10, 14, 18, \ldots]$ to $\frac{e+1}{e-1}$ by section 2.5.

4. $[2, 1, 2, 1, 1, 4, 1, \ldots, 1, 2n, 1, \ldots]$ can be proven to converge to e [42, Theorem 5.25.].

We will later see that every simple continued fraction converges, and that every real number can be represented uniquely by a simple continued fraction.

3.1.2 Properties of simple continued fractions

Proposition 3.1. For this kind of fraction we have that:

1. The **recurrence relations** become:

$$\begin{cases} p_n = b_n p_{n-1} + p_{n-2} \\ q_n = b_n q_{n-1} + q_{n-2} \end{cases} \tag{3.1}$$

2. The **determinant formulas** become:

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \qquad \Rightarrow \qquad \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1} q_n} \tag{3.2}$$

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n \, b_n \qquad \Rightarrow \qquad \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n \, b_n}{q_n q_{n-2}} \tag{3.3}$$

Proposition 3.2. For all $n \in \mathbb{N}$, $\frac{p_n}{q_n}$ is always an irreducible fraction.

Proof. Let $n \in \mathbb{N}$ and $d = \gcd(p_n, q_n)$. As $d \,|\, p_n$ and $d \,|\, q_n$, $d \,|\, (p_n q_{n-1} - p_{n-1} q_n)$ and so, $d \,|\, (-1)^n$ and $\frac{p_n}{q_n}$ is irreducible. □

Remark 3.1. As $b_n \geq 1$ for $n > 0$, equation (1.2) gives us the following bound on the growth of partial numerators and denominators of simple continued fractions: $p_n \geq F_n$ and $q_n \geq F_{n-1}$ where F_k is the k^{th} Fibonacci number, as described in section 2.1.1.

3.2 Unique representation of real numbers

3.2.1 Convergence of simple continued fractions

Theorem 3.1. *Every simple continued fraction converges to a real number ℓ. Furthermore, if $\{w_n\}_{n=0}^{\infty}$ is the sequence of its approximants, we have that*

$$w_0 \leq w_2 \leq w_4 \leq \cdots \to \ell \leftarrow \cdots \leq w_5 \leq w_3 \leq w_1 \tag{3.4}$$

Proof. As all the $b_n > 0$ for all $n \in \mathbb{N}^+$, it is easy to see that w_0 is the smallest approximant and w_1 is the greatest approximant.

For infinite simple continued fractions, by (3.3), $w_{2k} - w_{2k-2} > 0$ and $w_{2k+1} - w_{2k-1} < 0$ for $k > 0$. As the sequence $\{w_{2k}\}_{k=0}^{\infty}$ is monotonically increasing and bounded from above, it converges. Similarly, the sequence $\{w_{2k+1}\}_{k=0}^{\infty}$ is monotonically decreasing and bounded from below, so it also converges. As $\lim_{n\to\infty} q_n = \infty$, by (3.1), $\lim_{n\to\infty}(w_n - w_{n-1}) = 0$ and the limits of both sequences match.

Alternatively, the convergence can also be inferred from the Seidel-Stern theorem (2.1). □

3.2.2 The Euclidean algorithm

The computation of the continued fraction expansion of a rational number is related to the **Euclidean algorithm**, a process that is used in general for the computation of the greatest common divisor of elements of an Euclidean ring. This computation is done in the following way:

Let $\frac{a}{b}$ be a rational number. If it were negative, we can express it as $\frac{a}{b} = \lfloor \frac{a}{b} \rfloor + \frac{c}{d}$ where $\frac{c}{d}$ is positive, and proceed from there. If it is positive, there are two options. If $a < b$, the fraction is smaller than zero and so, we set $b_0 = 0$ and start by dividing b by a. If it is greater, we start by dividing a by b. After doing this division, we take the divisor and we divide it by the remainder of this division. We iterate this process -dividing the divisor by the remainder- until we get a division in which the remainder is zero (this eventually happens due to the fact that in the division algorithm the divisor is always greater than the remainder).

At the end of the process, the quotients of the division in each step are the coefficients of the continued fraction. Therefore, by means of this algorithm (A.4.1), every rational number can be represented as a terminating simple continued fraction.

For example, for the fraction $\frac{67}{24}$, the process would be the one shown at the right and hence, we would get that

$$\frac{67}{24} = [2, 1, 3, 1, 4].$$

$$\begin{aligned} 67 &= 2 \cdot 24 + 19 \\ 24 &= 1 \cdot 19 + 5 \\ 19 &= 3 \cdot 5 + 4 \\ 5 &= 1 \cdot 4 + 1 \\ 4 &= 4 \cdot 1 + 0 \end{aligned}$$

The reason why this works is because if we call D_1 the dividend, d_1 the divisor, c_1 the quotient and r_1 the remainder, we have that $d_k = D_{k+1}$, $r_k = d_{k+1}$ and so:

$$\begin{cases} \frac{a}{b} = \frac{D_1}{d_1} = c_1 + \frac{r_1}{d_1} \\ \frac{d_1}{r_1} = \frac{D_2}{d_2} = c_2 + \frac{r_2}{d_2} \\ \quad \vdots \\ \frac{d_{N-1}}{r_{N-1}} = \frac{D_N}{d_N} = c_N \end{cases} \Rightarrow \quad \frac{a}{b} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_N}}}$$

Remark 3.2. With a few changes, this procedure can be adapted for any Euclidean ring $\mathcal{R}$ to represent elements of its quotient field as simple continued fractions (understanding in this case that simple continued fractions are continued fractions with $a_i = 1$ and $b_i \in \mathcal{R}$). For example, it works for computing the simple continued fraction expansions of elements of $\mathbb{F}(x)$ (A.4.2) (the ring of rational functions over $\mathbb{F}$ in the indeterminate x) and $\mathbb{Q}(i)$ (A.4.1) (the quotient ring of the Gaussian integers).

3.2.3 Representation of rational numbers as simple continued fractions

There is an equivalence between terminating simple continued fractions and rational numbers:

Theorem 3.2. *A real number can be represented by a terminating simple continued fraction if and only if it is a rational number.*

Proof. $\Rightarrow$ As the coefficients of simple continued fractions are integers, so are all the partial numerators and denominators, including the N^{th} ones. Therefore, the value of the continued fraction, which is the N^{th} approximant, is the quotient of two integers, hence, a rational number.
$\Leftarrow$ This is a consequence of the application of the Euclidean algorithm. □

Remark 3.3. This representation is not necessarily unique, as we always have:

$$[b_0, b_1, \ldots, b_n] = [b_0, b_1, \ldots, b_n - 1, 1] \tag{3.5}$$

However, if we fix the condition that the last term of the continued fraction cannot be 1, then, the representation of every rational number as a finite simple continued fraction is unique.

Remark 3.4. This is generally not true for general continued fractions, as we have shown in our second example of a general continued fraction.

3.2.4 Representation of irrational numbers as simple continued fractions

Theorem 3.3. *Every irrational number can be uniquely expressed as an infinite simple continued fraction.*

Proof. **Uniqueness**

We are going to prove this in a constructive way. Let ξ be a real number and let us suppose that $[b_0, b_1, b_2, \dots]$ is its representation as a simple continued fraction. We are going to prove that this representation is the only possible one.

As b_0 and b_1 are the first coefficients of the simple continued fraction representation, we have that:

$$b_0 < \xi < b_0 + \frac{1}{b_1}.$$

The inequalities are strict as ξ is irrational. As $b_1 \geq 1$, $\frac{1}{b_1} \leq 1$, we infer that $b_0 = \lfloor \xi \rfloor$ so it is uniquely determined. It is clear that if we set $\xi_1 = \frac{1}{\xi - b_0}$, which is also an irrational number, then $\xi_1 = [b_1, b_2, b_3, \dots]$, and from the previous reasoning, we infer that $b_1 = \lfloor \xi_1 \rfloor$, so it is uniquely determined as well.

By setting $\xi_n = \frac{1}{\xi_{n-1} - b_{n-1}}$ repeatedly, we get that $b_n = \lfloor \xi_n \rfloor$ and so, the uniqueness of the coefficients of the simple continued fraction is proven.

This algorithm is known as the **simple continued fraction algorithm** (A.4.3) and the quantities ξ_n are called the n$^{\text{th}}$ **complete quotients** of ξ.

Existence

Let ξ be an irrational number. By applying the simple continued fraction algorithm on ξ, we get a simple continued fraction $[b_0, b_1, b_2, \dots]$. Let us check that this continued fraction indeed converges to ξ. To do so, we are going to first prove the following lemma:

Lemma 3.1. With the same notation as before, we have the following:

$$\xi = \frac{\xi_{n+1} p_n + p_{n-1}}{\xi_{n+1} q_n + q_{n-1}} \qquad \forall n \in \mathbb{N} \tag{3.6}$$

Proof. Let us prove it by induction on n. For $n = 0$,

$$\xi = b_0 + \frac{1}{\xi_1} = \frac{\xi_1 b_0 + 1}{\xi_1} = \frac{\xi_1 p_0 + p_{-1}}{\xi_1 q_0 + q_{-1}}$$

Let us suppose our statement is true for every $k \leq n - 1$ and let us prove it for $k = n$.

$$\xi = \frac{\xi_n p_{n-1} + p_{n-2}}{\xi_n q_{n-1} + q_{n-2}} = \frac{(b_n + \frac{1}{\xi_{n+1}}) p_{n-1} + p_{n-2}}{(b_n + \frac{1}{\xi_{n+1}}) q_{n-1} + q_{n-2}} = \frac{\xi_{n+1}(b_n p_{n-1} + p_{n-2}) + p_{n-1}}{\xi_{n+1}(b_n q_{n-1} + q_{n-2}) + q_{n-1}} \overset{(3.1)}{=} \frac{\xi_{n+1} p_n + p_{n-1}}{\xi_{n+1} q_n + q_{n-1}}$$

□

Now, with the help of this lemma, we get:

$$\xi - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - p_n q_{n-1}}{q_n(\xi_{n+1}q_n + q_{n-1})} = \frac{(-1)^n}{q_n(\xi_{n+1}q_n + q_{n-1})} \tag{3.7}$$

As $\xi_{n+1} > b_{n+1}$, we have that:

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n(b_{n+1}q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} \tag{3.8}$$

and so, as n tends to infinity, q_n also tends to infinity, and, therefore, $\xi = [b_0, b_1, b_2, \dots]$. □

Remark 3.5. Even though the simple continued fraction expansion for some irrational numbers such as e or ϕ is known, for most of irrational constants no expressions of the general term of its simple continued fraction are known. This is the case for π, whose simple continued fraction begins with $[3, 7, 15, 1, 292, 1, \dots]$ (sequence A001203 of OEIS) or the **Euler-Mascheroni constant** γ, which we do not even know if it is irrational, but whose continued fraction begins with $[0, 1, 1, 2, 1, 2, \dots]$ (sequence A002852 of OEIS).

3.3 Diophantine approximation

3.3.1 Best rational approximation

At the start of this section, I commented that the simple continued fractions were an important tool in number theory, and the main reason why this is the case is because of how they can be used to obtain precise rational approximations of real numbers. The theory of approximation of real numbers by rational numbers is known as **Diophantine approximation**.

Proposition 3.3. For every irrational number ξ we have the following properties:

1. For every $n \in \mathbb{N}$, $\dfrac{1}{q_n q_{n+2}} < \left|\xi - \dfrac{p_n}{q_n}\right| < \dfrac{1}{q_n q_{n+1}}$.
2. For every $n \in \mathbb{N}$, $|\xi q_{n+1} - p_{n+1}| < |\xi q_n - p_n|$ and therefore, $\left|\xi - \dfrac{p_{n+1}}{q_{n+1}}\right| < \left|\xi - \dfrac{p_n}{q_n}\right|$.
3. For every $n \in \mathbb{N}$ and $k < q_{n+1}$ such that $\dfrac{h}{k} \neq \dfrac{p_n}{q_n}$, we have that $|\xi k - h| > |\xi q_n - p_n|$.
4. For every $n \in \mathbb{N}^+$ and $k \leq q_n$ such that $\dfrac{h}{k} \neq \dfrac{p_n}{q_n}$, we have that $\left|\xi - \dfrac{h}{k}\right| > \left|\xi - \dfrac{p_n}{q_n}\right|$.

This last one can be expressed by saying that $\frac{p_n}{q_n}$ are the **best rational approximations** of ξ.

Proof. **1.** The upper bound was proven in the proof of theorem 3.3. For the lower bound, we consider equation (3.7). As $\xi_{n+1} < 1 + b_{n+1}$, we know that $\xi_{n+1}q_n + q_{n-1} < q_n + b_{n+1}q_n + q_{n-1}$ so, $\xi_{n+1}q_n + q_{n-1} < q_n + q_{n+1} \leq q_n + b_{n+2}\, q_{n+1} = q_{n+2}$ and hence,

$$\left|\xi - \frac{p_n}{q_n}\right| > \frac{1}{q_n q_{n+2}}.$$

2. By **1**, we know that

$$\left|\xi - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{q_{n+1}q_{n+2}} \quad \text{and} \quad \frac{1}{q_n q_{n+2}} < \left|\xi - \frac{p_n}{q_n}\right|.$$

.

Multiplying those inequalities by q_{n+1} and q_n respectively, we get that:

$$|\xi q_{n+1} - p_{n+1}| < \frac{1}{q_{n+2}} < |\xi q_n - p_n|,$$

and so, as $q_n \geq q_{n+1}$ for $n \in \mathbb{N}$

$$\left|\xi - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{q_{n+1}}|\xi q_n - p_n| \leq \left|\xi - \frac{p_n}{q_n}\right|.$$

3. Let $n \in \mathbb{N}$ and $k < q_{n+1}$ such that $\frac{h}{k} \neq \frac{p_n}{q_n}$. In order to prove that $|\xi k - h| > |\xi q_n - p_n|$, we are first going to express $\xi k - h$ as a linear combination of $\xi q_n - p_n$ and $\xi q_{n+1} - p_{n+1}$ and work from there.

Let $\xi k - h = \alpha(\xi q_n - p_n) + \beta(\xi q_{n+1} - p_{n+1})$. Then, α and β are the solutions of the linear system:

$$\begin{cases} k = \alpha\, q_n + \beta\, q_{n+1} \\ h = \alpha\, p_n + \beta\, p_{n+1} \end{cases}$$

Using Cramer's rule and the fact that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, we get that

$$\alpha = (-1)^n\,(k\,p_{n+1} - h\,q_{n+1})$$
$$\beta = (-1)^{n+1}\,(k\,p_n - h\,q_n)$$

It is clear that $\alpha, \beta \in \mathbb{Z}$. As $\frac{h}{k} \neq \frac{p_n}{q_n}$, $\beta \neq 0$ and as $\frac{p_{n+1}}{q_{n+1}}$ is irreducible and $k < q_{n+1}$, we know that $\frac{h}{k} \neq \frac{p_{n+1}}{q_{n+1}}$, so $\alpha \neq 0$.

Furthermore, α and β must have different signs because if that were not the case, $k = \alpha\, q_n + \beta\, q_{n+1}$ would imply that $k > q_n$ and that would contradict our hypothesis.

What's more, as odd approximants of the continued fraction of ξ are greater than ξ and even approximants are smaller than ξ, we get that $\xi - \frac{p_n}{q_n}$ and $\xi - \frac{p_{n+1}}{q_{n+1}}$ always have different signs and therefore, so do $\xi q_n - p_n$ and $\xi q_{n+1} - p_{n+1}$. Hence, $\alpha(\xi q_n - p_n)$ and $\beta(\xi q_{n+1} - p_{n+1})$ have the same sign and so,

$$|\xi k - h| = |\alpha(\xi q_n - p_n)| + |\beta(\xi q_{n+1} - p_{n+1})| > |\alpha||\xi q_n - p_n| \geq |\xi q_n - p_n|.$$

4. If $n \in \mathbb{N}^+$ and we have $k \leq q_n$ such that $\frac{h}{k} \neq \frac{p_n}{q_n}$, as $k \leq q_n < q_{n+1}$, it satisfies the hypothesis of **3** and we have:

$$\left|\xi - \frac{h}{k}\right| > \frac{1}{k}|\xi q_n - p_n| \geq \left|\xi - \frac{p_n}{q_n}\right| \qquad \square$$

Remark 3.6. The fact that approximants give best rational approximations has been known for thousands of years and it has been used by civilisations such as Greece, Egypt, Babylonia, India and China. For example, a procedure called *anthyphပaresis* based on the Euclidean algorithm was used by Archimedes to find the rational approximation $\frac{22}{7}$ for π [3, Section 1.1.].

3.3.2 Hurwitz's theorem

The closeness at which irrational numbers can be approximated by rational numbers is given by the following theorem:

Theorem 3.4 (Hurwitz's theorem). *Let ξ be an irrational number. Then, there exist infinitely many different rational numbers $\frac{p}{q}$ such that*

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2} \tag{3.9}$$

Using the theory of continued fractions we are going to prove an even more general result:

Theorem 3.5. *Let ξ be an irrational number and let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be its sequences of partial numerators and denominators. Then, at least one out of every three consecutive approximants satisfy*

$$0 < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{\sqrt{5}q_n^2}$$

Proof. Let $n \in \mathbb{N}^+$ and let us suppose that

$$\frac{1}{\sqrt{5}q_{n-1}^2} \leq \left|\xi - \frac{p_{n-1}}{q_{n-1}}\right|, \qquad \frac{1}{\sqrt{5}q_n^2} \leq \left|\xi - \frac{p_n}{q_n}\right|, \qquad \frac{1}{\sqrt{5}q_{n+1}^2} \leq \left|\xi - \frac{p_{n+1}}{q_{n+1}}\right|.$$

In the proof of proposition 3.3, we saw that $\xi - \frac{p_{n-1}}{q_{n-1}}$ and $\xi - \frac{p_n}{q_n}$ have different signs, so $\xi - \frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n} - \xi$ have the same sign. Therefore, by adding the first two inequalities, we have that:

$$\frac{1}{\sqrt{5}q_{n-1}^2} + \frac{1}{\sqrt{5}q_n^2} \leq \left|\frac{p_n}{q_n} - \xi\right| + \left|\xi - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_{n-1}q_n}$$

and so, multiplying by $\sqrt{5}q_{n-1}^2q_n^2$ both sides we get $q_{n-1}^2 + q_n^2 \leq \sqrt{5}q_{n-1}q_n$. By the same reasoning, $q_n^2 + q_{n+1}^2 \leq \sqrt{5}q_nq_{n+1}$.

If we substitute q_{n+1} via the relation (3.2), add both inequalities and move everything to the left side of the equation, we get:

$$2q_{n-1}^2 + (2b_n - 2\sqrt{5})q_{n-1}q_n + (b_n^2 - \sqrt{5}b_n + 2)q_n^2 \leq 0,$$

which can also be rewritten by completing the squares as:

$$q_{n-1}^2 + (\sqrt{5}b_n - 3)q_n^2 + (q_{n-1} + (b_n - \sqrt{5})q_n)^2 \leq 0.$$

If $b_n > 1$, then the left hand side would be the sum of three positive numbers and we would arrive at a contradiction, so b_n must be equal to one.

Nevertheless, if $b_n = 1$, we then have that

$$2q_{n-1}^2 + (2 - 2\sqrt{5})q_{n-1}q_n + (3 - \sqrt{5})q_n^2 \leq 0$$
$$\frac{1}{3+\sqrt{5}}((1+\sqrt{5})q_{n-1} - 2q_n)^2 \leq 0$$

This can only be possible if $(1+\sqrt{5})q_{n-1} - 2q_n = 0$, i.e., $\frac{q_n}{q_{n-1}} = \frac{1+\sqrt{5}}{2} = \phi$. As we know that ϕ is an irrational number (its simple continued fraction expansion is infinite) and q_{n-1} and q_n are integers, we arrive at a contradiction. Therefore, for some $k \in \{n-1, n, n+1\}$,

$$0 < \left|\xi - \frac{p_k}{q_k}\right| < \frac{1}{\sqrt{5}q_k^2}.$$ □

Remark 3.7. Hurwitz's theorem does not hold for any rational number. Let $\frac{h}{k}$ be a rational number in reduced form, i.e. h and k are coprimes. Then for all $\frac{p}{q} \neq \frac{h}{k}$,

$$\left|\frac{h}{k} - \frac{p}{q}\right| = \frac{|hq - pk|}{qk} \geq \frac{1}{qk} \implies \left|\frac{h}{k} - \frac{p}{q}\right| > \frac{1}{q^2} \quad \text{if} \quad q > k.$$

This shows that irrational numbers are characterised by how well they can be approximated by rationals. This property can be used to prove that numbers defined from infinite series such as e are irrational, as we can see in Varona Malumbre's book [42, Theorem 4.8.].

Once we have seen Hurwitz's theorem, it would be interesting to ask ourselves, is this the best we can do? Is there any $M \in \mathbb{R}$ greater than $\sqrt{5}$ such that for any irrational number ξ, we could find infinite rational numbers $\frac{p}{q}$ that satisfy this inequality?

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{Mq^2}. \tag{3.10}$$

The answer is **no**, as we will now see.

Theorem 3.6. *Let $\phi = \frac{1+\sqrt{5}}{2}$. Then, there is no $M > \sqrt{5}$ such that there are infinitely many rational numbers satisfying:*

$$0 < \left|\phi - \frac{p}{q}\right| < \frac{1}{Mq^2}. \tag{3.11}$$

Proof. We will later give a proof of this from the continued fraction expansion of ϕ. □

3.3.3 The Lagrange spectrum

Now that we have seen Hurwitz's theorem, let us ask ourselves a different question that it is related to it. Given a fixed irrational number ξ, is there a value $M(\xi)$ such that for $M \leq M(\xi)$, equation (3.10) holds for an infinite number of rationals but for $M > M(\xi)$ it only holds for a finite number of rationals?

This problem was first studied by Lagrange, and leads to the following definition:

Definition 3.2. *Let ξ be an irrational number. The* ***Lagrange constant*** *of ξ is:*

$$M(\xi) = \sup\{\lambda \in \mathbb{R} : \left|\xi - \frac{p}{q}\right| < \frac{1}{(\lambda - \epsilon)q^2} \text{ has infinitely many solutions } p, q \in \mathbb{N}, \forall \epsilon > 0\}$$
$$= \limsup_{q\to\infty} \frac{1}{q|\xi q - p|}.$$

By Hurwitz's theorem, we know that $M(\xi)$ is at least $\sqrt{5}$ for any irrational. While researching the possible values of the Lagrange constant of irrational numbers, Hurwitz found that if $M(\xi) \neq \sqrt{5}$, then $M(\xi) \geq \sqrt{8}$ and thus, Lagrange constants were never found in the interval $(\sqrt{5}, \sqrt{8})$. This motivated the following definition:

Definition 3.3. *The* ***Lagrange spectrum*** *is the set of possible values of the Lagrange constant, i.e.*

$$\mathcal{L} = \{M(\xi) : \xi \in \mathbb{R}/\mathbb{Q}\}$$

Remark 3.8. It is not easy to see the role that the ϵ plays in definition 3.2, but it is necessary for the two notions that we have given of $M(\xi)$ to be equivalent. In 2017 it was proven that there are a class of numbers called **not attainable numbers** consisting of the numbers ξ such that $|\xi - \frac{p}{q}| < \frac{1}{M(\xi)q^2}$ does not have infinitely many solutions [14]. The $M(\xi)$ for which ξ is not attainable receive the name of **not admissible numbers** and they are closely related to the topological structure of the spectrum.

In the study of the Lagrange spectrum, numerous connections with number theory problems were found, out of which, it is imperative that I mention the following:

In 1880, Andrey Markov found the following equation while studying the minimum of certain binary quadratic forms $f(x, y) = ax^2 + 2bxy + cy^2$:

$$u^2 + v^2 + w^2 = 3uvw \tag{3.12}$$

which is now known as **Markov's equation** . The solutions $u \in \mathbb{N}$ of that equation for which there exists $v, w \in \mathbb{N}$ with $u \geq v \geq w$ are known as **Markov numbers**. The first ones (they are infinite) are 1, 2, 5, 13, 29... (sequence A002559 in OEIS).

After deeply studying the equation, he realised that, whenever (u, v, w) was a solution of the equation, so were $(\overline{u}, v, w)$, $(u, \overline{v}, w)$ and $(u, v, \overline{w})$ with

$$\overline{u} = 3vw - u \qquad \overline{v} = 3wu - v \qquad \overline{w} = 3uv - w$$

and this process could be used to find all the solutions from the most basic one. This has been illustrated in appendix B.2.

In 1921, Oskar Perron proved that if u is a Markov number and we take

$$\alpha_u = \frac{1}{2u}\left(\sqrt{9u^2-4}+u+\frac{2v}{w}\right), \qquad \text{then} \qquad M(\alpha_u) = \frac{\sqrt{9u^2-4}}{u}. \tag{3.13}$$

What's more, if $M \in \mathcal{L}$ and $M < 3$, then $M = M(\alpha_u)$ for some Markov number. This implies that, as α_u grows to infinity, $M(\alpha_u)$ gathers around 3 and it can be easily seen that 3 is an accumulation point of the Lagrange spectrum (and, in fact, it is the smallest accumulation point [6]). The first 40 Markov numbers and their corresponding α_u and $M(\alpha_u)$ can be found in table C.3.

For values greater than 3, the structure of the Lagrange spectrum becomes way more complicated. As a brief summary, I remark that it is a closed set with fractal dimension, as it is linked to the Cantor sets defined by a dynamical operator called the **Gauss map** [30]. Furthermore, it contains the half line $[c_F, +\infty)$ known as **Hall's ray**, where the constant $c_F = 4 + \frac{253589820+283798\sqrt{462}}{491993569}$ is called **Freiman's constant**.

In some references, the Lagrange spectrum is also called the **Markov spectrum**, but we have decided not to used this term, as the Markov spectrum makes reference to a different set that contains the Lagrange spectrum [13]. The relation between the Markov and Lagrange spectrum shows that the theory of Diophantine approximation is closely related to the rich theory of quadratic forms.

There is an unexpected result by Perron which links both the theory of continued fractions with the study of the Lagrange spectrum.

Theorem 3.7. *Let $\xi = [b_0, b_1, b_2, \dots]$ an irrational number. Then,*

$$M(\xi) = \limsup_{n\to\infty} \left([b_{n+1}, b_{n+2}, b_{n+3}, \dots] + [0, b_n, b_{n-1}, \dots, b_1, b_0]\right).$$

Proof. As the approximants of ξ are all the best rational approximations of ξ by the proposition 3.3 part 3, we infer that

$$M(\xi) = \limsup_{q\to\infty} \frac{1}{q|\xi q - p|} = \limsup_{n\to\infty} \frac{1}{q_n|\xi q_n - p_n|}.$$

From equation (3.7), we know that

$$\left|\xi - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\xi_{n+1}q_n + q_{n-1})},$$

so

$$M(\xi) = \limsup_{n\to\infty} \frac{1}{q_n|\xi q_n - p_n|} = \limsup_{n\to\infty} \left(\xi_{n+1} + \frac{q_{n-1}}{q_n}\right).$$

We have already seen that $\xi_{n+1} = [b_{n+1}, b_{n+2}, b_{n+3}, \dots]$. Furthermore, we know that

$$\frac{q_{n-1}}{q_n} = \frac{1}{\dfrac{b_n q_{n-1} + q_{n-2}}{q_{n-1}}} = \frac{1}{b_n + \dfrac{q_{n-2}}{q_{n-1}}} = \frac{1}{b_n + \dfrac{1}{b_{n-1} + \dfrac{q_{n-3}}{q_{n-2}}}} = \frac{1}{b_n + \dfrac{1}{b_{n-1} + \dfrac{1}{\ddots + \dfrac{1}{b_0}}}}$$

Therefore, $\frac{q_{n-1}}{q_n} = [0, b_n, b_{n-1}, \dots, b_1, b_0]$ and we finish. □

With this theorem we can prove theorem 3.6. As $\phi = [1, 1, 1, 1, \dots]$, by this last theorem,

$$M(\phi) = \limsup_{n\to\infty}([1, 1, 1, \dots] + [0, 1, 1, \dots, 1]) = \phi + (\phi - 1) = \sqrt{5}.$$

3.3.4 Equivalent real numbers

There is an equivalence relation on the real numbers which is closely related to both simple continued fractions and other subfields of number theory.

Definition 3.4. *Two real numbers μ and ξ are **equivalent** if there exists $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that*

$$\xi = \frac{\alpha\mu + \beta}{\gamma\mu + \delta} \quad \textit{with} \quad \alpha\delta - \beta\gamma = \pm 1. \tag{3.14}$$

It is easy to see that this is an **equivalence relation** as it verifies the reflexive, symmetric and transitive properties. This is a consequence of the identification of $\mathcal{M}_{\mathbb{Z}}$ with $\mathrm{PGL}(2, \mathbb{Z})$ and the fact that, as the latter is a group, it has an identity (reflexive), every element has an inverse (symmetric) and it is closed by multiplication (transitive).

Proposition 3.4. The rational numbers form an equivalence class under this relation.

Proof. We are first going to see that every rational number is equivalent to 0. Let $\frac{p}{q}$ be a rational number in reduced form. Then, by Bézout's lemma, there exist $x, y \in \mathbb{Z}$ such that $px + qy = \gcd(p, q) = 1$. Therefore,

$$\frac{p}{q} = \frac{y\,0 + p}{-x\,0 + q} = \frac{\alpha\,0 + \beta}{\gamma\,0 + \delta}.$$

with $\alpha\delta - \beta\gamma = yq - p(-x) = 1$ and so, all rational number are in the same class. It is trivial to see that if a number is equivalent to a rational, then it must be rational, so we finish. □

Before we study the relation between this equivalence relation and simple continued fraction, let us introduce this lemma that will be used later on.

Lemma 3.2. Let $x = \frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$. If $\zeta > 1$, $\gamma > \delta > 0$ and $\alpha\delta - \beta\gamma = \pm 1$, then $\frac{\beta}{\delta}$ and $\frac{\alpha}{\gamma}$ are consecutive approximants of x.

Proof. The proof is not complicated but it does not add anything to our knowledge of continued fractions, so I prefer to omit it. It can be found in both Hardy's and Keng's books [17, Theorem 172; 22, Theorem 5.2.]. □

Now we have all the tools to prove the following result:

Theorem 3.8. *Two irrational numbers μ and ξ are equivalent under relation (3.14) if and only if their simple continued fraction expansions match from some term on, i.e. if*

$$\mu = [a_0, a_1, a_2, \dots], \qquad \xi = [b_0, b_1, b_2, \dots],$$

then $\mu \sim \xi$ if and only if $a_{k+m} = b_{n+m}$ for some $k, n \in \mathbb{N}$ and for all $m \in \mathbb{N}$.

Proof. $\Leftarrow$ Let $\mu = [a_0, a_1, \dots, a_{k-1}, c_0, c_1, \dots]$, let $\xi = [b_0, b_1, \dots, b_{n-1}, c_0, c_1, \dots]$ and let $\{\tilde{p}_n\}_{n=0}^{\infty}$, $\{\tilde{q}_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ be their partial numerators and denominators.

Let $\omega = [c_0, c_1, c_2, \dots]$. From the proof of theorem 3.3, we know that $\omega = \mu_k = \xi_n$ and by lemma 3.1, we then get:

$$\mu = \frac{\omega \tilde{p}_{k-1} + \tilde{p}_{k-2}}{\omega \tilde{q}_{k-1} + \tilde{q}_{k-2}} \qquad \xi = \frac{\omega p_{n-1} + p_{n-2}}{\omega q_{n-1} + q_{n-2}}$$

As $\tilde{p}_{k-1}\tilde{q}_{k-2} - \tilde{p}_{k-2}\tilde{q}_{k-1} = (-1)^{k-2}$ and $p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = (-1)^{n-2}$, $\mu \sim \omega$ and $\xi \sim \omega$, so $\mu \sim \xi$.

$\Rightarrow$ Let $\mu \sim \xi$, so $\mu = \dfrac{\alpha\xi + \beta}{\gamma\xi + \delta}$.

Let us assume that $\gamma\xi + \delta > 0$ (if this is not the case, we can multiply both the numerator and denominator by -1). Again, by lemma 3.1,

$$\xi = \frac{\xi_{n+1} p_n + p_{n-1}}{\xi_{n+1} q_n + q_{n-1}} \qquad \forall n \in \mathbb{N}$$

Hence,

$$\mu = \frac{\alpha_n \xi_{n+1} + \beta_n}{\gamma_n \xi_{n+1} + \delta_n} \qquad \text{with} \qquad \begin{cases} \alpha_n &= \alpha p_n + \beta q_n \\ \beta_n &= \alpha p_{n-1} + \beta q_{n-1} \\ \gamma_n &= \gamma p_n + \delta q_n \\ \delta_n &= \gamma p_{n-1} + \delta q_{n-1} \end{cases}$$

and $\alpha_n\delta_n - \beta_n\gamma_n = (\alpha\delta - \beta\gamma)(p_n q_{n-1} - p_{n-1} q_n) = (-1)^{n-1}$.

By equation (3.7) we know that

$$p_n = \xi q_n - \frac{\epsilon_n}{q_n} \quad \text{with} \quad \epsilon_n = \frac{(-1)^n q_{n-1}}{\xi_{n+1} q_n + q_{n-1}}$$

and $|\epsilon_n| < \frac{1}{q_{n+1}} \leq 1$ for all $n \in \mathbb{N}$. Therefore,

$$\gamma_n = (\gamma\xi + \delta) q_n - \frac{\gamma \epsilon_n}{q_n} \qquad \delta_n = (\gamma\xi + \delta) q_{n-1} - \frac{\gamma \epsilon_{n-1}}{q_{n-1}}$$

As $\gamma\xi + \delta > 0$, $q_n > q_{n-1} > 0$, and $\lim_{n\to\infty} q_n = \infty$; for some $m \in \mathbb{N}$, for all $m \geq n$, we have that $\gamma_m > \delta_m > 0$ and, by lemma 3.2, we then have $\frac{\beta_m}{\delta_m} = \frac{p_{k-1}}{q_{k-1}}$ and $\frac{\alpha_m}{\gamma_m} = \frac{p_k}{q_k}$ for some $k \in \mathbb{N}$. Therefore, we have $\xi_{m+1} = \mu_{k+1}$ and so, the tails of the continued fraction expansion of ξ and μ match. □

Corollary 3.1. Let μ and ξ two equivalent irrational numbers. Then $M(\mu) = M(\xi)$.

Proof. If $\mu \sim \xi$, then they must have the same continued fraction expansion from a point on, and so, the same Lagrange constant by theorem 3.7. □

3.3.5 Liouville's work on algebraic numbers

To finish this section on Diophantine approximation, it is important to mention some of the most important results about how well real numbers can be approximated by rational numbers and how this relates to whether they are algebraic or trascendental. This gives rise to a whole subfield in number theory known as **transcendental number theory**, whose main results I will now highlight.

Theorem 3.9 (Liouville's theorem). *If $\alpha \in \mathbb{R}$ is algebraic of order $n \geq 1$, then there exists a constant $C(\alpha) > 0$ such that, for any rational number $\frac{p}{q}$ with $\alpha \neq \frac{p}{q}$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha)}{q^n} \tag{3.15}$$

Proof. The proof can be found in Varona Malumbres' book [42, Theorem 4.10.]. □

This theorem has important applications. Firstly, for the case $n = 1$, it tells us that rational numbers are badly approximated by other rationals as we saw in remark 3.7. It also implies the two following results:

Corollary 3.2. If $\alpha \in \mathbb{R}$ is algebraic of order $n \geq 2$, for every $\epsilon > 0$ and every $C > 0$, the inequality

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{C}{q^{n+\epsilon}}, \tag{3.16}$$

has a finite number of solutions.

Proof. This is because for every $\epsilon > 0$ and every $C > 0$, there is a q_0 such that $C < C(\alpha)q_0^{\epsilon}$, and so, for all $q \geq q_0$, $|\alpha - \frac{p}{q}| > \frac{C}{q^{n+\epsilon}}$. □

Corollary 3.3 (Transcendence criterion). Let $\alpha \in \mathbb{R}$ and let us suppose that there is a constant $K(\alpha) > 0$ such that for all $m \in \mathbb{N}$, the inequality

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{K(\alpha)}{q^m}, \tag{3.17}$$

has a solution $\frac{p}{q}$ with $q \geq 2$. Then, α is transcendental.

Proof. Let α be such that equation (3.17) holds. If α were not transcendental, it would be algebraic of a certain order n and, by Liouville's theorem, there would exist $C(\alpha) > 0$ such that for all $\frac{p}{q}$ if $\alpha \neq \frac{p}{q}$ the equation (3.15) would hold. If $\frac{p}{q}$ also satisfied equation (3.17), then $C(\alpha)q^{-n} < K(\alpha)q^m$, so

$$q^{m-n} < \frac{K(\alpha)}{C(\alpha)}.$$

Let us focus on the $m \geq n$. As $q \geq 2$, we know that $2^{m-n} < \frac{K(\alpha)}{C(\alpha)}$ from which we deduce that

$$m < n + \log_2\left(\frac{K(\alpha)}{C(\alpha)}\right)$$

and we arrive at a contradiction. □

This last criterion allows us to construct and test transcendental numbers and it allowed Liouville in 1844 to give the first explicit examples of transcendental numbers. The numbers that satisfy this transcendence criteria are known as **Liouville numbers**.

These are two examples of families of Liouville numbers:

- If $a \in \mathbb{N}$, $a \geq 2$, the number $\alpha = \sum_{n=1}^{\infty} a^{-n!}$ is transcendental [42, Corollary 4.14.]. For $a = 10$, α is known as **Liouville's constant**.
- If $b \in \mathbb{N}$, $b \geq 2$, the number $\beta = \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{1}{b^{n!}}\right)$ is transcendental [17, Theorem 192.].

In 1874, Georg Cantor added more contributions to the theory of transcendental numbers with his study of the cardinality of real numbers. He proved that $\mathbb{R}$ was uncountable and the set of algebraic numbers was countable, from which it is deduced that almost all real numbers are transcendental [42, Section 6.4.1.].

Furthermore, thanks to the development of measure theory in the early 20^{th} century, mathematicians were able to prove results such as that the set of Liouville numbers form a dense subset of the set of real numbers or that their Lebesgue measure is equal to zero.

In the study of how well real numbers can be approximated by rational numbers, the following definition arises naturally:

Definition 3.5. *For a real number α, its **irrationality measure** or **Liouville-Roth constant** is defined as*

$$\begin{aligned}\mu(\alpha) &= \sup\{\lambda > 0 : 0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\lambda} \textit{ has infinitely many solutions}\} \\ &= \inf\{\lambda > 0 : 0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\lambda} \textit{ has a finite number of solutions}\}\end{aligned}$$

For rational numbers, we know that $\mu(\frac{p}{q}) = 1$ and for irrational numbers, $\mu(\alpha) \geq 2$. For algebraic numbers of order n, we have seen that Liouville's theorem implies that $\mu(\alpha) \leq n$. Many mathematicians tried to improve this bound:

- **Thue** proved in 1909 that $\mu(\alpha) \leq 1 + \frac{n}{2}$.
- **Siegel** proved in 1921 that $\mu(\alpha) < 2\sqrt{n}$.
- **Gelfond** and **Dyson** proved in 1921 that $\mu(\alpha) < \sqrt{2n}$.

The attempts to improve this bound finished in 1955 when Roth showed that all algebraic numbers had $\mu(\alpha) = 2$. The proof won him a Fields medal in 1958.

As a matter of fact, almost all real numbers have irrationality measure 2. For example, it can be proven that $\mu(e) = 2$ [42]. However, not all reals have irrationality measure 2: besides Liouville numbers, that are considered to have irrationality measure ∞, we can construct real numbers with any possible irrationality measure $\mu \in [2, \infty)$ by choosing any $a > 1$ and taking:

$$\alpha_a = \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{1}{\lfloor a^{(\mu-1)^n} \rfloor} \right). \qquad \text{Then, } \mu(\alpha_a) = \mu \text{ [5].}$$

The continued fraction expansion plays a central role in the study of the transcendence of numbers, as there are formulas that allow us to use these expansions to compute the irrationality measure and other quantifiers of transcendence. Some of them can be found in the article of Sondow [38, Theorem 1.].

3.4 Periodic simple continued fractions

Definition 3.6. *A simple (non-terminating) continued fraction* $[b_0, b_1, \dots]$ *is* ***periodic*** *if there exist integers* $k \geq 0$ *and* $m \geq 1$ *such that* $b_{n+m} = b_n$ *for all* $n \geq k$*. We will represent these periodic fractions with the notation* $[b_0, \dots, b_{k-1}, \overline{b_k, b_{k+1}, \dots, b_{k+m-1}}]$*. The number* m *is called the* ***period*** *of the fraction. If* $k = 0$*, then we say that the continued fraction is* ***purely periodic****.*

3.4.1 Examples of periodic simple continued fractions

- The golden ratio $\phi = [\overline{1}]$ is purely periodic.
- From $\sqrt{2} - 1 = \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}+1} = \frac{1}{\sqrt{2}+1} = \frac{1}{2+(\sqrt{2}-1)}$, we deduce that $\sqrt{2} - 1 = [0, \overline{2}]$ and so, $\sqrt{2} = [1, \overline{2}]$.

These two examples have in common that both are **algebraic numbers of degree 2** or **quadratic irrationals**. That is, both are solutions of irreducible quadratic equations in $\mathbb{Z}[x]$ ($x^2 - x - 1 = 0$ and $x^2 - 2 = 0$ respectively). This is not a coincidence, as we will now see.

Theorem 3.10 (Lagrange). *Every periodic simple continued fraction represents a quadratic irrationality. Conversely, the simple continued fraction representation of any real quadratic irrationality is periodic.*

Proof. For the first part, let $\theta = [\overline{b_0, b_1, \dots, b_{m-1}}]$ be a purely periodic continued fraction. Then,

$$\theta \overset{(3.6)}{=} \frac{\theta_m p_m + p_{m-1}}{\theta_m q_m + q_{m-1}} = \frac{\theta p_m + p_{m-1}}{\theta q_m + q_{m-1}}$$

so θ satisfies $q_m \theta^2 + (q_{m-1} - p_m)\theta - p_{m-1} = 0$. If $\theta = [b_0, \dots, b_{k-1}, \overline{b_k, b_{k+1}, \dots, b_{k+m-1}}]$, θ_k is purely periodic, hence quadratic irrational and, by rationalizing the expression given by lemma 3.1, we conclude that θ must be quadratic irrational as well.

For the second part, let ξ be the root of an irreducible polynomial $\alpha x^2+\beta x+\gamma$ with integer coefficients and $\alpha \neq 0$. If we substitute ξ in the equation $\alpha\xi^2+\beta\xi+\gamma=0$, by the equality (3.6), for any $n \in \mathbb{N}^+$ we get the equation

$$\alpha_n \xi_n^2 + \beta_n \xi_n + \gamma_n = 0$$

where

$$\begin{cases} \alpha_n = \alpha p_{n-1}^2 + \beta p_{n-1} q_{n-1} + \gamma q_{n-1}^2 \\ \beta_n = 2\alpha p_{n-1} p_{n-2} + \beta(p_{n-1} q_{n-2} + p_{n-2} q_{n-1}) + 2\gamma q_{n-1} q_{n-2} \\ \gamma_n = \alpha p_{n-2}^2 + \beta p_{n-2} q_{n-2} + \gamma q_{n-2}^2 \end{cases}$$

and it can be seen that $\alpha_{n-1} = \gamma_n$ and $\beta_n^2 - 4\alpha_n\gamma_n = \beta^2 - \alpha\gamma$ for all $n \in \mathbb{N}^+$ [42].

Now, from equation (3.8), we get that

$$p_{n-1} = \xi q_{n-1} + \frac{\epsilon_n}{q_{n-1}}$$

for some ϵ_n such that $|\epsilon_n| < 1$. If we substitute p_{n-1} in the expression of α_n and use that $\alpha\xi^2+\beta\xi+\gamma=0$, we get:

$$\begin{aligned} \alpha_n &= \alpha\Big(\xi q_{n-1} + \frac{\epsilon_n}{q_{n-1}}\Big)^2 + \beta\Big(\xi q_{n-1} + \frac{\epsilon_n}{q_{n-1}}\Big) q_{n-1} + \gamma q_{n-1}^2 \\ &= (\alpha\xi^2+\beta\xi+\gamma) q_{n-1}^2 + 2\alpha\epsilon_n\xi + \alpha\frac{\epsilon_n^2}{q_{n-1}^2} + \beta\epsilon_n \\ &= 2\alpha\epsilon_n\xi + \alpha\frac{\epsilon_n^2}{q_{n-1}^2} + \beta\epsilon_n. \end{aligned}$$

Therefore, $|\alpha_n| \le 2|\alpha\xi| + |\alpha| + |\beta|$ and so, the α_n are bounded for all $n \in \mathbb{N}^+$. From what we saw before, the γ_n and the β_n are also bounded (as $\gamma_n = \alpha_{n-1}$ and $\beta_n^2 - 4\alpha_n\gamma_n = \beta^2 - \alpha\gamma$). Thus, as α_n, β_n and γ_n are bounded integers, there is a finite number of possible triples $\{\alpha_n, \beta_n, \gamma_n\}$ and so, a finite number of possible polynomials $\alpha_n x^2 + \beta_n x + \gamma_n$ and a finite number of possible solutions ξ_n such that $\alpha_n\xi_n^2 + \beta_n\xi_n + \gamma_n = 0$.

Hence, for some $n, m \in \mathbb{N}^+$, we deduce that $\xi_n = \xi_{n+m}$ and so, by theorem 3.3, they must have the same continued fraction expansion $[b_n, b_{n+1}, b_{n+2}, \ldots] = [b_{n+m}, b_{n+m+1}, b_{n+m+2}, \ldots]$ and from this we deduce that ξ is periodic. □

Remark 3.9. The table C.4 represents the periodic expansion of the square root of the first 80 natural numbers. As we can see, the period of the numbers follows interesting patterns that have been widely studied.

Remark 3.10. The fact that a real quadratic irrationality has a periodic continued fraction implies that it has finite complete quotients. These can be computed with the function (A.4.4) and be used to calculate the Lagrange constant of that quadratic irrationality (A.4.5).

It would also be interesting to analyse what conditions quadratic irrationals need to satisfy in order to have a purely periodic continued fraction expansion. The answer to this question is known to have been given in Galois' first article in 1828, though the result was implicit in the earlier work of Lagrange.

Theorem 3.11 (Galois). *A purely periodic simple continued fraction represents a quadratic irrationality ξ if and only if its algebraic conjugate $\overline{\xi}$ satisfies $-1 < \overline{\xi} < 0$. In that case $\overline{\xi} = -\frac{1}{\mu}$ where μ is the valued of the continued fraction you get when you reverse the period.*

Proof. Niven, Zuckerman and Montgomery proved this theorem in a clever way by expressing any purely periodic ξ as the solution of a quadratic equation $f(x)$ and showing that $f(-1)$ and $f(0)$ have different signs [31, Theorem 7.20.]. □

Corollary 3.4. Let n be a natural number that is not a perfect square. Then, the simple continued fraction of $\sqrt{d}$ is of the form:

$$\sqrt{d} = [b_0, \overline{b_1, b_2, \ldots, b_{n-1}, 2b_0}\,]$$

where $b_1, b_2, \ldots, b_{n-1}$ is a palindromic expression, that is, $b_k = b_{n-k}$ for all $k \leq n$.

Proof. Let us consider $\xi = b_0 + \sqrt{d} = \lfloor\sqrt{d}\rfloor + \sqrt{d}$, so the first term of the simple continued fraction expansion of ξ is $2b_0$. Then, it is easy to see that $-1 < \overline{\xi} < 0$, so, by theorem 3.11, ξ is a purely periodic simple continued fraction $\xi = [\overline{2b_0, b_1, \ldots, b_{n-1}}]$ and so, $\sqrt{d} = [b_0, \overline{b_1, \ldots, b_{n-1}, 2b_0}]$.

Furthermore, $\overline{\xi} = \lfloor\sqrt{d}\rfloor - \sqrt{d} = -\frac{1}{\mu}$ where μ is the valued of the continued fraction that you get when you reverse the period, so $\sqrt{d} - \lfloor\sqrt{d}\rfloor = \frac{1}{\mu} = [0, \overline{b_{n-1}, \ldots, 2b_0}]$.

Thus, $\sqrt{d} = \sqrt{d} - \lfloor\sqrt{d}\rfloor + \lfloor\sqrt{d}\rfloor = [b_0, \overline{b_{n-1}, \ldots, 2b_0}]$ and comparing this expression with the one we had obtained before, we finish. □

3.5 Pell's equation

This is the name given to the Diophantine equation

$$x^2 - dy^2 = 1, \tag{3.18}$$

where d is an integer and not a perfect square[1].

There are several references to this equation scattered throughout the history of number theory. Interestingly enough, even though mathematicians such as Archimedes, Fermat and Euler studied this equation, the one after whom the equation is named, Pell, did not. In the 17^{th} century, Euler mistakenly credited Pell with some results involving this equation that were due to Lord Brouncker. Despite Euler's mistake, Pell's name remains associated to this equation ever since then [3, Section 2.4.].

There is an interesting connection between the solutions of this equation and the simple continued fraction expansion of $\sqrt{d}$ that we will know see. Let us begin by considering the more general equation

$$x^2 - dy^2 = l \quad \text{where} \quad 0 < |l| < \sqrt{d}.$$

[1]Otherwise, $x^2 - dy^2 = (x - \sqrt{d}y)(x + \sqrt{d}y) = 1$ and the only possible solutions would be the trivial ones $(x, y) = (1, 0)$ and $(x, y) = (-1, 0)$

3.5.1 Solutions of Pell's equation by continued fractions

This equation is closely related to the continued fraction expansion of $\sqrt{d}$ as we will now see:

Lemma 3.3. Let $\xi = \sqrt{d}$. Then, for all $n \in \mathbb{N}$, the ξ_n take the form

$$\xi_n = \frac{\sqrt{d} + P_n}{Q_n} \tag{3.19}$$

where P_n and Q_n are integers such that $P_n^2 \equiv d \mod Q_n$.

Proof. We will prove it by induction on n. For the case $n = 1$, we have that

$$\xi_1 = \frac{1}{\sqrt{d} - \lfloor\sqrt{d}\rfloor} = \frac{\sqrt{d} + \lfloor\sqrt{d}\rfloor}{d - \lfloor\sqrt{d}\rfloor^2},$$

so the result holds for $p_1 = \lfloor\sqrt{d}\rfloor$ and $Q_1 = d - \lfloor\sqrt{d}\rfloor^2$.

Let us assume that the induction hypothesis is true. As $\xi_{n-1} = b_{n-1} + \frac{1}{\xi_n}$, in order to prove the result, we have to find $P_n, Q_n \in \mathbb{N}$ such that

$$\frac{\sqrt{d} + P_{n-1}}{Q_{n-1}} = b_{n-1} + \frac{Q_n}{\sqrt{d} + P_n}.$$

This is equivalent to the system of equations:

$$\begin{cases} d + P_{n-1}P_n = b_{n-1}Q_{n-1}P_n + Q_{n-1}Q_n \\ P_{n-1} + P_n = b_{n-1}Q_{n-1} \end{cases} \xLeftrightarrow[(2')=(2)]{(1')=(1)-P_n(2)} \begin{cases} d - P_n^2 = Q_{n-1}Q_n \\ P_{n-1} + P_n = b_{n-1}Q_{n-1} \end{cases}$$

The system on the right can easily be solved by first finding P_n from the second equation and then using it to find Q_n in the first equation. As this first equation implies that $P_n^2 \equiv d \mod Q_n$, we finish. □

Now we can give the main result:

Theorem 3.12. *The equation $x^2 - dy^2 = (-1)^n Q_n$ is always soluble.*

Proof. We know from lemma 3.1 that

$$\sqrt{d} = \frac{\xi_n p_{n-1} + p_{n-2}}{\xi_n q_{n-1} + q_{n-2}} \overset{(3.19)}{=} \frac{(\sqrt{d} + P_n)p_{n-1} + Q_n p_{n-2}}{(\sqrt{d} + P_n)q_{n-1} + Q_n q_{n-2}} = \frac{p_{n-1}\sqrt{d} + (P_n p_{n-1} + Q_n p_{n-2})}{q_{n-1}\sqrt{d} + (P_n q_{n-1} + Q_n q_{n-2})},$$

so,

$$1 = \frac{p_{n-1}\sqrt{d} + (P_n p_{n-1} + Q_n p_{n-2})}{(P_n q_{n-1} + Q_n q_{n-2})\sqrt{d} + d q_{n-1}}.$$

As $\sqrt{d}$ is an irrational number, we know that the following equations must hold:

$$p_{n-1} = P_n q_{n-1} + Q_n q_{n-2}, \qquad d q_{n-1} = P_n p_{n-1} + Q_n p_{n-2}.$$

Multiplying by p_{n-1} the first equation and subtracting q_{n-1} times the second one, we get:

$$p_{n-1}^2 - d q_{n-1}^2 = Q_n(q_{n-2}p_{n-1} - p_{n-2}q_{n-1}) \overset{(3.2)}{=} (-1)^{n-2} Q_n = (-1)^n Q_n.$$ □

Remark 3.11. From Keng's book, it can be deduced that if $l \neq (-1)^n Q_n m^2$ where $m \in \mathbb{Z}$ and $|l| < \sqrt{d}$, then the equation $x^2 - dy^2 = l$ has no solution. This is a consequence[2] of the following result [22, Theorem 7.3.] for the case $\xi = \sqrt{d}$:

Proposition 3.5. If $\frac{p}{q}$ satisfies that $|p^2 - \xi^2 q^2| < \xi$, then $\frac{p}{q}$ is an approximant of ξ.

Furthermore, in Keng's book a method is described to transform $x^2 - dy^2 = l$ for any $l \in \mathbb{N}^+$ into a Pell's equation where $|l| < \sqrt{d}$ [22, Section 11.5.].

Remark 3.12. The table C.5 contains the possible values of $(-1)^n Q_n$ for $d \leq 80$.

Corollary 3.5. Let m be the period of the continued fraction expansion of $\sqrt{d}$. Let $n > 1$ and $p_{n-1}^2 - dq_{n-1}^2 = (-1)^n Q_n$. Then, $p_{n-1+km}^2 - dq_{n-1+km}^2 = (-1)^{n+km} Q_{n+km}$ for all $k \in \mathbb{N}$.

Proof. This follows from the previous theorem and the fact that $\xi_n = \xi_{n+km}$ so,

$$\frac{\sqrt{d} + P_{n-1}}{Q_{n-1}} = \frac{\sqrt{d} + P_{n-1+km}}{Q_{n-1+km}}. \qquad \square$$

Let us now focus on the original Pell's equation (3.18).

Theorem 3.13. *Pell's equation $x^2 - dy^2 = 1$ has a non-trivial solution for every integer d that is not a perfect square. Furthermore, if n is the smallest integer such that $(-1)^n Q_n = 1$, all the solutions of $x^2 - dy^2 = 1$ are given by*

$$x + \sqrt{d}y = \pm(p_{n-1} + \sqrt{d}q_{n-1})^s, \quad \text{with } s \in \mathbb{Z}. \tag{3.20}$$

Proof. Let $l = (-1)^n Q_n$ for some $n \in \mathbb{N}^+$. Then, by corollary 3.5, the equation $x^2 - dy^2 = l$ has infinitely many solutions. If we consider the classes of $x \mod |l|$ and $y \mod |l|$, we can partition this set of solutions into l^2 classes, in which there must be at least one class with at least two solutions. Therefore, there exist (x_1, y_1), (x_2, y_2) such that

$$x_1^2 - dy_1^2 = x_2 - dy_2^2 = l, \qquad x_1 \equiv x_2 \mod |l|, \qquad y_1 \equiv y_2 \mod |l|.$$

It is easy to see that if we set

$$x = \frac{x_1x_2 - dy_1y_2}{l}, \qquad y = \frac{x_1y_2 - y_1x_2}{l},$$

these are both **integers**, $(x, y) \neq (\pm 1, 0)$ and

$$x^2 - dy^2 = \left(\frac{x_1x_2 - dy_1y_2}{l}\right)^2 - d\left(\frac{x_1y_2 - y_1x_2}{l}\right)^2 = \frac{(x_1^2 - dy_1^2)(x_2 - dy_2^2)}{l^2} = 1.$$

From proposition 3.5 we infer that $\frac{x}{y} = \frac{p_{n-1}}{q_{n-1}}$ is an approximant of $\sqrt{d}$, so $1 = (-1)^k Q_k$ for some $k \in \mathbb{N}$. Let us now prove that for the smallest $n \in \mathbb{N}$ such that $1 = (-1)^k Q_k$, we generate all solutions of the equation.

[2] The m in the condition comes from the fact that if $x^2 - dy^2 = l$ has a solution, a trivial solution of $x^2 - dy^2 = lm^2$ is (mx, my).

Let $\epsilon = p_n + \sqrt{d}q_n > 1$. As $\pm(p_n + \sqrt{d}q_n)^{-1} = \pm(p_n - \sqrt{d}q_n)$, it suffices to prove that all positive solutions of $x^2 - dy^2 = 1$ are given by $x + \sqrt{d}y = \epsilon^s$ (where $s > 0$). Let (x, y) be one of those positive solutions, so $x + \sqrt{y} > 1$. Then, for some $s > 0$:

$$\epsilon^s \leq x + \sqrt{d}y < \epsilon^{s+1} \qquad \Rightarrow \qquad 1 \leq \epsilon^{-s}(x + \sqrt{d}y) < \epsilon.$$

It is easy to see that $\epsilon^{-s}(x + \sqrt{d}y) = x_0 + \sqrt{d}y_0$ for some integers (x_0, y_0) and that $x_0^2 - dy_0^2 = 1$, as $(x_0 + \sqrt{d}y_0)^{-1} = \epsilon^s(x - \sqrt{d}y) = x_0 - dy_0$. To finish the proof, it suffices to prove that $(x_0, y_0) = (1, 0)$, as this would imply that $x + \sqrt{d}y = \epsilon^s$ for some $s > 0$. Let us suppose that $1 < x_0 + \sqrt{d}y_0 < \epsilon$. Then, $0 < \epsilon^{-1} < x_0 + \sqrt{d}y_0 < 1$, so we have:

$$\begin{aligned} 2x_0 &= (x_0 + \sqrt{d}y_0) + (x_0 - \sqrt{d}y_0) > 1 + \epsilon^{-1} > 0 \\ 2\sqrt{d}y_0 &= (x_0 + \sqrt{d}y_0) - (x_0 - \sqrt{d}y_0) > 1 - 1 \quad > 0 \end{aligned}$$

We know that $1 < x_0 + \sqrt{d}y_0 < p_{n-1} + \sqrt{d}q_{n-1}$. As $x_0 = \sqrt{1 + dy_0^2}$ and we also know that $p_{n-1} = \sqrt{1 + dq_{n-1}^2}$, so we get that $1 < \sqrt{1 + dy_0^2} + \sqrt{d}y_0 < \sqrt{1 + dq_{n-1}^2} + \sqrt{d}q_{n-1}$.

The function $f_d(t) = \sqrt{1 + dt^2} + \sqrt{d}t$ is continuous and monotonically increasing for $t > 0$, so we infer that $0 < y_0 < q_{n-1}$. As $x_0^2 - dy_0^2 = p_{n-1}^2 - dq_{n-1}$, and $x_0 > 0$, we also get that $0 < x_0 < p_{n-1}$. By proposition 3.5, $\frac{x_0}{y_0}$ must be a convergent of $\sqrt{d}$, and we arrive at a contradiction as we had taken the smallest n such that $p_{n-1} + dq_{n-1} = 1$, but the denominator of the convergent y_0 is less than q_{n-1}. □

Remark 3.13. In the table C.6, I have displayed the smallest solutions of $x^2 - dy^2 = 1$ for $d < 132$ with the help of the function A.4.6. It is very interesting to see how different values of d give solutions of such difference of scale. This same phenomenon can be appreciated in much more detail in figure B.3, where the y of the smallest solution have been plotted for every $d < 10000$. We can observe how large the size of the solution is compared to the value of d, to the point that for the largest y the size is comparable to the function $100^{\sqrt{d}}$.

This gives us some perspective on the reason why mathematicians in the past were unable to solve Pell's equation for some values of d; for example, the smallest integer solution of $x^2 - 9959y^2 = 1$ has an order of magnitude of 10^{208}, making it impossible to find without the help of a theory as the one given by continued fractions, and a computer.

Remark 3.14. We have seen in the previous theorem that $x^2 - dy^2 = 1$ always has a solution. However, this is not the case for the equation $x^2 - dy^2 = -1$. It is easy to see that for $d = 3$, we have that $x^2 - 3y^2 \equiv x^2 + y^2 \equiv 3 \mod 4$, but x^2 and y^2 can only be equivalent to either 0 or $1 \bmod 4$. Therefore, the equation $x^2 - 3y^2 = -1$ does not have a solution.

In fact, it can be proven that a sufficient and necessary condition for $x^2 - dy^2 = -1$ to have solutions is that the continued fraction expansion of $\sqrt{d}$ has an odd period. As a downside, we do not know any infallible criterion yet to determine if the square root of a number has an odd period without computing its continued fraction expansion [39, Exercise 6.7.5.].

3.5.2 Connections of Pell's equation with the theory of quadratic fields

Pell's equation appears naturally in a multitude of problems of Diophantine analysis, out of which perhaps one of the first was **Archimedes' cattle problem**, a problem from 251 B.C. which involves computing the number of cattle in a herd from a given set of restrictions [41].

But, what really made this equation relevant in number theory were its applications to the theory of quadratic fields and, more specifically, to the study of the fundamental units of the ring of integers of those fields.

Definition 3.7. *A* ***quadratic field*** *is a field of the form:*

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}$$

where $d \in \mathbb{Z}$ is not a perfect square. When $d < 0$, $\mathbb{Q}(\sqrt{d})$ is called ***imaginary*** *and when $d > 0$, $\mathbb{Q}(\sqrt{d})$ is said to be* ***real****. As $\mathbb{Q}(\sqrt{d}) \cong \mathbb{Q}(\sqrt{d'})$, if and only if $d = k^2 d'$ for some $k \in \mathbb{Z}$, we often assume that d is* ***square-free****, that is, that d is equal to the product of different primes.*

Definition 3.8. *Let $\alpha \in \mathbb{Q}(\sqrt{d})$. The* ***conjugate*** *of $\alpha = a + b\sqrt{d}$ is defined as $\overline{\alpha} = a - b\sqrt{d}$, its trace is defined as $\operatorname{Tr}\alpha = \alpha + \overline{\alpha} = 2a$ and its* ***norm*** *is defined as $\operatorname{N}\alpha = \alpha\overline{\alpha} = a^2 - db^2$.*

Definition 3.9. *The* ***ring of integers*** *of $\mathbb{Q}(\sqrt{d})$ is*

$$\begin{aligned}\mathcal{O}_d &= \{\alpha \in \mathbb{Q}(\sqrt{d}) : \alpha^2 - t\alpha + n = 0 \text{ for some } t, n \in \mathbb{Z}\} \\ &= \{\alpha \in \mathbb{Q}(\sqrt{d}) : \operatorname{Tr}\alpha, \operatorname{N}\alpha \in \mathbb{Z}\}\end{aligned}$$

Proposition 3.6. The norm has the following properties:

1. $\operatorname{N}(\alpha\beta) = \operatorname{N}(\alpha)\operatorname{N}(\beta)$ for all $\alpha, \beta \in \mathbb{Q}(\sqrt{d})$.
2. An element $\epsilon \in \mathcal{O}_d$ is a unit of the ring of integers if and only if $N(\epsilon) = \pm 1$.

Proof. For the first one, if $\alpha = a_1 + a_2\sqrt{d}$ and $\beta = b_1 + b_2\sqrt{d}$, we therefore have that $\alpha\beta = (a_1b_1 + da_2b_2) + (a_2b_1 + a_1b_2)\sqrt{d}$, so

$$N(\alpha\beta) = (a_1b_1 + da_2b_2)^2 - d(a_2b_1 + a_1b_2)^2 = (a_1^2 + da_2^2)(b_1^2 + db_2^2) = N(\alpha)N(\beta)$$

The second one can easily be deduced from the fact that by 1, N is a group homomorphism between $\mathcal{O}_d$ and $\mathbb{Z}$ (with respect to the product), and so, $N(\alpha^{-1}) = N(\alpha)^{-1}$. Thus, N maps the units of $\mathcal{O}_d$ to the units of $\mathbb{Z}$, so $N(\epsilon) = \pm 1$ for all units of $\mathcal{O}_d$. □

Theorem 3.14. *Let $d \in \mathbb{Z}$ be square-free.*
The ring of integers in $\mathbb{Q}(\sqrt{d})$ is $\mathcal{O}_d = \mathbb{Z}[\delta_0] = \mathbb{Z} + \delta_0\mathbb{Z}$ where δ_0 is:

$$\delta_0 = \begin{cases} \sqrt{d} & \text{for } d \equiv 2, 3 \mod 4 \\ \frac{1+\sqrt{d}}{2} & \text{for } d \equiv 1 \mod 4 \end{cases} \tag{3.21}$$

Proof. It can be found in Trifkovic's book [39, Theorem 4.2.2.]. □

Definition 3.10. *The **fundamental unit** of $\mathcal{O}_d$ is a unit $\epsilon_d \in \mathcal{O}_d$ satisfying the following conditions:*

- *For any unit $\epsilon \in \mathcal{O}_d$, $\epsilon = \pm\epsilon_d^k$ for some $k \in \mathbb{Z}$.*
- $\epsilon_d > 1$.

Theorem 3.15. *The fundamental unit of $\mathcal{O}_d$ can be computed by the means of the simple continued fraction expansion of $\sqrt{d}$.*

Proof. I will only prove the case where $d \equiv 2, 3 \mod 4$. However, it is worth noting that for the case $d \equiv 1 \bmod 4$, the way of computing the fundamental unit is also by computing a simple continued fraction expansion: instead of computing the expansion of $\sqrt{d}$, it is necessary to compute the expansion of δ_0. This method is described in Trifkovic's book [39, Theorem 6.7.4.] and it proves the general case of how the simple continued fraction expansion of a quadratic irrational δ may be used to compute the fundamental unit of a ring of the form $\mathbb{Z}[\delta]$.

Let us now proceed to prove the case where $d \equiv 2, 3 \mod 4$. By proposition 3.6, we know that $N(\epsilon_d) = \pm 1$ so, if $\epsilon_d = x + y\sqrt{d}$, then $x, y \in \mathbb{Z}$ must be a solution of one of these Pell's equations:

$$x^2 - dy^2 = -1 \qquad\qquad x^2 - dy^2 = 1$$

Let us suppose that for this d there are no solutions to the first equation. Then, all units must have norm 1 and, by theorem 3.13, if n is the smallest integer such that $(-1)Q_n = 1$, then $\epsilon_d = p_{n-1} + \sqrt{d}q_{n-1}$ is the fundamental unit of $\mathcal{O}_d$.

Now let us suppose that there exists $x_0, y_0 \in \mathbb{N}$ such that $N(\epsilon_0) = N(x_0 + \sqrt{d}y_0) = -1$. By theorem 3.13, once again, if n is the smallest integer such that $(-1)Q_n = 1$, then every unit of norm 1 is $\pm\epsilon_1^k$ with $k \in \mathbb{N}$, $\epsilon_1 = p_{n-1} + \sqrt{d}q_{n-1}$. Let $m \in \mathbb{N}$ such that $\epsilon_1^m < \epsilon_0 < \epsilon_1^{m+1}$ and let $\epsilon_d = \epsilon_0\epsilon_1^{-m}$. We claim that ϵ_d is the fundamental unit of $\mathcal{O}_d$. It is easy to see that $\epsilon_d > 1$ and $N(\epsilon_d) = -1$, so the only thing left to prove is that for any unit ϵ, $\epsilon = \pm\epsilon_d^k$ for some $k \in \mathbb{Z}$.

It is clear that $\epsilon_d^2 = \epsilon_1$, because $N(\epsilon_d^2) = 1$ and $1 < \epsilon_d^2 < \epsilon_1^2$. Without any loss of generality, let $\epsilon > 0$ a unit. If $N(\epsilon) = 1$, $\epsilon = \epsilon_1^k = \epsilon_d^{2k}$ for some $k \in \mathbb{Z}$ and if $N(\epsilon) = -1$, $N(\epsilon\,\epsilon_d) = 1$ and so, $\epsilon = \epsilon_d^{2k-1}$ for some $k \in \mathbb{Z}$ as we wanted to prove.

Furthermore, by proposition 3.5, it is easy to see that if $\epsilon_d = x + \sqrt{d}y$, then x/y has to be an approximant of $\sqrt{d}$ and, in fact, it is easy to prove that $(x, y) = (p_{n-1}, q_{n-1})$ where $n \in \mathbb{N}^+$ is the smallest such that $(-1)^n Q_n = -1$. □

Remark 3.15. In a similar fashion to remark 3.13, I have displayed in the table C.7 the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for the first 60 non-square integers. Once again, different values of d give completely different solutions in terms of size. However, in figure B.4, we can see that the size of the fundamental units is much smaller than the size of the smallest solution of $x^2 - dy^2 = 1$, as the first is comparable to $10^{\sqrt{d}}$. This is because fundamental units do not necessarily have to be solutions of $x^2 - dy^2 = 1$, they can also be solutions of $x^2 - dy^2 = -1$ and also, if $d \equiv 1 \bmod 4$, the x and y can be half integers.

3.6 Open problems and lines of research

To close this dissertation, I will address some of the topics of research in number theory in which continued fractions are involved:

The efforts to find general expressions of the continued fractions of famous constants have not worn down. The focus nowadays [23, 24] is mainly on:

- **Hurwitz continued fractions**, which are periodic continued fraction of the kind:

  $$[b_0, \dots, b_{r-1}, \overline{f_1(k), \dots, f_s(k)}]_{k=h}^{\infty} = [b_0, \dots, f_1(h), \dots, f_s(h), f_1(h{+}1), \dots, f_s(h{+}1), \dots]$$

  for certain $r, s, h \in \mathbb{N}$, where $f_1(k), \dots, f_s(k)$ are polynomials in k with rational coefficients. The continued fractions of $\frac{e^2+1}{e^2-1}$, $\frac{e+1}{e-1}$ and e are of this kind.

- **Tasoev continued fractions**, which are periodic continued fraction of the kind:

  $$[b_0, \dots, b_{r-1}, \overline{g_1(k), \dots, g_s(k)}]_{k=h}^{\infty} = [b_0, \dots, g_1(h), \dots, g_s(h), g_1(h{+}1), \dots, g_s(h{+}1), \dots]$$

  for certain $r, s, h \in \mathbb{N}$, where $g_1(k), \dots, g_s(k)$ are exponentials in k with rational coefficients.

An ongoing problem in number theory is the search for bounds for the **irrationality measure** of famous constants [5]. Since Apèry used the fact that $\mu(\zeta(3)) > 1$, to prove the irrationality of the constant named after him [40], multiple papers have been written on that topic. One of the latest results has been a prominent paper by Zeilberger and Zudilin that claims that the irrationality measure of π is at most $7.103205334137\dots$ [45]. In particular, the study of the irrationality measure of π is linked to many other open problems, such as the convergence of the **generalised Flint Hill series** $\mathcal{S}_{u,v} = \sum_{n=1}^{\infty} \frac{1}{n^u |\sin(n)|^v}$, which has been proven to converge if $\mu(\pi) < 1 + \frac{u}{v}$ [1].

Another field of research are the simple continued fractions with coefficients in $\mathbb{F}[x]$. With a few changes, some of the theorems that we have proven in this chapter can be adapted to this new setup in order to study topics such as **purely periodic polynomial continued fractions**, **rational approximation** in $\mathbb{F}[x]$ and **Pell's polynomial equation** [2, 26, 29].

As for the Lagrange spectrum, there are two main lines of research:
On the one hand, there is the study of the relation between the **Lagrange and Markov spectra**, which is not fully known yet. The last articles on the topic indicate that in order to crack this problem, techniques from both **dynamical systems** and **geometry** are needed [30].
On the other hand, mathematicians are becoming more and more interested in studying the **structure of the Lagrange spectrum** for simple continued fractions with coefficients **in other rings**. Regarding this topic, I refer to Asmus L. Schmidt, who has done significant work on computing the limit points of the Lagrange spectrum in fields such as $\mathbb{Q}(i\sqrt{2})$ [36] or $\mathbb{Q}(i\sqrt{11})$ [35].

Appendices

A | Code

A.1 Transformations of continued fractions

A.1.1 Continued fraction from its convergents

```
FromConvergents::usage="FromConvergents[n_,listP_,listQ_] takes as
   parameters two lists of length at least n+1 and computes the lists of
    coefficients of the continued fraction whose partial numerators and
   denominators are the arguments of the function. If the lists P and Q
   satisfy P[[i-1]]Q[[i]]-P[[i]]Q[[i-1]] for some i<n+2, it will return
   ComplexInfinity as part of the lists of coefficients."

FromConvergents[n_,listP_,listQ_]:=
Module[{P=listP, Q=listQ,A={-listP[[1]]listQ[[2]]+listP[[2]]listQ[[1]]},
   B={listP[[1]],listQ[[2]]}},(*We initialize the variables*)
If[Length[P]<(n+1)||Length[Q]<(n+1),Message[WrongLength::error,n+1,n+1];
   Return[]]; (*Error message*)
A=Join[A,Table[-(P[[i]]Q[[i+1]]-P[[i+1]]Q[[i]])/(P[[i-1]]Q[[i]]
   -P[[i]]Q[[i-1]]),{i,2,n}]]; (*We apply the formulas*)
B=Join[B,Table[(P[[i-1]]Q[[i+1]]-P[[i+1]]Q[[i-1]])/(P[[i-1]]Q[[i]]
   -P[[i]]Q[[i-1]]),{i,2,n}]];
 {A,B} (*We return lists A and B*)]


FromConvergentsK::usage="FromConvergentsK[P_,Q_] takes as parameters two
    pure functions and it returns two functions that describe the
   coefficients of the continued fraction whose partial numerators and
   denominators are P and Q."

FromConvergentsK[P_,Q_]:={Function[n,If[n==1,-P[0] Q[1]+P[1]Q[0],
   -(P[n-1]Q[n]-P[n]Q[n-1])/(P[n-2]Q[n-1]-P[n-1]Q[n-2])]],
Function[n,If[n==1,Q[1],(P[n-2]Q[n]-P[n]Q[n-2])/(P[n-2]Q[n-1]
   -P[n-1]Q[n-2])]]}(*The definition of pure functions inside a module
      is done through the Function command*)
```

A.1.2 Transformation to continued fraction with partial denominator 1

```
OnesDown::usage="OnesDown[n_,listA_,listB_] receives as parameters two
   lists of length n and n+1 with the coefficients of a continued
   fraction and returns the partial numerators of the equivalent
   continued fraction such that all the partial denominators are 1.";

OnesDown[n_,listA_,listB_]:=
Module[{A=listA, B=listB,C={listA[[1]]/listB[[2]]}},
If[Length[A]<n||Length[B]<(n+1),Message[WrongLength::error,n,n+1];
   Return[]];(*Error message*)
C=Join[C,Table[A[[i]]/(B[[i]]B[[i+1]]),{i,2,n}]]] (*This is the
   transform*)

OnesDownK::usage="OnesDownK[A_,B_] receives as parameters two pure
   functions that describe the coefficients of a continued fraction and
   returns a pure function with the partial numerators of the equivalent
    continued fraction such that all the partial denominators are 1.";

OnesDownK[A_,B_]:=Function[n,If[n==1,A[n]/B[n],A[n]/(B [n-1]B[n])]]
```

A.1.3 Transformation to continued fraction with partial numerator 1

```
OnesUp::usage="OnesUp[n_,listA_,listB_] receives as parameters two lists
    of length n and n+1 with the coefficients of a continued fraction
   and returns the partial denominators of the equivalent continued
   fraction such that all the partial numerators are 1.";

OnesUp[n_,listA_,listB_]:=
Module[{A=listA, B=listB, Z={1,1/listA[[1]]},D={listB[[1]]}},
If[Length[A]<n||Length[B]<(n+1),Message[WrongLength::error,n,n+1];
   Return[]];(*Error message*)
Z=Join[Z,Table[If[EvenQ[i],Product[A[[2j-1]]/A[[2j]],{j,1,i/2}],
   Product[A[[2j]]/A[[2j+1]],{j,1,(i-1)/2}]/A[[1]]],{i,2,n}]];
D=Join[D,Table[Z[[i+1]]B[[i+1]],{i,1,n}]] (*To define this, we treat
   differently the case when i is odd from when it is even*)]

OnesUpK::usage="OnesUpK[A_,B_] receives as parameters two pure functions
    that describe the coefficients of a continued fraction and returns a
    pure function with the partial denominators of the equivalent
   continued fraction such that all the partial numerators are 1.";

OnesUpK[A_,B_]:=Function[n,B[n]Product[A[k]^(-1)^(n-k+1),{k,1,n}]]
```

A.1.4 Transformation from sum to continued fraction

```
SumToCF::usage="SumToCF[F_] takes the pure function F that determines
   the general term of a series and returns the expression of the
   coefficients of the continued fraction whose convergents are the
   partial sums of that series, via Euler's identity."

SumToCF[F_]:={Function[n,If[n==1,F[1],-F[n]/F[n-1]]],
Function[n,If[n==1,1,1+F[n]/F[n-1]]]}
```

A.1.5 Canonical even part of a continued fraction

```
CanonicalEvenPart::usage="CanonicalEvenPart[n_,listA_,listB_] takes as
   parameters two lists of length at least 2n and 2n+1 that are the
   coefficients of a continued fraction and computes the lists of the
   first n coefficients of its canonical even part.";

CanonicalEvenPart[n_,listA_,listB_]:=
Module[{A=listA,B=listB,C={listA[[1]]listB[[3]]},
  D={listB[[1]],listA[[2]]+listB[[2]]listB[[3]]}},(*We initialize the
     lists C and D*)
If[Length[A]<2n||Length[B]<(2n+1),Message[WrongLength::error,2n,2n+1];
   Return[]];(*Error message*)
C=Join[C,Table[-A[[2i-2]]A[[2i-1]]B[[2i+1]]/B[[2i-1]],{i,2,n}]];
D=Join[D,Table[A[[2i]]+B[[2i]]B[[2i+1]]+A[[2i-1]]B[[2i+1]]/B[[2i-1]],
   {i,2,n}]];{C,D}]

CanonicalEvenPartK::usage="CanonicalEvenPartK[A_,B_] takes as parameters
    two pure functions that are the coefficients of a continued fraction
    and returns the coefficients of its canonical even part (as pure
   functions).";

CanonicalEvenPartK[A_,B_]:={Function[n,If[n==1,A[1]B[2],
   (A[2n-2]A[2n-1]B[2n])/B[2n-2]]],
Function[n,If[n==1,A[2]+B[1]B[2],A[2n]+B[2n-1]B[2n]+
   (A[2n-1]B[2n])/B[2n-2]]]}
```

A.1.6 Canonical odd part of a continued fraction

```
CanonicalOddPart::usage="CanonicalOddPart[n_,listA_,listB_] takes as
    parameters two lists of length at least 2n+1 and 2n+2 that are the
    coefficients of a continued fraction and computes the lists of the
    first n coefficients of its canonical odd part.";

CanonicalOddPart[n_,listA_,listB_]:=(*You need lists of length 2n+1 and
    2n+2*)
Module[{A=listA,B=listB,C={-listA[[1]]listA[[2]]listB[[4]]/listB[[2]],
   -listA[[3]]listA[[4]]listB[[6]]listB[[2]]/listB[[4]]},
   D={listB[[1]]+listA[[1]]/listB[[2]],listA[[3]]listB[[2]]+
   listB[[2]]listB[[3]]listB[[4]]+listA[[2]]listB[[4]]}},
   (*We initialize the lists C and D*)
If[Length[A]<(2n+1)||Length[B]<(2n+2),Message[WrongLength::error,2n+1,
   2n+2];Return[]];(*Error message*)
C=Join[C,Table[-A[[2i-1]]A[[2i]]B[[2i+2]]/B[[2i]],{i,3,n}]];
D=Join[D,Table[A[[2i+1]]+B[[2i+1]]B[[2i+2]]+A[[2i]]B[[2i+2]]/B[[2i]],
   {i,2,n}]];{C,D}]

CanonicalOddPartK::usage="CanonicalOddPartK[A_,B_] takes as parameters
    two pure functions that are the coefficients of a continued fraction
    and returns the coefficients of its canonical odd part (as pure
    functions).";

CanonicalOddPartK[A_,B_]:={Function[n,If[n==2,-A[3]A[4]B[1]B[5]/B[3],
   -A[2n-1]A[2n]B[2n+1]/B[2n-1]]],
Function[n,If[n==0,B[0]+A[1]/B[1],If[n==1,A[3]B[1]+B[1]B[2]B[3]
   +A[2]B[3],A[2n+1]+B[2n]B[2n+1]+A[2n]B[2n+1]/B[2n-1]]]]}
```

A.1.7 Extension of a continued fraction by an element

```
Extend::usage="Extend[r_,p_,n_,listA_,listB_] takes two lists of length
    at least n and n+1 that are the coefficients of a continued fraction
    and returns the coefficients of the continued fraction whose
    approximants are the same until the position p, where the approximant
     is the element r, and after that the k+1-th approximant is the k-th
    of the original continued fraction.";

Extend[r_, p_,n_,listA_,listB_]:=Module[{A=listA,B=listB, ND=PartialND[p
    ,listA,listB],rho,C,D},
If[Length[A]<n||Length[B]<(n+1),Message[WrongLength::error,n,n+1];
   Return[]];(*First error message*)
If[p>n,Message[WrongPosition::error,p,n];Return[]]; (*Second error
   message*)
rho=(ND[[1,p+1]]-ND[[2,p+1]]*r)/(ND[[1,p]]-ND[[2,p]]*r);
C=Table[Which[i<=p,A[[i]],i==p+1,rho,i==p+2,-A[[p+1]]/rho,i>=p+3,
   A[[i-1]]],{i,1,n+1}];
D=Table[Which[i<=p,B[[i]],i==p+1,B[[p+1]]-rho,i==p+2,1,i==p+3,B[[p+2]]
   +A[[p+1]]/rho,i>=p+4,B[[i-1]]],{i,1,n+2}];
{C,D}]
```

A.1.8 Bauer-Muir transformation

```
BauerMuir::usage="BauerMuir[n_,listA_,listB_,listG_] takes as parameters
     three lists of lengths n, n+1 and n+1 with the coefficients of a
    continued fraction and the coefficients for the Bauer-Muir
    transformation and returns a list with the coefficients of the
    transformed continued fraction.";

BauerMuir[n_,listA_,listB_,listG_]:=
Module[{A=listA,B=listB,G=listG,L,C={listA[[1]]-listG[[1]](listB[[2]]
   +listG[[2]])},D={listB[[1]]+listG[[1]],listB[[2]]listG[[2]]}},(*We
      set the first and the first two elements of C and D respectively*)
If[Length[A]<n||Length[B]<(n+1)||Length[G]<(n+1),
   Message[WrongLength2::error,n,n+1,n+1];Return[]];(*Error message*)
L=Table[A[[i]]-G[[i]](B[[i+1]]+G[[i+1]]),{i,1,n}]; (*Creates a list with
     the lambdas*)
C=Join[C,Table[A[[i-1]]*L[[i]]/L[[i-1]],{i,2,n}]];
D=Join[D,Table[B[[i+1]]+G[[i+1]]-G[[i-1]]*L[[i]]/L[[i-1]],{i,2,n}]];(*
    Applies the definition of th Bauer-Muir transformation*)
{C,D}]

BauerMuirK::usage=" BauerMuirK[A_,B_,G_] takes as parameters three pure
    functions that describe the coefficients of a continued fraction and
    the coefficients for the Bauer-Muir transformation and returns a list
     with two pure functions that describe the coefficients of the
    transformed continued fraction.";

BauerMuirK[A_,B_,G_]:={Function[n,If[n==1,A[1]-G[0](B[1]+G[1]),
   A[n-1]*(A[n]-G[n-1](B[n]+G[n]))/(A[n-1]-G[n-2](B[n-1]+G[n-1]))]],
Function[n,If[n<2,B[n]+G[n],B[n]+G[n]-G[n-2]*(A[n]-G[n-1](B[n]+G[n]))
    *1/(A[n-1]-G[n-2](B[n-1]+G[n-1]))]]}(*It has the same implementation
    as the version for lists ith the exception of the indices of the
    functions that change due to the fact that in Mathematica the lists
    begin with 1 and our coefficients begin with 0*)
```

A.2 Evaluation of continued fractions

A.2.1 Forward recurrence algorithm

```
ForwardRecurrence::usage="ForwardRecurrence[n_, listA_, listB_] takes as
    parameters two lists of length at least n and n+1 and computes the n
   -th approximant via the forward recurrence method."

ForwardRecurrence[n_,listA_,listB_]:=
Module[{A=Prepend[listA,1],B=listB,P={listB[[1]],listB[[2]]listB[[1]]
   +listA[[1]]},Q={1,listB[[2]]}},(*Initialization of variables*)
If[Length[A]<(n+1)||Length[B]<(n+1),Message[WrongLength::error,n,n+1];
    Return[]];(*Error message*)
For[i=3,i<=n+1,i++,AppendTo[P,B[[i]]P[[i-1]]+A[[i]]P[[i-2]]];
   AppendTo[Q,B[[i]]Q[[i-1]]+A[[i]]Q[[i-2]]]];(*Sequential updating of
      the lists of partial numerators and denominators*)
Last[P/Q](*Returns n-th approximant*)]
```

A.2.2 Euler-Minding algorithm

```
EulerMinding::usage="EulerMinding[n_, listA_, listB_] takes as
    parameters two lists of length at least n and n+1 and computes the n-
    th approximant via the Euler-Minding method."

EulerMinding[n_,listA_,listB_]:=
Module[{A=Prepend[listA,1],B=listB,Q={1,listB[[2]]},W={listB[[1]]},
   det=1},(*Initialization of variables*)
If[Length[A]<(n+1)||Length[B]<(n+1),Message[WrongLength::error,n,n+1];
    Return[]];(*Error message*)
For[i=3,i<=n+1,i++,AppendTo[Q,B[[i]]Q[[i-1]]+A[[i]]Q[[i-2]]]];(*We first
     create a list of the partial denominators*)
For[i=2,i<=n+1,i++,det=det*(-A[[i]]); AppendTo[W,W[[i-1]]
   -(det)/(Q[[i]]Q[[i-1]])]];(*From that, we compute the approximants
      using Euler Minding's formula*) Last[W]]
```

A.2.3 Backward recurrence algorithm

```
BackwardRecurrence::usage="BackwardRecurrence[n_, listA_, listB_] takes
    as parameters two lists of length at least n and n+1 and computes the
     n-th approximant via the backward recurrence method."

BackwardRecurrence[n_,listA_,listB_]:=
Module[{A=Prepend[listA,1],B=listB,L={0}},
If[Length[A]<(n+1)||Length[B]<(n+1),Message[WrongLength::error,n,n+1];
    Return[]];(*Error message*)
For[i=n+1,i>1,i--,L=A[[i]]/(B[[i]]+L)]; (*We create a list backwards by
    applying sequentially the iteration L*)
First[L+B[[1]]]]
```

A.3 Continued fractions in the complex numbers

A.3.1 Chordal distance

```
ChordalD::usage="ChordalD[z1_,z2_] takes as parameters two complex
   numbers and returns its distance according to the chordal metric.";

ChordalD[z1_,z2_]:=Which[z1==Infinity&&z2==Infinity,0,z1==Infinity,
   2/Sqrt[1+Norm[z2]^2],z2==Infinity,2/Sqrt[1+Norm[z1]^2],True,
   2Norm[z1-z2]/(Sqrt[1+Norm[z1]^2]*Sqrt[1+Norm[z2]^2])]
```

A.3.2 Convergence of the Stern-Stolz series of a continued fraction

```
SternStolz::usage="SternStolz[A_,B_] takes as parameters two pure
   functions that describe the coefficients of a continued fraction and
   returns the conditions under which the Stern-Stolz series of that
   continued fraction converges. True means the Stern-Stolz series
   converges so, the continued fraction diverges; whereas False means
   that the Stern-Stolz series diverge, and this may or may not imply
   the convergence of the continued fraction.";

SternStolz[A_,B_]:=Simplify[
   SumConvergence[Abs[B[2 n]Product[A[2k-1]/A[2k],{k,1,n}]],n]&&
   SumConvergence[Abs[B[2n+1]/A[1]Product[A[2k]/A[2k+1],{k,1,n}]],n]]
   (*For this function, we divide into even and odd parts to make the
      study of the convergence of the series easier for Mathematica.*)
```

A.3.3 Representation of a function as a regular C-fraction

```
CFraction::usage="CFraction[n_,Fx_,x_] takes a natural number n and a
   expression defining a function Fx of x and returns a list with the n
   first terms of the regular C-fraction equivalent to Fx. It will
   likely fail if it is not given a normal function."

CFraction[n_,Fx_,x_]:=Module[{k,pade1,pade2,pade,cf,a0=Fx/.{x->0}},
pade1=Table[PadeApproximant[Fx,{x,0,{k,k}}],{k,0,Ceiling[(n+1)/2]}];
   (*Computes the terms in the diagonal of the Pade table*)
If[a0==0,pade1[[1]]=0]; (*The function PadeApproximant fails to return
   the approximant (0,0) if this is zero, so we have implemented this
   case separadetly*)
pade2=Table[PadeApproximant[Fx,{x,0,{k+1,k}}],{k,0,Floor[(n+1)/2]}];
   (*Computes the terms under the diagonal*)
pade=Table[If[OddQ[k],pade1[[(k+1)/2]],pade2[[k/2]]],{k,1,n+1}]; (*Join
   both to form the staircase sequence*)
cf=Simplify[FromConvergents[n,Numerator[pade],Denominator[pade]]];
Prepend[OnesDown[n,cf[[1]],cf[[2]]],a0] (*Uses FromConvergents to get
   the continued fraction and OnesDown to get the partial denominators
   equal 1*)]
```

A.4 Simple continued fractions and number theory

A.4.1 Euclidean algorithm to represent simple continued fractions

```
EuclideanSCF::usage="EuclideanSCF[rat_] takes as argument a rational
    number (or a quotient between two Gaussian integers) and returns a
    list of the coefficients of the simple continued fraction expansion
    in the natural numbers (or in Gaussian integers) through the
    Euclidean algorithm. The reason why this algorithm works for Gaussian
     integers is that the functions Quotient and Mod are programmed to
    work with them"

EuclideanSCF[rat_]:=Module[{num=Numerator[Simplify[rat]],den=Denominator
    [Simplify[rat]],k,Q={}}, (*The following piece of code implements the
     Euclidean algorithm. The variable k is a dummy variable used to
    store and upgrade the values of num and den after every iteration*)
While[den!=0,
   AppendTo[Q,Quotient[num,den]];k=Mod[num,den];num=den;
   den=k]; Q (*This is the list with the coefficients*)]
```

A.4.2 Euclidean algorithm for simple continued fractions with polynomial coefficients

```
EuclideanSCFPolynomial::usage="EuclideanSCF[rat_,x_,mod_] takes as
    argument a rational function over x (a quotient of two polynomials
    over x) and returns a list of the coefficients of the simple
    continued fraction expansion (polynomials) through the Euclidean
    algorithm. The third argument is an optional one that allows us to
    work with polynomials with coefficients in Zn by adding the option:
    Modulus->n"

EuclideanSCFPolynomial[rat_,x_]:=Module[{num=Numerator[Simplify[rat]],
    den=Denominator[Simplify[rat]],r,Q={}},
While[Exponent[den,x]!=-Infinity,AppendTo[Q,
   PolynomialQuotient[num,den,x]];(*As polynomial=0 does not have a
      logical value, if we used den!=0 the code would not work, that is
      why we use Exponent[den,x]!=-Infinity, as 0 is the only polynomial
       with degree -Infinity (according to Mathematica)*)
r=PolynomialRemainder[num,den,x];num=den;den=r];
Q (*The rest of the implementation is the same as EuclideanSCF, with the
     only difference that PolynomialQuotient and PolynomialReminder are
    used instead of Quotient and Mod*)]

EuclideanSCFPolynomial[rat_,x_,mod_]:=Module[
   {num=Numerator[Simplify[rat]],den=Denominator[Simplify[rat]],r,Q={}},
While[Exponent[den,x]!=-Infinity,AppendTo[Q,
   PolynomialQuotient[num,den,x,mod]];
r=PolynomialRemainder[num,den,x,mod];num=den;den=r]; (*This includes the
     option to work in finite fields*)
Q]
```

A.4.3 Simple continued fraction algorithm

```
SCFAlgorithm[real_,n_]:=Module[{B={},r=real},(*This piece of code
   computes the complete quotients r and stores their integer parts in
   the list B*)
For[i=0,i<=n,i++,B=AppendTo[B,Floor[r]];r=1/(r-B[[i+1]])]; B]
```

A.4.4 Complete quotients of a quadratic irrational

```
CompleteQuotients::usage="CompleteQuotients[r_] takes a quadratic
   irrational as an argument and returns a finite list with all the
   possible complete quotients of r. The complete quotients that are
   found inside the inner brackets are those which repeat periodically.
   If it is given something different from a quadratic irrational, it
   will return a error message instead."

CompleteQuotients[r_]:=Module[{q,l,period,lperiod,list={},x,rep},
If[!QuadraticIrrationalQ[r],Message[NotAQuadraticIrrational::error];
   Return[]]; (*Error message*)
q=ContinuedFraction[r]; (*Stores the continued fraction expansion of r*)
period=Last[q]; (*Stores the periodic part of that expansion*)
l=Length[q]-1;
lperiod=Length[period];x=q;
For[i=1,i<=l-1,i++,AppendTo[list,Delete[x,1]];x=Last[list]]; (*This loop
    produces a list where the elements are the result of removing the
   first digits of the continued fraction expansion of r (this
   corresponds to the definition of complete quotients)*)
rep=Table[{RotateLeft[period,n]},{n,0,lperiod-1}]; (*This produces a
   list in which we find all the possible cyclic permutations of the
   periodic part of the expansion. This corresponds to the expansion of
   all the complete quotients that repeat periodically*)
list=FromContinuedFraction/@list;
rep=FromContinuedFraction/@rep; (*These past two lines generate the real
    numbers associated to the lists that we have created*)
Join[list,{rep}] (*The union of these lists (the second one inside
   brackets) give us our desired list as a result*)]
```

A.4.5 Lagrange constant of a quadratic irrationality

```
LagrangeConstant::usage= "LagrangeConstant[r_] takes a quadratic
   irrational as an argument and returns its Lagrange constant. If it is
    given something different from a quadratic irrational, it will
   return a error message instead."

LagrangeConstant[r_]:=Module[{q,period,rep,lperiod,repinv}, (*To compute
    this, we will use the formula that calculates it from the continued
   fraction expansion*)
If[!QuadraticIrrationalQ[r],Message[NotAQuadraticIrrational::error];
   Return[]]; (*Error message*)
q=ContinuedFraction[Abs[r]];(*Stores the continued fraction expansion of
    the absolute value of r. The absolute value is necessary for the
   function to be able to deal with negative numbers*)
period=Last[q]; (*Stores the periodic part of that expansion*)
lperiod=Length[period];
rep=Table[{RotateLeft[period,n]},{n,0,lperiod-1}]; (*This produces a
   list of the expansions of all complete quotients that repeat
   periodically [b_n,b_n+1,...]*)
repinv=Reverse[rep,3];
repinv=Prepend[0]/@repinv ; (*This reverses the previous list and adds a
    zero before it, so we get the possible limits when n->Infinity of
   [0,b_n,...,b_1,b_0]*)
rep=FromContinuedFraction/@rep;
repinv=FromContinuedFraction/@repinv;(*We transform the continued
   fractions into real numbers*)
Max[Simplify[rep+repinv]] (*The Lagrange constant is the greatest
   element in the list that results when we add the elements of the
   lists rep and repinv*)]
```

A.4.6 Smallest solution of $x^2 - dy^2 = 1$

```
SmallestPellSolution1::usage="SmallerPellSolution1[d] takes a non-square
    integer and returns the list {x,y}, where x and y are the smallest
   positive solutions of x^2-dy^2=1.";

SmallestPellSolution1[d_]:=Module[{rd,q,period,k=0,n=0,u},
If[IntegerQ[Sqrt[d]],Message[NonSquareInteger::error];Return[]]; (*Error
    message*)
q= ContinuedFraction[Sqrt[d]];
If[Floor[Sqrt[d]]^2-d==1,u={Floor[Sqrt[d]],1},period=Last[q]; (*If the
   0-th approximant is the solution, it stops*)
While[k!=1,k=(-1)^(n+1) Denominator[FromContinuedFraction[{period}]];
   period=RotateLeft[period,1];n=n+1];
u=NumeratorDenominator[FromContinuedFraction[ContinuedFraction[Sqrt[d],
  n]]]](*If not, it rotates the period until the (-1)^(n+1)Q_(n+1)=1
      and it returns the n-th partial numerator and n-th partial
      denominators which are the smallest solutions*)]
```

A.4.7 Values of $(-1)^n Q_n$ for $\sqrt{d}$

```
PellsPossibilities::usage="PellsPossibilities[d_] takes a non-square
   integer and returns a list of all possible values of the form (-1)^n
   Q_n for which Pell's equation x^2-dy^2=(-1)^n Q_n has integer
   solutions. If it is given something different than an integer that is
    not a perfect square, it will return a error message instead.";

PellsPossibilities[d_]:=Module[{sq=Sqrt[d],period,lperiod,rep,rep2},
If[IntegerQ[Sqrt[d]],Message[NonSquareInteger::error];Return[]]; (*Error
    message*)
period=Last[ContinuedFraction[sq]];
lperiod=Length[period];
rep=Table[{RotateLeft[period,i]},{i,0,lperiod-1}]; (*This produces a
   list of the expansions of all complete quotients of sqrt(d) that
   repeat periodically *)
rep=FromContinuedFraction/@rep; (*We transform the continued fractions
   into real numbers*)
rep2=Join[rep,rep];
Sort[DeleteDuplicates[Table[(-1)^k*Denominator[rep2[[k]]], {k,1,2lperiod
   }]]] (*As some continued fraction expansions have a periodic part of
   odd length, to compute all possibles (-1)^n Q_n, we need to iterate
   twice the length of the period. After that, we return the solutions
   sorted and without duplicates*)]
```

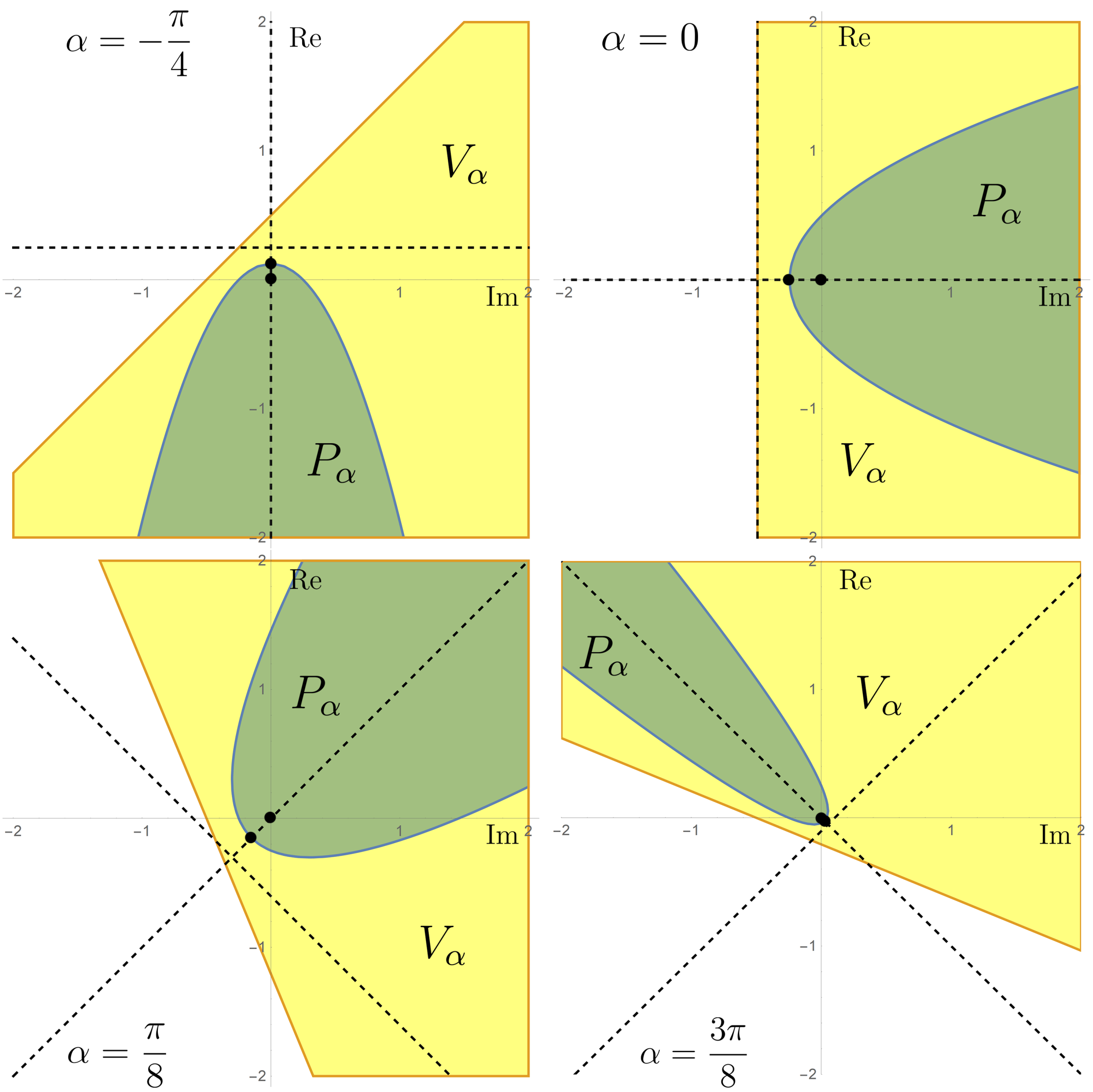
A.4.8 Graph of the *n*-th first solutions to Markov's equation

```
MarkovSolutions::usage="MarkovSolutions[n_] generates n triples of
   solutions to Markov's equation from the initial triple {1,2,5}.";

MarkovSolutions[n_]:=Module[{list={{1,2,5}}},For[i=2,i<=n,i++,
   AppendTo[list,If[EvenQ[i],{list[[Floor[i/2],1]],list[[Floor[i/2],3]],
   3*list[[Floor[i/2],1]]*list[[Floor[i/2],3]]-list[[Floor[i/2],2]],
   {list[[Floor[i/2],2]],list[[Floor[i/2],3]],3*list[[Floor[i/2],2]]*
   list[[Floor[i/2],3]]-list[[Floor[i/2],1]]}]]];list]

MarkovGraph::usage="MarkovGraph[n_,fontColor_,fontSize_,edgeColor_,
   ratio_] generates a graph with n triples of solutions to Markov's
   equation with the set colour of font for the numbers, that size of
   font, that color of edges and that aspect ratio";

WhitePanel[label_]:=Panel[label,Background->White,FrameMargins->-2]
(*This is an auxiliary function to add a white panel under the numbers
   of the graph so it looks better*)
```

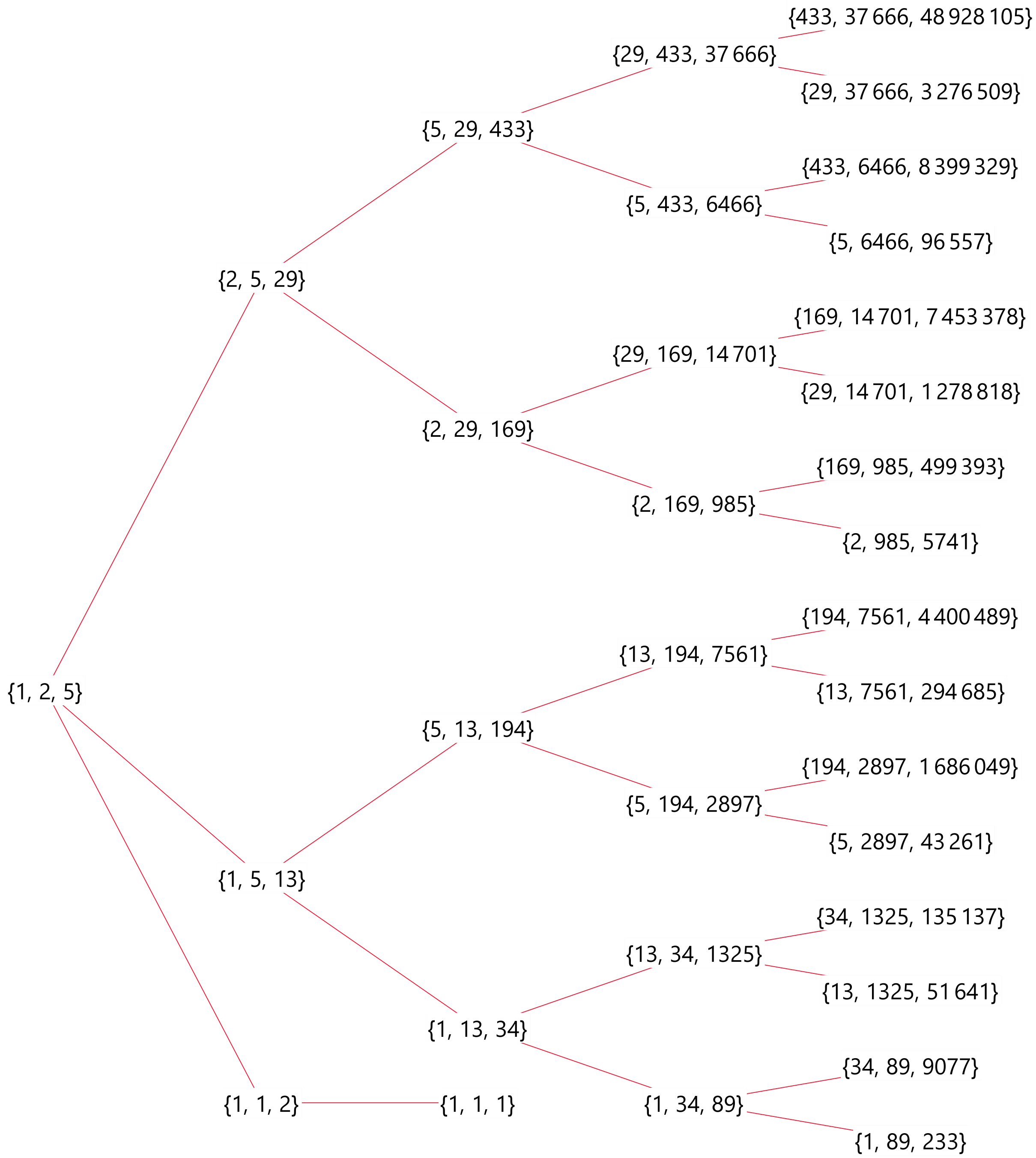
```
MarkovGraph[n_,fontColor_,fontSize_,edgeColor_,ratio_]:=Module[
   {M=MarkovSolutions[n-2],G},(*We write MarkovSolutions[n-2], because
      the two solutions {1,1,1},{1,1,2} will be added later*)
If[n<3,Message[MarkovGraphError::error];Return[]];
G=Graph[Join[{{1,1,1}\[UndirectedEdge]{1,1,2},
   {1,1,2}\[UndirectedEdge]{1,2,5}},
   Table[M[[i]]\[UndirectedEdge]M[[2i]],{i,1,Floor[(n-2)/2]}],
   Table[M[[i]]\[UndirectedEdge]M[[2i+1]],{i,1,Floor[(n-3)/2]}]],
   (*We create the graph by joining the triples*)
   VertexLabels->{"Name",Placed[Automatic,Center,WhitePanel]},
   VertexSize->Tiny,
   GraphLayout->{"LayeredEmbedding","Orientation"->Left},
   VertexLabelStyle->Directive[fontColor,fontSize],
   VertexStyle->White, EdgeStyle->edgeColor, AspectRatio->ratio]
   (*We set up the vertices labels and the rest of stylistic options*)]
```

B | Figures

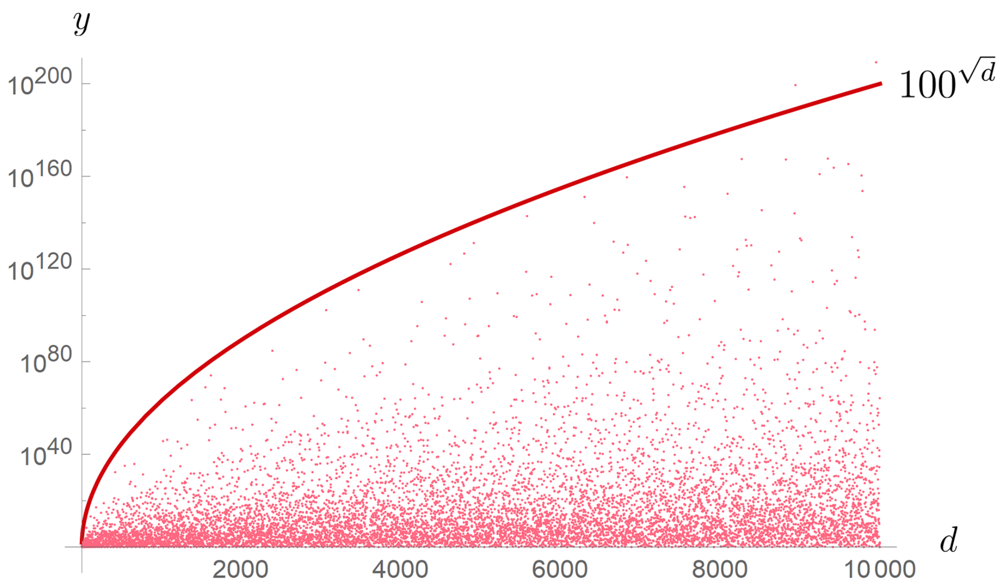
B.1 Regions of convergence of the parabola theorem for multiple values of α



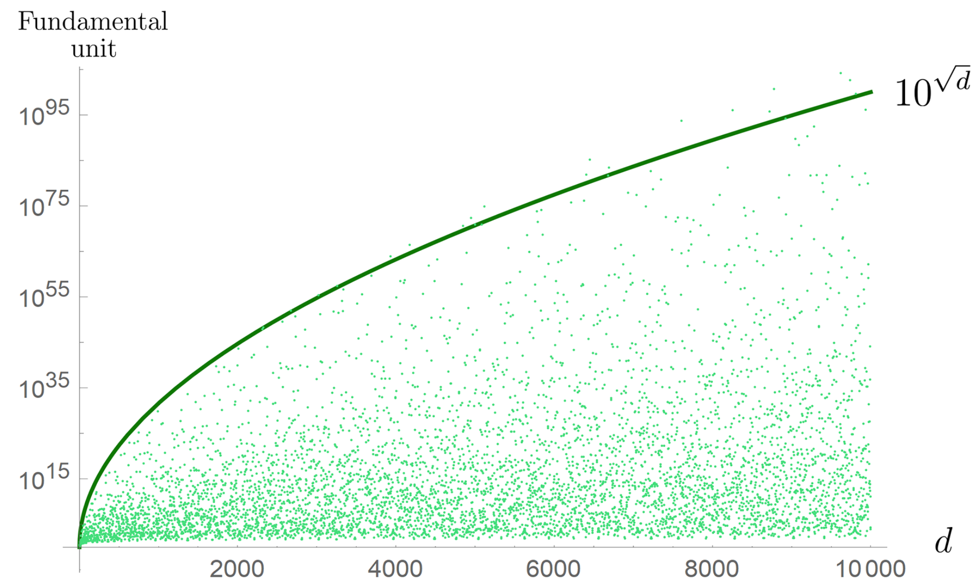
B.2 First solutions of Markov's equation



B.3 Size of y of the smallest solution of $x^2 - dy^2 = 1$



B.4 Size of the fundamental unit of $\mathbb{Q}(\sqrt{d})$



C Tables

C.1 Padé table of e^z

1	$\frac{1}{-z+1}$	$\frac{1}{\frac{z^2}{2}-z+1}$	$\frac{1}{-\frac{z^3}{6}+\frac{z^2}{2}-z+1}$	$\cdots$
$z+1$	$\frac{\frac{z}{2}+1}{-\frac{z}{2}+1}$	$\frac{\frac{z}{3}+1}{\frac{z^2}{6}-\frac{2z}{3}+1}$	$\frac{\frac{z}{4}+1}{-\frac{z^3}{24}+\frac{z^2}{4}-\frac{3z}{4}+1}$	$\cdots$
$\frac{z^2}{2}+z+1$	$\frac{\frac{z^2}{6}+\frac{2z}{3}+1}{-\frac{z}{3}+1}$	$\frac{\frac{z^2}{12}+\frac{z}{2}+1}{\frac{z^2}{12}-\frac{z}{2}+1}$	$\frac{\frac{z^2}{20}+\frac{2z}{5}+1}{-\frac{z^3}{60}+\frac{3z^2}{20}-\frac{3z}{5}+1}$	$\cdots$
$\frac{z^3}{6}+\frac{z^2}{2}+z+1$	$\frac{\frac{z^3}{24}+\frac{z^2}{4}+\frac{3z}{4}+1}{-\frac{z}{4}+1}$	$\frac{\frac{z^3}{60}+\frac{3z^2}{20}+\frac{3z}{5}+1}{\frac{z^2}{20}-\frac{2z}{5}+1}$	$\frac{\frac{z^3}{120}+\frac{z^2}{10}+\frac{z}{2}+1}{-\frac{z^3}{120}+\frac{z^2}{10}-\frac{z}{2}+1}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

C.2 Padé table of $\frac{z^2-1}{z^2+1}$

-1	-1	$\frac{-1}{2z^2+1}$	$\frac{-1}{2z^2+1}$	$\cdots$
-1	-1	$\frac{-1}{2z^2+1}$	$\frac{-1}{2z^2+1}$	$\cdots$
$2z^2-1$	$2z^2-1$	$\frac{z^2-1}{z^2+1}$	$\frac{z^2-1}{z^2+1}$	$\cdots$
$2z^2-1$	$2z^2-1$	$\frac{z^2-1}{z^2+1}$	$\frac{z^2-1}{z^2+1}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

C.3 First 40 Markov numbers and their associated points of the Lagrange spectrum

u	α_u	$M(\alpha_u)$	u	α_u	$M(\alpha_u)$
1	$\frac{3+\sqrt{5}}{2}$	$\sqrt{5}$	9077	$\frac{154398+17\sqrt{741527357}}{308618}$	$\frac{\sqrt{741527357}}{9077}$
2	$1+\sqrt{2}$	$2\sqrt{2}$	10946	$\frac{4827+\sqrt{67395890}}{5473}$	$\frac{2\sqrt{67395890}}{5473}$
5	$\frac{9+\sqrt{221}}{10}$	$\frac{\sqrt{221}}{5}$	14701	$\frac{426667+29\sqrt{1945074605}}{852658}$	$\frac{\sqrt{1945074605}}{14701}$
13	$\frac{23+\sqrt{1517}}{26}$	$\frac{\sqrt{1517}}{13}$	28657	$\frac{50549+\sqrt{7391012837}}{57314}$	$\frac{\sqrt{7391012837}}{28657}$
29	$\frac{34+\sqrt{7565}}{58}$	$\frac{\sqrt{7565}}{29}$	33461	$\frac{39202+\sqrt{10076746685}}{66922}$	$\frac{\sqrt{10076746685}}{33461}$
34	$\frac{15+5\sqrt{26}}{17}$	$\frac{10\sqrt{26}}{17}$	37666	$\frac{554659+29\sqrt{31921370}}{546157}$	$\frac{10\sqrt{31921370}}{18833}$
89	$\frac{157+\sqrt{71285}}{178}$	$\frac{\sqrt{71285}}{89}$	43261	$\frac{222099+5\sqrt{16843627085}}{432610}$	$\frac{\sqrt{16843627085}}{43261}$
169	$\frac{198+\sqrt{257045}}{338}$	$\frac{\sqrt{257045}}{169}$	51641	$\frac{673983+65\sqrt{960045437}}{1342666}$	$\frac{5\sqrt{960045437}}{51641}$
194	$\frac{249+5\sqrt{21170}}{485}$	$\frac{2\sqrt{21170}}{97}$	62210	$\frac{1384289+89\sqrt{2176922306}}{2768345}$	$\frac{2\sqrt{2176922306}}{31105}$
233	$\frac{411+\sqrt{488597}}{466}$	$\frac{\sqrt{488597}}{233}$	75025	$\frac{132339+\sqrt{50658755621}}{150050}$	$\frac{\sqrt{50658755621}}{75025}$
433	$\frac{2223+5\sqrt{1687397}}{4330}$	$\frac{\sqrt{1687397}}{433}$	96557	$\frac{495717+5\sqrt{83909288237}}{965570}$	$\frac{\sqrt{83909288237}}{96557}$
610	$\frac{269+\sqrt{209306}}{305}$	$\frac{2\sqrt{209306}}{305}$	135137	$\frac{2298654+17\sqrt{164358078917}}{4594658}$	$\frac{\sqrt{164358078917}}{135137}$
985	$\frac{1154+\sqrt{8732021}}{1970}$	$\frac{\sqrt{8732021}}{985}$	195025	$\frac{228486+\sqrt{342312755621}}{390050}$	$\frac{\sqrt{342312755621}}{195025}$
1325	$\frac{17293+13\sqrt{15800621}}{34450}$	$\frac{\sqrt{15800621}}{1325}$	196418	$\frac{86617+\sqrt{21701267282}}{98209}$	$\frac{2\sqrt{21701267282}}{98209}$
1597	$\frac{2817+\sqrt{22953677}}{3194}$	$\frac{\sqrt{22953677}}{1597}$	294685	$\frac{3846027+13\sqrt{781553243021}}{7661810}$	$\frac{\sqrt{781553243021}}{294685}$
2897	$\frac{14873+5\sqrt{75533477}}{28970}$	$\frac{\sqrt{75533477}}{2897}$	426389	$\frac{99349857+233\sqrt{1636268213885}}{198697274}$	$\frac{\sqrt{1636268213885}}{426389}$
4181	$\frac{7375+\sqrt{157326845}}{8362}$	$\frac{\sqrt{157326845}}{4181}$	499393	$\frac{84399387+169\sqrt{2244540316037}}{168794834}$	$\frac{\sqrt{2244540316037}}{499393}$
5741	$\frac{6726+5\sqrt{11865269}}{11482}$	$\frac{5\sqrt{11865269}}{5741}$	514229	$\frac{907065+\sqrt{2379883179965}}{1028458}$	$\frac{\sqrt{2379883179965}}{514229}$
6466	$\frac{8299+25\sqrt{940706}}{16165}$	$\frac{10\sqrt{940706}}{3233}$	646018	$\frac{829153+5\sqrt{234753331682}}{1615045}$	$\frac{2\sqrt{234753331682}}{323009}$
7561	$\frac{98681+13\sqrt{514518485}}{196586}$	$\frac{\sqrt{514518485}}{7561}$	925765	$\frac{15747082+17\sqrt{7713367517021}}{31476010}$	$\frac{\sqrt{7713367517021}}{925765}$

C.4 Continued fraction expansion of the first 80 square roots

$\sqrt{d}$	Continued fraction	$\sqrt{d}$	Continued fraction
$\sqrt{2}$	$1,\ \overline{2}$	$\sqrt{43}$	$6,\ \overline{1,1,3,1,5,1,3,1,1,12}$
$\sqrt{3}$	$1,\ \overline{1,2}$	$\sqrt{44}$	$6,\ \overline{1,1,1,2,1,1,1,12}$
$\sqrt{5}$	$2,\ \overline{4}$	$\sqrt{45}$	$6,\ \overline{1,2,2,2,1,12}$
$\sqrt{6}$	$2,\ \overline{2,4}$	$\sqrt{46}$	$6,\ \overline{1,3,1,1,2,6,2,1,1,3,1,12}$
$\sqrt{7}$	$2,\ \overline{1,1,1,4}$	$\sqrt{47}$	$6,\ \overline{1,5,1,12}$
$\sqrt{8}$	$2,\ \overline{1,4}$	$\sqrt{48}$	$6,\ \overline{1,12}$
$\sqrt{10}$	$3,\ \overline{6}$	$\sqrt{50}$	$7,\ \overline{14}$
$\sqrt{11}$	$3,\ \overline{3,6}$	$\sqrt{51}$	$7,\ \overline{7,14}$
$\sqrt{12}$	$3,\ \overline{2,6}$	$\sqrt{52}$	$7,\ \overline{4,1,2,1,4,14}$
$\sqrt{13}$	$3,\ \overline{1,1,1,1,6}$	$\sqrt{53}$	$7,\ \overline{3,1,1,3,14}$
$\sqrt{14}$	$3,\ \overline{1,2,1,6}$	$\sqrt{54}$	$7,\ \overline{2,1,6,1,2,14}$
$\sqrt{15}$	$3,\ \overline{1,6}$	$\sqrt{55}$	$7,\ \overline{2,2,2,14}$
$\sqrt{17}$	$4,\ \overline{8}$	$\sqrt{56}$	$7,\ \overline{2,14}$
$\sqrt{18}$	$4,\ \overline{4,8}$	$\sqrt{57}$	$7,\ \overline{1,1,4,1,1,14}$
$\sqrt{19}$	$4,\ \overline{2,1,3,1,2,8}$	$\sqrt{58}$	$7,\ \overline{1,1,1,1,1,1,14}$
$\sqrt{20}$	$4,\ \overline{2,8}$	$\sqrt{59}$	$7,\ \overline{1,2,7,2,1,14}$
$\sqrt{21}$	$4,\ \overline{1,1,2,1,1,8}$	$\sqrt{60}$	$7,\ \overline{1,2,1,14}$
$\sqrt{22}$	$4,\ \overline{1,2,4,2,1,8}$	$\sqrt{61}$	$7,\ \overline{1,4,3,1,2,2,1,3,4,1,14}$
$\sqrt{23}$	$4,\ \overline{1,3,1,8}$	$\sqrt{62}$	$7,\ \overline{1,6,1,14}$
$\sqrt{24}$	$4,\ \overline{1,8}$	$\sqrt{63}$	$7,\ \overline{1,14}$
$\sqrt{26}$	$5,\ \overline{10}$	$\sqrt{65}$	$8,\ \overline{16}$
$\sqrt{27}$	$5,\ \overline{5,10}$	$\sqrt{66}$	$8,\ \overline{8,16}$
$\sqrt{28}$	$5,\ \overline{3,2,3,10}$	$\sqrt{67}$	$8,\ \overline{5,2,1,1,7,1,1,2,5,16}$
$\sqrt{29}$	$5,\ \overline{2,1,1,2,10}$	$\sqrt{68}$	$8,\ \overline{4,16}$
$\sqrt{30}$	$5,\ \overline{2,10}$	$\sqrt{69}$	$8,\ \overline{3,3,1,4,1,3,3,16}$
$\sqrt{31}$	$5,\ \overline{1,1,3,5,3,1,1,10}$	$\sqrt{70}$	$8,\ \overline{2,1,2,1,2,16}$
$\sqrt{32}$	$5,\ \overline{1,1,1,10}$	$\sqrt{71}$	$8,\ \overline{2,2,1,7,1,2,2,16}$
$\sqrt{33}$	$5,\ \overline{1,2,1,10}$	$\sqrt{72}$	$8,\ \overline{2,16}$
$\sqrt{34}$	$5,\ \overline{1,4,1,10}$	$\sqrt{73}$	$8,\ \overline{1,1,5,5,1,1,16}$
$\sqrt{35}$	$5,\ \overline{1,10}$	$\sqrt{74}$	$8,\ \overline{1,1,1,1,16}$
$\sqrt{37}$	$6,\ \overline{12}$	$\sqrt{75}$	$8,\ \overline{1,1,1,16}$
$\sqrt{38}$	$6,\ \overline{6,12}$	$\sqrt{76}$	$8,\ \overline{1,2,1,1,5,4,5,1,1,2,1,16}$
$\sqrt{39}$	$6,\ \overline{4,12}$	$\sqrt{77}$	$8,\ \overline{1,3,2,3,1,16}$
$\sqrt{40}$	$6,\ \overline{3,12}$	$\sqrt{78}$	$8,\ \overline{1,4,1,16}$
$\sqrt{41}$	$6,\ \overline{2,2,12}$	$\sqrt{79}$	$8,\ \overline{1,7,1,16}$
$\sqrt{42}$	$6,\ \overline{2,12}$	$\sqrt{80}$	$8,\ \overline{1,16}$

C.5 Possible values of $(-1)^n Q_n$ for the first 80 non square integers

d	$(-1)^n Q_n$	d	$(-1)^n Q_n$
2	$-1, 1$	43	$-7, -3, -2, 1, 6, 9$
3	$-2, 1$	44	$-7, -4, 1, 2, 5$
5	$-1, 1$	45	$-5, -3, 1, 4$
6	$-2, 1$	46	$-10, -7, -5, 1, 2, 3, 6$
7	$-3, 1, 2$	47	$-11, 1, 2$
8	$-2, 1$	48	$-6, 1$
10	$-1, 1$	50	$-1, 1$
11	$-2, 1$	51	$-2, 1$
12	$-3, 1$	52	$-3, -2, 1, 9$
13	$-4, -3, -1, 1, 3, 4$	53	$-7, -4, -1, 1, 4, 7$
14	$-5, 1, 2$	54	$-5, -2, 1, 3$
15	$-6, 1$	55	$-6, 1, 5$
17	$-1, 1$	56	$-7, 1$
18	$-2, 1$	57	$-8, -3, 1, 7$
19	$-3, -2, 1, 5$	58	$-9, -7, -6, -1, 1, 6, 7, 9$
20	$-2, 1$	59	$-10, -2, 1, 5$
21	$-5, -3, 1, 4$	60	$-11, 1, 2$
22	$-6, -2, 1, 3$	61	$-12, -9, -5, -4, -3, -1, 1, 3, 4, 5, 9, 12$
23	$-7, 1, 2$	62	$-13, 1, 2$
24	$-4, 1$	63	$-14, 1$
26	$-1, 1$	65	$-1, 1$
27	$-2, 1$	66	$-2, 1$
28	$-3, 1, 2$	67	$-7, -3, -2, 1, 6, 9$
29	$-5, -4, -1, 1, 4, 5$	68	$-2, 1$
30	$-5, 1$	69	$-11, -5, 1, 3, 4$
31	$-6, -3, 1, 2, 5$	70	$-6, -5, 1, 9$
32	$-7, 1, 2$	71	$-11, -7, 1, 2, 5$
33	$-8, 1, 3$	72	$-4, 1$
34	$-9, 1, 2$	73	$-9, -8, -3, -1, 1, 3, 8, 9$
35	$-10, 1$	74	$-10, -7, -1, 1, 7, 10$
37	$-1, 1$	75	$-11, 1, 6$
38	$-2, 1$	76	$-6, -4, -3, 1, 2, 5, 9$
39	$-3, 1$	77	$-13, -7, 1, 4$
40	$-2, 1$	78	$-14, 1, 3$
41	$-5, -1, 1, 5$	79	$-15, 1, 2$
42	$-6, 1$	80	$-4, 1$

C.6 Smallest solutions to Pell's equation $x^2 - dy^2 = 1$

d	x	y	d	x	y	d	x	y
2	3	2	47	48	7	90	19	2
3	2	1	48	7	1	91	1574	165
5	9	4	50	99	14	92	1151	120
6	5	2	51	50	7	93	12151	1260
7	8	3	52	649	90	94	2143295	221064
8	3	1	53	66249	9100	95	39	4
10	19	6	54	485	66	96	49	5
11	10	3	55	89	12	97	62809633	6377352
12	7	2	56	15	2	98	99	10
13	649	180	57	151	20	99	10	1
14	15	4	58	19603	2574	101	201	20
15	4	1	59	530	69	102	101	10
17	33	8	60	31	4	103	227528	22419
18	17	4	61	1766319049	226153980	104	51	5
19	170	39	62	63	8	105	41	4
20	9	2	63	8	1	106	32080051	3115890
21	55	12	65	129	16	107	962	93
22	197	42	66	65	8	108	1351	130
23	24	5	67	48842	5967	109	158070671986249	15140424455100
24	5	1	68	33	4	110	21	2
26	51	10	69	7775	936	111	295	28
27	26	5	70	251	30	112	127	12
28	127	24	71	3480	413	113	1204353	113296
29	9801	1820	72	17	2	114	1025	96
30	11	2	73	2281249	267000	115	1126	105
31	1520	273	74	3699	430	116	9801	910
32	17	3	75	26	3	117	649	60
33	23	4	76	57799	6630	118	306917	28254
34	35	6	77	351	40	119	120	11
35	6	1	78	53	6	120	11	1
37	73	12	79	80	9	122	243	22
38	37	6	80	9	1	123	122	11
39	25	4	82	163	18	124	4620799	414960
40	19	3	83	82	9	125	930249	83204
41	2049	320	84	55	6	126	449	40
42	13	2	85	285769	30996	127	4730624	419775
43	3482	531	86	10405	1122	128	577	51
44	199	30	87	28	3	129	16855	1484
45	161	24	88	197	21	130	6499	570
46	24335	3588	89	500001	53000	131	10610	927

C.7 Fundamental units for the first 60 square free integers

d	Fundamental unit of $\mathbb{Q}(\sqrt{d})$	d	Fundamental unit of $\mathbb{Q}(\sqrt{d})$
2	$1+\sqrt{2}$	51	$50+7\sqrt{51}$
3	$2+\sqrt{3}$	53	$\frac{1}{2}\left(7+\sqrt{53}\right)$
5	$\frac{1}{2}\left(1+\sqrt{5}\right)$	55	$89+12\sqrt{55}$
6	$5+2\sqrt{6}$	57	$151+20\sqrt{57}$
7	$8+3\sqrt{7}$	58	$99+13\sqrt{58}$
10	$3+\sqrt{10}$	59	$530+69\sqrt{59}$
11	$10+3\sqrt{11}$	61	$\frac{1}{2}\left(39+5\sqrt{61}\right)$
13	$\frac{1}{2}\left(3+\sqrt{13}\right)$	62	$63+8\sqrt{62}$
14	$15+4\sqrt{14}$	65	$8+\sqrt{65}$
15	$4+\sqrt{15}$	66	$65+8\sqrt{66}$
17	$4+\sqrt{17}$	67	$48842+5967\sqrt{67}$
19	$170+39\sqrt{19}$	69	$\frac{1}{2}\left(25+3\sqrt{69}\right)$
21	$\frac{1}{2}\left(5+\sqrt{21}\right)$	70	$251+30\sqrt{70}$
22	$197+42\sqrt{22}$	71	$3480+413\sqrt{71}$
23	$24+5\sqrt{23}$	73	$1068+125\sqrt{73}$
26	$5+\sqrt{26}$	74	$43+5\sqrt{74}$
29	$\frac{1}{2}\left(5+\sqrt{29}\right)$	77	$\frac{1}{2}\left(9+\sqrt{77}\right)$
30	$11+2\sqrt{30}$	78	$53+6\sqrt{78}$
31	$1520+273\sqrt{31}$	79	$80+9\sqrt{79}$
33	$23+4\sqrt{33}$	82	$9+\sqrt{82}$
34	$35+6\sqrt{34}$	83	$82+9\sqrt{83}$
35	$6+\sqrt{35}$	85	$\frac{1}{2}\left(9+\sqrt{85}\right)$
37	$6+\sqrt{37}$	86	$10405+1122\sqrt{86}$
38	$37+6\sqrt{38}$	87	$28+3\sqrt{87}$
39	$25+4\sqrt{39}$	89	$500+53\sqrt{89}$
41	$32+5\sqrt{41}$	91	$1574+165\sqrt{91}$
42	$13+2\sqrt{42}$	93	$\frac{1}{2}\left(29+3\sqrt{93}\right)$
43	$3482+531\sqrt{43}$	94	$2143295+221064\sqrt{94}$
46	$24335+3588\sqrt{46}$	95	$39+4\sqrt{95}$
47	$48+7\sqrt{47}$	97	$5604+569\sqrt{97}$

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