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Elliptic curves over discrete valuation rings

Study group on Mazur's Torsion Theorem

Plan for this session

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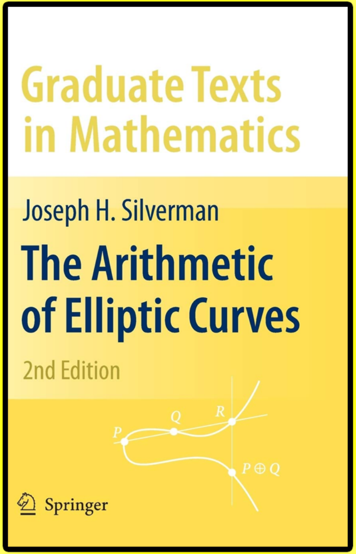
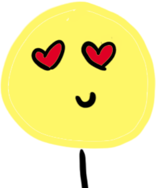
1. Introduction to the theory of elliptic curves over DVRs.
2. Discussion of the types of reduction.
3. Reduction of torsion points.
4. The Néron-Ogg-Shaferevich criterion.

If in doubt... Chapter VII





Discrete valuation ring

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DVR

A **discrete valuation ring** is a principal ideal domain R with exactly one non-zero maximal ideal $\mathfrak{p}$.

Discrete valuation ring

DVR

A **discrete valuation ring** is a principal ideal domain R with exactly one non-zero maximal ideal $\mathfrak{p}$. Alternatively, it is a PID endowed with a discrete valuation v on the field of fractions K of R such that

$$R = \{0\} \cup \{x \in K : v(x) \geq 0\}$$

Residue field

The **residue field** k of R is defined as $k = R/\mathfrak{p}$

Uniformiser

A **uniformiser** of R is an element $\pi \in R$ satisfying that $v(\pi) = 1$.

My fave example of DVR ($p \neq 2, 3$)

$$R = \mathbb{Z}_p$$

$$K = \mathbb{Q}_p$$

$$k = \mathbb{F}_p$$

$$\pi = p$$

Minimal Weierstrass equation

Let E/K be an elliptic curve given by a Weierstrass equation $y^2 = x^3 + ax + b$.

$$y^2 = x^3 + ax + b \quad \xleftrightarrow{x'=u^2x \qquad y'=u^3y} \quad (y')^2 = (x')^3 + u^{-4}ax' + u^{-6}b$$

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Minimal Weierstrass equation

We say that a Weierstrass equation is **minimal** if a and b belong to R and $v(a) < 4$ or $v(b) < 6$ (equivalently $v(\Delta)$ is minimal).

Over $K = \mathbb{Q}_5$,

$$y^2 = x^3 + 30\,000\,x - 4\,000\,000 \quad \xrightarrow{u=10} \quad y^2 = x^3 + 3x - 4$$

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Over $K = \mathbb{Q}_5$,

$$y^2 = x^3 + 30\,000\,x - 4\,000\,000 \quad \xrightarrow{u=5} \quad y^2 = x^3 + 48x - 256$$

Definition time!

Minimal Weierstrass model

The **minimal Weierstrass model** for E is the projective scheme $\mathcal{E}$ over R defined by a minimal Weierstrass equation.

Definition time!

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Minimal Weierstrass model

The **minimal Weierstrass model** for E is the projective scheme $\mathcal{E}$ over R defined by a minimal Weierstrass equation.

Reduction of E modulo $\mathfrak{p}$

The **reduction of E modulo $\mathfrak{p}$**, which we will denote by $\overline{E}$, is the special fiber $\mathcal{E}_k$ of $\mathcal{E}$. We will denote by $\overline{E}_{sm}$ the **smooth locus** of $\overline{E}$.

It is easy to see that $\overline{E}$ is an irreducible projective curve over k (maybe singular) and that we can define a group law on $\overline{E}_{sm}$ using the secant line construction.

Definition time!

Reduction map

The **reduction map** is the map

$$\begin{aligned} E(K) &\to \overline{E}(k) \\ P &\longmapsto \overline{P} \end{aligned}$$

induced by the natural morphism $R \to R/\mathfrak{p} = k$.

Definition time!

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The **reduction map** is the map

$$E(K) \to \overline{E}(k)$$
$$P \longmapsto \overline{P}$$

induced by the natural morphism $R \to R/\mathfrak{p} = k$.
We also have,

$$E_0(K) = \{P \in E(K) : \overline{P} \in \overline{E}_{sm}(k)\}.$$

Definition time!

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$$E_1(K) = \{P \in E(K) : \overline{P} = \overline{O}\}.$$

Example

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Over $\mathbb{Q}_5$,

$$E : y^2 = x^3 + 3x - 4 \quad \xrightarrow{\text{mod } 5} \quad \overline{E} : y^2 = x^3 - \overline{2}x + \overline{1} = (x - \overline{2})^2(x - \overline{1})$$

Then,

$$\begin{aligned}
\overline{E}(\mathbb{F}_5) &= \{\overline{O}, (\overline{0}, -\overline{1}), (\overline{0}, \overline{1}), (\overline{1}, \overline{0}), (\overline{2}, \overline{0})\} \\
\overline{E}_{sm}(\mathbb{F}_5) &= \{\overline{O}, (\overline{0}, -\overline{1}), (\overline{0}, \overline{1}), (\overline{1}, \overline{0})\} \cong \mathbb{Z}/4\mathbb{Z} \\
E_0(\mathbb{Q}_5) &= \{(x, y) \in E(\mathbb{Q}_5) : x \not\equiv 2 \pmod 5\}
\end{aligned}$$

For example, $(1, 0)$ and $(0, \pm 2i)$ all belong to $E_0(\mathbb{Q}_5)$ (where i is considered to be one of the roots of the polynomial $x^2 + 1$).

Example

Over $\mathbb{Q}_5$,

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Then,

$$E_1(\mathbb{Q}_5) = \{(x, y) \in E(\mathbb{Q}_5) : v_5(y) < 0\} = \{(x, y) \in E(\mathbb{Q}_5) : v_5(x) < 0\}$$

For example, $[4](0, 2i) = \left(\frac{707679}{1537600}, \frac{3027727631}{1906624000} i\right) \in E_1(\mathbb{Q}_5)$.

Example

Over $\mathbb{Q}_5$,

$$E : y^2 = x^3 + 3x - 4 \quad \xrightarrow{\text{mod } 5} \quad \overline{E} : y^2 = x^3 - \overline{2}x + \overline{1} = (x - \overline{2})^2(x - \overline{1})$$

It is easy to check that if $x \equiv 2 \pmod 5$, then, $v(x^3 + 3x - 4) = 1$ and so, as $v(y^2)$ is even, this shows that there cannot be any point in $E(\mathbb{Q}_5)$ that reduces to $(\overline{2}, \overline{0})$. Therefore, $E(\mathbb{Q}_5) = E_0(\mathbb{Q}_5)$ and $\#E(\mathbb{Q}_5)/E_0(\mathbb{Q}_5) = 1$.

The quotient $E(K)/E_0(K)$

Finiteness of $E(K)/E_0(K)$

The group $E(K)/E_0(K)$ is finite. Furthermore, if $\overline{E}_{sm}(k) \cong k^*$ (i.e. E has split multiplicative reduction) then the group is cyclic of order $-v(j)$; otherwise, it has cardinality at most 4.

Types of reduction

As $\overline{E}$ is given by the equation $y^2 = x^3 + \overline{a}x + b$, so it is non-singular (hence an elliptic curve) if and only if $\overline{\Delta} \neq 0$.

Good reduction 😇

We say that E has **good reduction** if $\overline{\Delta} \neq 0$.

Example: $y^2 = x^3 + 1$ in $\mathbb{Q}_p$ for all $p \geq 5$ ($\Delta = -2^4\,3^3$).

Types of reduction

As $\overline{E}$ is given by the equation $y^2 = x^3 + \overline{a}x + b$, so it is non-singular (hence an elliptic curve) if and only if $\overline{\Delta} \neq 0$.

Bad reduction 😈

We say that E has **bad reduction** if $\overline{\Delta} = 0$.

If $\overline{\Delta} = 0$ and either $\overline{a} \neq 0$ or $\overline{b} \neq 0$, we say that E has **multiplicative reduction**.

If $\overline{\Delta} = 0$ and $\overline{a} = \overline{b} = 0$, we say that E has **additive reduction**.

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Examples:

$y^2 = x^3 + 3x - 4$ over $\mathbb{Q}_5$ has (split) multiplicative reduction.

$y^2 = x^3 + 5$ over $\mathbb{Q}_5$ has additive reduction.

Types of reduction

Good Reduction

Bad Reduction

Semistable Reduction

Good Reduction

$\Delta \in R^\times$

Multiplicative Reduction

$\Delta \notin R^\times$ but either $\begin{cases} a \in R^\times \\ b \in R^\times \end{cases}$

Additive Reduction

$\Delta \notin R^\times,\ a \notin R^\times,\ b \notin R^\times$

Behaviour of reduction type under extensions

Preservation of reduction type under extension

Let K'/K be a finite extension. Suppose that either K'/K is unramified or E has semi-stable reduction over K. Then a minimal Weierstrass equation for E over K is still minimal over K'. Therefore the reduction type of E over K is the same as that over K'.

Behaviour of reduction type under extensions

Preservation of reduction type under extension

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Semi-stable reduction theorem

There exists a finite extension K'/K such that E has semi-stable reduction over K'.

$$y^2 = x^3 + ax + b \quad \xleftrightarrow{x'=u^2x \quad y'=u^3y} \quad (y')^2 = (x')^3 + u^{-4}ax' + u^{-6}b$$

Behaviour of reduction type under extensions

Over $\mathbb{Q}_5$, $y^2 = x^3 + 5$ has additive reduction but over $\mathbb{Q}_5(5^{1/6})$, it has **good reduction**.

$$y^2 = x^3 + 5 \quad \xleftrightarrow{x'=(5^{1/6})^2 x \quad y'=(5^{1/6})^3 y} \quad (y')^2 = (x')^3 + 1$$

Behaviour of reduction type under extensions

Over $\mathbb{Q}_5$, $y^2 = x^3 + 5$ has additive reduction but over $\mathbb{Q}_5(5^{1/6})$, it has **good reduction**.

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Over $\mathbb{Q}_5$, $y^2 = x^3 + 75x - 500$ has additive reduction but over $\mathbb{Q}_5(5^{1/2})$ it has **multiplicative reduction**.

$$y^2 = x^3 + 75x - 500 \quad \xleftrightarrow{x'=(5^{1/2})^2 x \quad y'=(5^{1/2})^3 y} \quad (y')^2 = (x')^3 + 3x - 4$$

Behaviour of reduction type under extensions

For all sufficiently large extensions K'/K, the curve E over K' has either good or multiplicative reduction.

Potential reduction

We say that E has **potentially good** or **potentially multiplicative** reduction according to the type of semistable reduction that E has over an extension of K.

Test of potentially good reduction

E has potentially good reduction if and only if $j(E) = -1728(4a)^3/\Delta$ is integral.

Reduction of torsion points

Suppose that E has good reduction. Since $\mathcal{E}$ is a proper smooth group over R, for any n its n-torsion $\mathcal{E}[n]$ is a finite flat group scheme over R.

Reduction of torsion points

Suppose E has good reduction

- If n is prime to the residue characteristic, then:
  - The reduction map $E[n](K) \to \overline{E}[n](k)$ is injective.
  - The reduction map $E_0[n](K) \to \overline{E}_{sm}[n](k)$ is injective.
  - The reduction map $E[n](\overline{K}) \to \overline{E}[n](\overline{k})$ is an isomorphism of Galois modules.
- If K is an extension of $\mathbb{Q}_p$ with $e < p - 1$, then it is also true that the reduction map $E[n](K) \to \overline{E}[n](k)$ is injective.

The Néron-Ogg-Shaferevich criterion

Let G_K be the absolute Galois group of K and I_K the inertia subgroup.

The Néron-Ogg-Shaferevich criterion

Let ℓ be a prime different from the residue characteristic. Then:

- E has good reduction if and only if I_K acts trivially on $T_\ell(E)$.
- E has semi-stable reduction if and only if I_K acts unipotently on $T_\ell(E)$.

Let's recall some definitions

As unramified extensions of K correspond to extensions of the residue field k, we have,

Inertia subgroup I_K

The **inertia subgroup** I_K of G_K is the set of elements of G_K that act trivially on $\overline{k}$.

Tate module

The ℓ-adic **Tate module** of E, denoted $T_\ell(E)$, is the inverse limit of the groups $E[\ell^n](\overline{K})$, where the transition maps are multiplication by ℓ. Explicitly, an element of $T_\ell(E)$ is a sequence $(x_0, x_1, \cdots)$ of $\overline{K}$-points of E, where $x_0 = O$ and $\ell x_i = x_{i-1}$ for $i > 0$.

The fact that $E[\ell^n](\overline{K}) = (\mathbb{Z}/\ell^n\mathbb{Z})^2$ allows us to define an isomorphism between $T_\ell(E)$ and $\mathbb{Z}_\ell^2$ through the inverse limit.

Example: Action of the inertia subgroup on the 4-torsion of E

$$
P_{(0,0)} = O \begin{cases}
P_{(0,2)} = (2,0) \begin{cases} P_{(0,1)} = (2-\sqrt{5}, -5+\sqrt{5}) \\ P_{(0,3)} = (2-\sqrt{5}, \quad 5-\sqrt{5}) \\ P_{(2,1)} = (2+\sqrt{5}, -5-\sqrt{5}) \\ P_{(2,3)} = (2-\sqrt{5}, \quad 5+\sqrt{5}) \end{cases} \\
P_{(2,0)} = (1,0) \begin{cases} P_{(1,0)} = (1-2i, -2-4i) \\ P_{(3,0)} = (1-2i, \quad 2+4i) \\ P_{(1,2)} = (1+2i, -2+4i) \\ P_{(3,2)} = (1+2i, \quad 2-4i) \end{cases} \\
P_{(2,2)} = (-3,0) \begin{cases} P_{(1,1)} = (-3-2\sqrt{5}, \quad 2i(5+2\sqrt{5})) \\ P_{(3,3)} = (-3-2\sqrt{5}, -2i(5+2\sqrt{5})) \\ P_{(1,3)} = (-3+2\sqrt{5}, -2i(5-2\sqrt{5})) \\ P_{(3,1)} = (-3+2\sqrt{5}, \quad 2i(5-2\sqrt{5})) \end{cases}
\end{cases}
$$

$$E : y^2 = x^3 - 7x + 6$$

$$\Delta = 2^8\, 5^2$$

$$(\mathbb{Z}/4\mathbb{Z})^2 \longrightarrow E[4](K)$$

$$(n,m) \longmapsto P_{(n,m)}$$

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$$\Delta = 2^8\, 5^2$$

$$(\mathbb{Z}/4\mathbb{Z})^2 \longrightarrow E[4](K)$$
$$(n,m) \longmapsto P_{(n,m)}$$

Over $K = \mathbb{Q}_7$ the curve has good reduction.

Indeed, $i, \sqrt{5} \notin \mathbb{Q}_7$, but they are both in the extension $\mathbb{Q}_7(i)$.

Example: Action of the inertia subgroup on the 4-torsion of E

$$
P_{(0,0)} = O \begin{cases}
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\end{cases}
$$

Over $K = \mathbb{Q}_7$,

As $\mathbb{Q}_7(i)$ is unramified, we deduce that $I_{\mathbb{Q}_7(i)/\mathbb{Q}_7} = \{id\}$ and therefore the inertia group acts trivially on $E[4](\overline{\mathbb{Q}_7})$.

Example: Action of the inertia subgroup on the 4-torsion of E

$$
P_{(0,0)} = O
\begin{cases}
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P_{(2,1)} = (2+\sqrt{5}, -5-\sqrt{5}) \\
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\end{cases} \\
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P_{(3,1)} = (-3+2\sqrt{5}, \quad 2i(5-2\sqrt{5}))
\end{cases}
\end{cases}
$$

Over $K = \mathbb{Q}_5$ the curve has multiplicative reduction.

We know that $i \in \mathbb{Q}_5$, but $\sqrt{5} \notin \mathbb{Q}_5$. As the extension $\mathbb{Q}_5(\sqrt{5})/\mathbb{Q}_5$ is ramified, the element $\sigma \in \mathrm{Gal}(\mathbb{Q}_5(\sqrt{5})/\mathbb{Q}_5)$ such that $\sigma(\sqrt{5}) = -\sqrt{5}$ is in $I_{\mathbb{Q}_5}$.

Example: Action of the inertia subgroup on the 4-torsion of E

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\end{cases}
$$

Over the basis $\{(1,0),(0,1)\}$, the action of σ over $(\mathbb{Z}/4\mathbb{Z})^2$ is given by the matrix

$$M_\sigma = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

showing that the action of $I_{\mathbb{Q}_5}$ on $E[4](\overline{\mathbb{Q}_5})$ is unipotent.

Corollaries of the Néron-Ogg-Shaferevich criterion

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Corollary 1

If I_k acts trivially (or unipotently) on one $T_\ell(E)$ then it does so on all of them.

Corollary 2

E has potentially good reduction if and only if I_K acts through a finite quotient on $T_\ell(E)$.

Corollary 3

Isogenous curves have the same reduction type.

Thank you!

Any questions?

Questions

Proof of the Néron-Ogg-Shafarevich criterion I

First, note that I_K acts trivially on $T_\ell(E)$ if and only if it does so on $E[\ell^n](\overline{K})$ for all n. Thus, if E has good reduction then I_K acts trivially on $T_\ell(E)$ by what we've already shown.

Proof of the Néron-Ogg-Shaferevich criterion I

First, note that I_K acts trivially on $T_\ell(E)$ if and only if it does so on $E[\ell^n](\overline{K})$ for all n. Thus, if E has good reduction then I_K acts trivially on $T_\ell(E)$ by what we've already shown.

Conversely, suppose I_K acts trivially on $T_\ell(E)$. Thus all ℓ^n torsion points belong to $E(K^{un})$. Let d be the order of $E(K^{un})/E_0(K^{un})$, which is finite. Then $E_0(K^{un})[\ell^n]$ is the kernel of the map $E(K^{un})[\ell^n] \to E(K^{un})/E_0(K^{un})$, and thus has cardinality at least ℓ^{2n}/d.

Since the reduction map $E_0(K^{un}) \to \overline{E}_{sm}(\overline{K})$ is injective on ℓ-power torsion, it follows that $\overline{E}_{sm}(\overline{k})[\ell^n]$ has cardinality at least ℓ^{2n}/d. But this is not true for G_m (where the cardinality is ℓ^n) or G_a (where the cardinality is 1), and so E cannot have multiplicative or additive reduction. Thus E has good reduction.

Proof of the Néron-Ogg-Shaferevich criterion II

Now suppose that I_K acts unipotently on $T_\ell(E)$. It thus fixes some vector in $T_\ell(E)$, which implies that $E(K^{un})[\ell^n]$ has cardinality at least ℓ^n. Arguing as in the previous slide, we see that $\overline{E}_{sm}$ cannot be G_a, and so E has semi-stable reduction.

Proof of the Néron-Ogg-Shaferevich criterion II

Now suppose that I_K acts unipotently on $T_\ell(E)$. It thus fixes some vector in $T_\ell(E)$, which implies that $E(K^{un})[\ell^n]$ has cardinality at least ℓ^n. Arguing as in the previous slide, we see that $\overline{E}_{sm}$ cannot be G_a, and so E has semi-stable reduction.

Finally, suppose that E has semi-stable reduction. The multiplication-by-ℓ^n map on the smooth locus $\overline{E}_{sm}$ of E is flat, and so $\overline{E}_{sm}[\ell^n]$ is a flat group scheme over R. Let G be the scheme-theoretic closure in $\overline{E}_{sm}[\ell^n]$ of the set of $\overline{K}$-points which extend to $\overline{R}$-points. Then G is finite and flat, and $G_k = \overline{E}_{sm}[\ell^n]$.

Since G has ℓ-power order, it is étale, and so $G(K^{un}) = \overline{E}_{sm}[\ell^n](\overline{k})$, which contains $\mathbb{Z}/\ell^n\mathbb{Z}$ (since E is semi-stable). Thus $E[\ell^n](K^{un})$ contains $\mathbb{Z}/\ell^n\mathbb{Z}$ for all n, which shows that I_K fixes a vector in $T_\ell(E)$. Since the determinant of $T_\ell(E)$ is the ℓ-cyclotomic character, which is trivial on I_K, the result follows.