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Crazy for 2

Genus 2 curves in characteristic 2 via Kummer surfaces

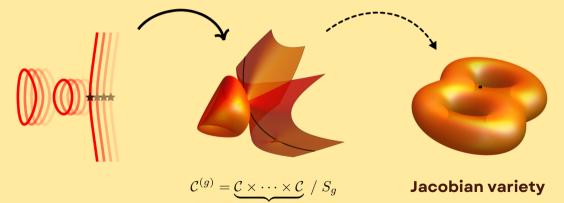


The Jacobian variety associated to the curve



Given a hyperelliptic curve, how can we compute an explicit model of its Jacobian as a projective variety?

The idea is





Let

$$\mathcal{C}: y^2 + h(x)y = f(x)$$

be a hyperelliptic curve of genus $g \ge 1$ where $f(x), h(x) \in k[x]$, $\deg f(x) = 2g + 2$ and $\deg h(x) \le g + 1$.

The curve has two different points at infinity that I will denote by ∞_+ and $\infty_-.$

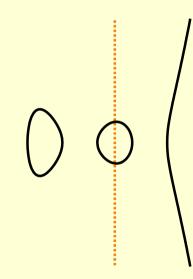
The hyperelliptic involution

The curve

$$\mathcal{C}: y^2 + h(x)y = f(x)$$

has a natural involution defined by

$$\iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$$
$$(x, y) \longmapsto (x, -y - h(x))$$
$$\infty_{+} \longmapsto \infty_{-}$$
$$\infty_{-} \longmapsto \infty_{+}$$





The following:

$$\Theta_{+} = \underbrace{C \times \cdots \times C}_{g-1} \times \{\infty_{+}\} \text{ and } \Theta_{-} = \underbrace{C \times \cdots \times C}_{g-1} \times \{\infty_{-}\}$$

define divisors of $C^{(g)}$ and an embedding of the Jacobian into projective space is given by $\mathcal{L}(2(\Theta_+ + \Theta_-))$.

(These are functions in the function field of $C^{(g)}$ that at worst can only possibly have poles in $2(\Theta_+ + \Theta_-)$ of the "right" multiplicity.)

Example



For g = 2, let's consider two copies of a curve C

$$y_1^2 + h(x_1)y_1 = f(x_1)$$
 $y_2^2 + h(x_2)y_2 = f(x_2)$

Then, some independent functions of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ are

1,
$$x_1 + x_2$$
, $x_1 x_2$, $(x_1 + x_2)^2$, $\frac{(2y_1 + h(x_1)) - (2y_2 + h(x_2))}{x_1 - x_2}$, ...

In this case $|\mathcal{L}(2(\Theta_+ + \Theta_-))| = 16$.



The embedding would be given by considering

$$[1:x_1+x_2:x_1x_2:(x_1+x_2)^2:\frac{(2y_1+h(x_1))-(2y_2+h(x_2))}{x_1-x_2},\dots] \hookrightarrow \mathbb{P}^{15}$$

where $(x_1,y_1), (x_2,y_2) \in \mathcal{C}.$



The embedding by $\mathcal{L}(2(\Theta_++\Theta_-))$ is given by the intersection of many conics:

| Genus | 1 | 2 | 3 | • • • | g |
|-----------------------------------|---|----|------|-------|---------------------|
| \mathbb{P}^n in which it embeds | 3 | 15 | 63 | • • • | $4^{g} - 1$ |
| Number of conics | 2 | 72 | 1568 | ••• | $2^{2g-1}(2^g-1)^2$ |



 $\iota_{\mathcal{C}}$ extends to an involution on $\mathcal{C}^{(g)}$, such that $\iota_{\mathcal{C}}$ acts linearly on the elements of $\mathcal{L}(2(\Theta_{+} + \Theta_{-}))$. If the field of definition has characteristic different than 2, we can "diagonalise" this action to obtain a decomposition:

$$\mathcal{L}(2(\Theta_{+}+\Theta_{-})) = \{\text{even functions}\} \oplus \{\text{odd functions}\}$$

where

$$\iota_{\mathcal{C}}(\mathsf{even}) = \mathsf{even}$$
 $\iota_{\mathcal{C}}(\mathsf{odd}) = -\mathsf{odd}$



The functions

$$\{1, x_1 + x_2, x_1 x_2, (x_1 + x_2)^2, \dots\}$$

are even and $|\{\text{even functions}\}| = 10.$

The functions

$$\left\{\frac{(2y_1+h(x_1))-(2y_2+h(x_2))}{x_1-x_2},\frac{(2y_1+h(x_1)x_2)-(2y_2+h(x_2))x_1}{x_1-x_2},\dots\right\}$$
 are odd and $\left|\left\{\text{odd functions}\right\}\right| = 6.$



Kummer variety

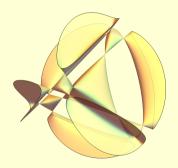
Let \mathcal{A} be an Abelian variety (e.g. the Jacobian of a hyperelliptic curve) and let ι be the involution in \mathcal{A} that sends an element to its inverse. Then, the **Kummer variety** associated to \mathcal{A} , Kum(\mathcal{A}) is the quotient variety \mathcal{A}/ι .

Fact

For g > 1, $\mathcal{A}[2]$ is the set of all fixed points under the action of ι and these points are singular points of $\operatorname{Kum}(\mathcal{A})$.

Examples of Kummer varieties





Suppose that the field of definition has characteristic different than 2.

- If the dimension of A is 2, Kum(A) is a surface described by a quartic in P³ with 16 nodal singularities.
- Generally, if the dimension of \mathcal{A} is g, $\operatorname{Kum}(\mathcal{A})$ can be found as an intersection in \mathbb{P}^{2^g-1} .



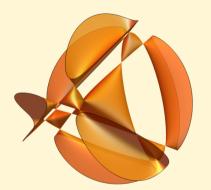
- Their models are considerably easier.
- They are **not** Abelian varieties, so they do not have a group law. However, they inherit a *pseudo-group law*.
- For a hyperelliptic curve C, the projective embedding of the Kummer variety associated to the Jacobian of C is given by $\mathcal{L}(\Theta_+ + \Theta_-)$.

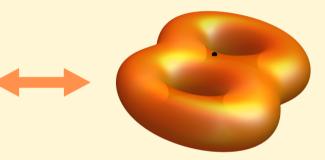


$\mathcal{L}(\Theta_+ + \Theta_-) \subset \{ \text{even functions of } \mathcal{L}(2(\Theta_+ + \Theta_-)) \}$

In fact, the space of even functions of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ is generated as a vector space by the products of every two functions of $\mathcal{L}(\Theta_+ + \Theta_-)$.

Furthermore, the space of odd functions of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ defines a model of the desingularisation of the Kummer surface as the intersection of 3 quadrics in \mathbb{P}^5 .





Desingularisation of the Kummer surface Explicit projective models of the Jacobian of a genus 2 curve 1 **Canonical form.** We shall normally suppose that the characteristic¶ of the ground field is not 2 and consider curves C of genus 2 in the shape

$$C: Y^2 = F(X),$$
 (1.1.1)

where

$$F(X) = f_0 + f_1 X + \ldots + f_6 X^6 \in k[X]$$
(1.1.2)

1. The Jacobian variety

We shall work with a general curve \mathscr{C} of genus 2, over a ground field K of characteristic not equal to 2, 3 or 5, which may be taken to have hyperelliptic form

$$\mathscr{C}: Y^2 = F(X) = f_6 X^6 + f_5 X^5 + f_4 X^4 + f_3 X^3 + f_2 X^2 + f_1 X + f_0$$
(1)

with f_0, \ldots, f_6 in $K, f_6 \neq 0$, and $\Delta(F) \neq 0$, where $\Delta(F)$ is the discriminant of F. In \mathbb{F}_5 there is, for example, the curve $Y^2 = X^5 - X$ which is not birationally equivalent to the above form.

1 **Canonical form.** We shall normally suppose that the characteristic¶ of the ground field is not 2 and consider curves C of genus 2 in the shape

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2. Set-up

Let k be a field of characteristic not equal to two, k^{s} a separable closure of k, and $f = \sum_{i=0}^{6} f_{i}X^{i} \in k[X]$ a separable polynomial with $f_{6} \neq 0$. Denote by Ω the set of the six roots of f in k^{s} , so that $k(\Omega)$ is the splitting field of f over k in k^{s} . Let C be the smooth projective



In algebraically closed fields of characteristic 2, the 2-torsion of the Jacobian of a curve ${\cal C}$ of genus g is

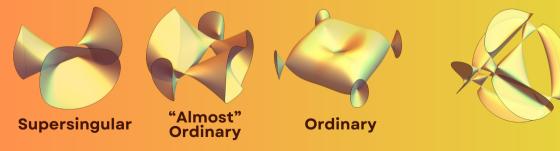
 $\mathcal{J}(\mathcal{C})[2] \cong (\mathbb{Z}/2\mathbb{Z})^r$

for some $0 \le r \le g$.

| Characteristic | cteristic 2 | | | Not 2 |
|-------------------------|-------------|-------|-------|-------|
| 2-rank | 0 | 1 | 2 | |
| Number of singularities | 1 | 2 | 4 | 16 |
| Singularity type | Elliptic | D_8 | D_4 | A_1 |

Characteristic 2

Characteristic different than 2





- In characteristic 2 we cannot diagonalise the action of $\iota_{\mathcal{C}}$, so it does no longer makes sense to talk about even and odd functions.
- So what can be said about the desingularisation of Kummer surfaces in characteristic 2?

Work in progress

Computing $\mathcal{L}(2(\Theta_+ + \Theta_-))$

for genus 2 curves in characteristic 2

Studying the desingularisation of Kummer surfaces Finding explicit models that can be used for computation

2*f1*f6*g2*g3*k1*k2*k3*k4 + f6*g0*g1*g2*g3*k1*k2*k3*k4 - 2*f5*g0*g2*2*g3*k1*k2*k3*k4 - 2*f1*f5*g3*2*k1*k2*k3*k4 + 4*f0*f6*g3*2*k1*k2*k3*k4 + 4*f0*f6*g3*2*k1*k2*k3*k4 - 2*f1*f6*g3*2*k1*k2*k3*k4 + 4*f0*f6*g3*2*k1*k2*k3*k4 + 4*f0*f6*g3*f1*k2*k3*k4 + 4*f0*f6*g3*f1*k2*k3*k4 + 4*f0*f6*g3*f1*k2*k3*k4 + 4*f0*f6*g3*f1*k2*k3*k4 + 4*f0*f6*g3*f1*k2*k3*k4 + 4*f0*f6*g3*f1*k2*k3*k4 + 4*f0*f1*k2*k3*k4 + 4*f0*f1*k2*k3*k4 + 4*f0*f1*k2*k3*k4 + 4*f1*k2*k3*k4 + 4*f1*k2*k3*k4*k4*k2*k3*k4*k2*k3*k4*k2*k3*k4*k2*k3*k4*k2*k3*k4*k2*k3*k f6*e0^2*e3^2*k1*k2*k3*k4 - 3*f5*e0*e1*e3^2*k1*k2*k3*k4 + 2*f4*e0*e2*e3^2*k1*k2*k3*k4 - e0*e2^3*e3^2*k1*k2*k3*k4 - f1*e2*e3^3*k1*k2*k3*k4 - f1*e2*e3^3*k1*k2*k3*k4 - f1*e2*e3*a*k1*k2*k3*k4 -2*f1*f6*g3*2*k2*2*k3 *f 3*2*k2*2*k3*k4 - 2*f5*g0*g2*g3*2* 2*k3*k4 + f4*g0*g3*3*k2*2*k3*k4 - g0*g2*2*g3*3*k2 3*k4 - 8*f3*f4*f6*k1*k3^2*k4 + 8^2 ** g ** g ** 3 **ft *g ** ** ** ** **g: k1 ** 2* g2* 3* 8*f4*f6*g1*g2*k1*k3^2*k4 f3*f6* f6*g1^2*g2*g3* 3^2*k 2*f4*g1*g2*g3^2*k1*k3^2* 2*f3* + f6*g1^2*g3^2*k2*k3^2*k4 - f6*g0*g2*g3^2*k2*k3^2*k4 - 2*f5*g1*g2*g3^2*k2*k3^2*k4 - 3*f5_____g3^3*k2*k3^2*k4 + f4*g1*g3^3*k2*k3^2*k4 - 2*g0*g2*g3^4*k2*k3^2*k4 - 8*f4*f5*f6*k3^3*k4 - 2*f5*f6*g2^2*k3^3*k4 - 2*f5*f6*g1*g3*k3^3*k4 + 4*f4*f6*g2*g3*k3^3*k4 + f6*g2^3*g3*k3^3*k4 - 2*f4*f5*g3^2*k3^3*k4 - 2*f3*f6*g3^2*k3^3*k4 - 2*f4*f5*g3^2*k3^3*k4 - 2*f5*f6*g3^2*k3^3*k4 - 2*f5*f6*g3^2*k3^3*k4 - 2*f4*f5*g3^2*k3^3*k4 - 2*f4*f5*g3^2*k3^3*k4 - 2*f4*f5*g3^2*k3^3*k4 - 2*f5*f6*g3^2*k3^3*k4 - 2*f5*f6*g3^3*k4 - 2*f5*f6*g3^3*k4 - 2*f5*f6*g3^3*k4 - 2*f5*f6*g3^3*k4 - 2*f5*f6*g3^2*k3^3*k4 - 2*f5*f6*g3^3*k4 - 2*f 2*f5*g2^2*g3^2*k3^3*k4 - f6*g0*g3^3*k3^3*k4 - 4*f5*g1*g3^3*k3^3*k4 + f4*g2*g3^3*k3^3*k4 - f3*g3^4*k3^3*k4 - 2*g1*g2*g3^3*k3^3*k4 - g0*g3^5*k3^3*k4 - g0*g3^5*k3^5*k4 - g0*g3^5*k4 - g0*g3^5*k3^5*k4 - g0*g3^5*k4 - g0* 4*f1*f6*k1^2*k4^2 + 4*f6*g0*g1*k1/ 4*2 - f1*g3*2*k1*2*k4*2 + g0*g1*g3*2*k1*2*k4*2 + 4*fr g0_2*k1*k2*k4*2 + g0*g2*g3*_ *k2*k4*2 + 4*f6*g0*g3*k2*2*k4*2 + 2*f6*g0*g1*g3*k3*k4*v1 - 2*f5*g0*g2*g3*k3*k4*v1 + 2*f4*g0*g3*k3*k4*v1 - g0*g2*2*g3*2*k3*k4*v1 + 4*f6*g0*k4*2*v1 + g0*g3*2*k4*v1 + 4*f1*f6*g3*k2*k4*v2 + f1*g3*3*k2*k4*v2 + 8*f4*f6*g1*k3*k4*v2 + 2*f6*g1*g2*2*k3*k4*v2 + 2*f6*g1*2*g3*k3*k4*v2 - 2*f6*g0*g2*g3*k3*k4*v2 + 2*f5*g1*g2*g3*k3*k4*v2 -2*f5*g0*g3*2*k3*k4*v2 + 2*f4*g1*g3*2*k3*k4*v2 + 2*g1*g2*2*g3*2*k3*k4*v2 + g1*2*g3*3*k3*k4*v2 - 2*g0*g2*g3*3*k3*k4*v2 + 4*f6*g1*k4*2*v2 + g1*g3*2*k4*v2 + g1*g3*2*k4*v2*k4*v2*k4*k4*v2 + g1*g3*2*k4*v2 + g1*g3* 2*f4*g2*g3^2*k3*k4*v3 + 2*g2^3*g3^2*k3*k4*v3 - f3*g3^3*k3*k4*v3 - g1*g2*g3^3*k3*k4*v3 - g0*g3^4*k3*k4*v3 + 4*f6*g2*k4^2*v3 + g2*g3^2*k4^2*v3 + 4*f6*g0*g3*k2*k4*v4 + g0*g3^3*k2*k4*v4 - 16*f4*f6*k3*k4*v4 - 4*f6*g2^2*k3*k4*v4 - 4*f6*g1*g3*k3*k4*v4 - 4*f4*g3^2*k3*k4*v4 - g2^2*g3^2*k3*k4*v4 g1*g3^3*k3*k4*v4 - 8*f6*k4^2*v4 - 2*g3^2*k4^2*v4 - 4*f5*g3*k3*k4*v5 - 2*g2*g3^2*k3*k4*v5 - g3*k3*k4*v6 4*f1*f6*g0*g3*k1^2*k2*k3 + 4*f0*f6*g1*g3*k1^2*k2*k3 + f1*g0*g3^3*k1^2*k2*k3 + f0*g1*g3^3*k1^2*k2*k3 + 4*f0*f6*g2*g3*k1*k2^2*k3 + f0*g2*g3^3*k1*k2^2*k3 + f1*g0*g3*k1*k2^2*k3 + f0*g2*g3*k1*k2^2*k3 + f1*g0*g3*k1*k2^2*k3 + f0*g1*g3*k1*k2^2*k3 + f1*g0*g3*k1*k2^2*k3 + f1*g0*g3*k1*k2*k3 + f0*g1*g3*k1*k2*k3 + f1*g0*g3*k1*k2*k3 + f1*g0*g3*k1*k2*k3 + f0*g1*g3*k1*k2*k3 + f1*g0*g3*k1*k2*k3 + f0*g1*g3*k1*k2*k3 + f1*g0*g3*k1*k2*k3 + f1*g0*g3*k1*k2*k3 + f0*g1*g3*k1*k2*k3 + f0*g1*g1*k2*k3 + f0*g1*g1*k2*k3*k2*k3*k2*k3*k2*k3*k2*k3*k2*k3*k2*k3*k2*k3*k2*k3*k 4*f0*f6*g3^2*k2^3*k3 + f0*g3^4*k2^3*k3 - 8*f1*f4*f6*k1^2*k3^2 + 8*f4*f6*g0*g1*k1^2*k3^2 - 2*f1*f6*g2^2*k1^2*k3^2 + 2*f6*g0*g1*g2^2*k1^2*k3^2 - 2*f1*f6*g2*c2*k1^2*k3^2 + 2*f6*g2*c2*k1^2*k3^2 - 2*f1*f6*g2*c2*k1^2*k3^2 + 2*f6*g2*c2*k1*c2*k3^2 + 2*f1*f6*g2*c2*k1*c2*k3*c2 + 2*f1*f6*g2*c2*k1*c2*k3*c2 + 2*f1*f0*g2*c2*k1*c2*k3*c2 + 2*f1*g2*c2*k1*c2*k3*c2 + 2*f1*g2*c2*c2*k1*c2*k3*c2 + 2*f1*g2*c2*k1*c2*k3*c2*k1*c2*k1*c2*k3*c2*c2*k1*c2*k1*c2*k1*c2*k3*c2*c2*k1*c2*k1*c2*k3*c2*c2*k1*c

2*g1*g2*g3*k1*k3*v2 + g0*g3^2*k1*k3*v2 + g0*g3^2*k1*k2*v3 - 2*f5*g2*k1*k3*v3 - 2*g2^2*g3*k1*k3*v3 + 2*g1*g3^2*k1*k3*v3 - 2*g3*k1*k3*v3 + 4*f5*k1*k3*v4 +