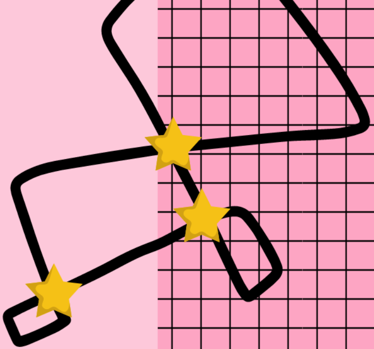
ALVARO GONZALEZ HERNANDEZ

Study group on the Hasse-Weil L-functions and the Weil conjectures



Intersection Theory on Surfaces

Context: zeta functions of schemes

Let X be a scheme of finite type over $\mathbb{F}_q$, let $|X|$ denote the set of closed points of X and let $\mathbb{k}_x$ denote the residue field of X at a point $x \in X$.

Zeta function of X

The **zeta function** of X is defined as

$$Z(X,t) = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}$$

where if $|\mathbb{k}_x| = q^n$, we define the degree of x to be $\deg(x) = n$.

The zeta function of a scheme

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- $Z(X, q^{-s})$ defines an analytic function on compact subsets $U \subset \mathbb{C}$.
- With a change of variables $Z(X,t)$ can be related to the generating function for the numbers $|X(\mathbb{F}_{q^n})|$ of points of X over finite extensions of $\mathbb{F}_q$.

Let's assume that X is a smooth projective curve over $\mathbb{F}_q$ (more generally, let X be a projective scheme over $\mathbb{F}_q$ with $X \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(\overline{\mathbb{F}_q})$ irreducible and non-singular). Then, $Z(X, t)$ has the following properties:

The Weil conjectures

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Rationality

$Z(X,t)$ is the formal power series expansion of a rational function of t.

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Functional equation

$Z(X,t)$ satisfies an identity of the form

$$Z(X, q^{-1}t^{-1}) = q^{1-g}t^{2-2g}Z(X,t)$$

The Weil conjectures

Riemann Hypothesis

$Z(X,t)$ is of the form

$$Z(X,t) = \frac{\prod_{i=1}^{2g}(1-\omega_i t)}{(1-t)(1-qt)}$$

where $|\omega_i| = q^{1/2}$ for all i.

Proving the Riemann Hypothesis

Sketch of the proof of the Riemann Hypothesis

1. Prove that this conjecture is equivalent to a statement about the growth of $|X(\mathbb{F}_{q^n})|$ as $n \to \infty$.

2. Relate the quantity $|X(\mathbb{F}_{q^n})|$ to the intersection number of two divisors of $Y = X \times_{\mathrm{Spec}(\overline{\mathbb{F}_q})} X$ defined as pull-backs of divisors along the Frobenius endomorphism of X.

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Objectives of today

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1. Understand how divisors transform under morphisms between schemes.
2. Discuss how to define an intersection number for curves in surfaces.

Ramification index

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Let $\varphi : Y \longrightarrow Z$ be a finite surjective morphism between two integral schemes of finite type over a field k.

Let $y \in Y$ be a codimension 1 point such that $\varphi(y) = z$ has codimension 1 in Z. We naturally have an inclusion of the discrete valuation rings $\mathcal{O}_{Z,z} \hookrightarrow \mathcal{O}_{Y,y}$.

Ramification index

The **ramification index** at y is defined as

$$e(y) = \begin{cases} 0 & \text{if } z = \varphi(y) \text{ has codimension greater than } 1 \\ v_z(t) & \text{if } z = \varphi(y) \text{ has codimension } 1 \end{cases}$$

where t is a uniformiser of $\mathcal{O}_{Y,y}$ and v_z is the natural valuation defined in $\mathcal{O}_{Z,z}$.

Example of ramification

Let's consider the map $\varphi : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ induced in $\operatorname{Spec}(k[x]) \longrightarrow \operatorname{Spec}(k[t])$ from the ring homomorphism

$$\begin{aligned} \phi\colon\, k[t] &\longrightarrow k[x] \\ t &\longmapsto x^2 \end{aligned}$$

Example of ramification

$$\begin{aligned} \mathrm{Spec}(k[x]) &\longrightarrow \mathrm{Spec}(k[t]) \\ (x) &\longmapsto (t) \\ (x-2) &\longmapsto (t-4) \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{\mathrm{Spec}(k[t]),(t)} = k[t]_{(t)} &\longrightarrow k[x]_{(x)} = \mathcal{O}_{\mathrm{Spec}(k[x]),(x)} \\ t &\longmapsto x^2 \end{aligned}$$

Therefore $e([x]) = v_x(x^2) = 2$.

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Therefore $e([x-2]) = v_{x-2}(x^2-4) = 1$.

Pull-back

For y a point of codimension 1, we denote by $[y] = \overline{\{y\}}$ the corresponding prime divisor.

Pull-backs

Let $\varphi : Y \longrightarrow Z$ as before and let $D = \sum n_z[z]$ be a divisor of Z. The **pull-back** of D along φ is defined as:

$$\varphi^*(D) = \sum n_z\, \varphi^*([z]) = \sum n_z \sum_{\varphi(y)=z} e(y)[y].$$

Example of pullback

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Let $\varphi : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ as before, and let $D = -3[t] + [t-4] \in \text{Div})(\mathbb{A}^1_k)$. Then,

$$\begin{aligned}\varphi^*(D) &= -3\varphi^*([t]) + \varphi^*([t-4]) \\ &= -3e([x])[x] + e([x-2])[x-2] + e([x+2])[x+2] \\ &= -6[x] + [x-2] + [x+2]\end{aligned}$$

Pull-backs

Let $k(Z) = \mathcal{O}_{Z,(0)}$ denote the field of rational functions of Z. Then φ induces a morphism of sheaf of rings $\varphi : \mathcal{O}_Z \to \varphi_* \mathcal{O}_Y$, which, by localising, gives us a map $k(Z) \to k(Y)$.

Let $f \in k(Z)^*$, we will denote by (f) the divisor defined by f.

Proposition

Let $\tilde{f}$ be the image of f under the inclusion $k(Z) \to k(Y)$ induced by φ. Then, $(\tilde{f}) = \varphi^*((f))$.

Pull-backs

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Corollary

Pull-backs send principal divisors to principal divisors and therefore define a map $\mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(Y)$.

In our previous example the induced map between function fields is given by

$$\begin{aligned} k(t) &\longrightarrow k(x) \\ \frac{f(t)}{g(t)} &\longmapsto \frac{f(x^2)}{g(x^2)} \end{aligned}$$

As $\mathrm{Pic}(\mathbb{A}^2) = \{0\}$, the map between $\mathrm{Pic}(\mathbb{A}^2) \to \mathrm{Pic}(\mathbb{A}^2)$ is uninteresting. A similar map over $\mathrm{Pic}(\mathbb{P}^2) \to \mathrm{Pic}(\mathbb{P}^2)$, would give us a group homomorphism $\mathbb{Z} \to \mathbb{Z}$ that can be checked to be multiplication by 2.

Push-forwards

Degree

The **degree** of the morphism φ is the degree of the field extension $k(Y)$ over $k(Z)$, that is,

$$\deg(\varphi) = [k(Y) : k(Z)]$$

Push-forwards

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In our example, $k(t) \hookrightarrow k(x)$ as the set of rational functions over x^2. This shows that $k(x)$ can be understood as the extension of $k(t)$ by an element x satisfying $x^2 - t = 0$. This is a degree 2 extension of $k(t)$, so

$$\deg(\varphi) = [k(x) : k(t)] = 2$$

Push-forwards

Let $\mathbb{k}_y$ and $\mathbb{k}_z$ denote the residue fields of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$ respectively

Push-forward

Let $D = \sum n_y[y] \in \mathrm{Div}(Y)$. Then the push-forward of D by φ is the divisor $D = \sum n_y \varphi_*([y])$ where

$$\varphi_*([y]) = \begin{cases} 0 & \text{if } z = \varphi(y) \text{ has codimension greater than } 1 \\ [\mathbb{k}_z : \mathbb{k}_y][z] & \text{if } z = \varphi(y) \text{ has codimension } 1 \end{cases}$$

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In our example the residue fields of $k[x]_{(x-\alpha)}$, $k[t]_{(t-\beta)}$ are k for all points, so $\varphi_*([x-\alpha]) = [t-\alpha^2]$ for every α.

Push-forwards

Proposition

Let $g \in k(Y)$. Then, $\varphi_*((g)) = (N_{k(Y)/k(Z)}(g))$, where $N_{k(Y)/k(Z)} \colon k(Y) \longrightarrow k(Z)$ denotes the norm map.

Corollary

Push forwards preserve linear equivalence and therefore descend to give a map $\mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Z)$.

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In our example, for instance,

$$\begin{aligned}\varphi_*\left(\left(\tfrac{x}{1+x}\right)\right) &= \left(N_{k(x)/k(t)}\left(\tfrac{x}{1+x}\right)\right) = \left(N_{k(x)/k(t)}\left(\tfrac{x(1-x)}{(1+x)(1-x)}\right)\right) \\ &= \left(N_{k(x)/k(t)}\left(\tfrac{-t}{1-t} + x\,\tfrac{1}{1-t}\right)\right) = \left(\left(\tfrac{-t}{1-t}\right)^2 - t\left(\tfrac{1}{1-t}\right)^2\right) = \left(\tfrac{-t}{1-t}\right)\end{aligned}$$

Combining pull-backs and push-forwards

Combining φ_* and φ^* gives us endomorphisms of $\mathrm{Div}(Y)$ and $\mathrm{Div}(Z)$.

Proposition

The endomorphism $\varphi_*\varphi^*$ of $\mathrm{Div}(Z)$ acts by $D \mapsto \deg(\varphi) \cdot D$.

Why do we need an intersection theory for surfaces?

- Cohomological techniques were shown to be very strong tools to prove analogue results to the Riemann hypothesis for Kahler manifolds.
- Intersection theory represents *something* about the cohomological structure of a variety.

Intuition behind intersection numbers

Given Y a non-singular projective surface, we are interested in finding a way to *count* the number of points of intersection of two irreducible curves $C, D \subset S$ in a **consistent way**.

What is the easiest case of intersection that we can consider?

Transversal intersection

Two smooth curves C and D in Y **intersect transversely at $P \in Y$** if there are regular functions f and g defined in a neighborhood of P in Y, so that C is given locally near P as the zeros of f and D by g, and such that the images of f and g in $\mathcal{O}_{Y,P}$ generate the maximal ideal.

Transversal divisor

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Transversal divisor

Two smooth curves C and D in Y **intersect transversely at $P \in Y$** if there are uniformisers $f \in \mathcal{O}_{C,P}$, $g \in \mathcal{O}_{D,P}$ such that the images of f and g in $\mathcal{O}_{Y,P}$ generate the maximal ideal.

We say that C and D **intersect transversely** if they do so at every point of their intersection.

Intersection numbers

Let Y be a smooth projective surface. Then, there exists a pairing

$$\begin{aligned} \mathrm{Div}(Y) \times \mathrm{Div}(Y) &\longrightarrow \mathbb{Z} \\ (D_1,\, D_2) &\longmapsto D_1 \cdot D_2 \end{aligned}$$

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1. $D_1 \cdot D_2$ is a bilinear and symmetric pairing.
2. $D_1 \cdot D_2$ depends on D_1 and D_2 only up to linear equivalence, i.e.

$$D_1 \overset{\text{lin}}{\sim} D_1 \quad \Longrightarrow \quad D_1 \cdot D_2 = D_1' \cdot D_2$$

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3. If C and D are smooth curves that do not have any common components,

$$C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P$$

where $(C \cdot D)_P = \dim_k \mathcal{O}_{Y,P}/(f, g)$.

Example

In $\mathbb{P}^2$

$$C: \quad ZY^2 = X^3 \qquad\qquad D: \quad ZY = X^2$$

$$\begin{aligned}(C\cdot D)_{[0:0:1]} &= \dim_k \frac{k[X,Y,Z]_{[0:0:1]}}{(ZY - X^3, ZY - X^2)} = \dim_k \frac{k[x,y]_{(0,0)}}{(y - x^2, y^2 - x^3)} \\ &= \dim_k \frac{k[x]_{(0)}}{((x-1)x^3)} = \dim_k \frac{k[x]_{(0)}}{(x^3)} = \dim_k(k \oplus kx \oplus kx^2) = 3\end{aligned}$$

Similarly $(C\cdot D)_{[1:1:1]} = 1$ and $(C\cdot D)_{[0:1:0]} = 2$. Therefore $C \cdot D = 6$.

Case where $Y = \mathbb{P}^2$

On $\mathbb{P}^2$, the divisor class of a divisor is given by its degree.

Bezout's theorem

If two plane algebraic curves C and D of degrees n_1 and n_2 have no component in common, they have $n_1 n_2$ intersection points, counted with their multiplicity,

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Therefore, the intersection pairing in $\mathbb{P}^2$ is given by

$$\begin{aligned} \mathrm{Div}(\mathbb{P}^2) \times \mathrm{Div}(\mathbb{P}^2) &\longrightarrow \mathbb{Z} \\ (D_1, D_2) &\longmapsto \deg(D_1)\deg(D_2) \end{aligned}$$

and therefore, this pairing over $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ is multiplication.

How can we define an intersection pair in a general case?

Intersection number

Let C and D be two curves that intersect transversely. Then,

$$C \cdot D = \deg_C(\mathcal{O}(D)|_C).$$

In general, this allows us to extend the definition of intersection numbers to curves that do not intersect transversely, by generally defining

$$C \cdot D = \deg_C(\mathcal{O}(D)|_C)$$

How can we define an intersection pair in a general case?

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Proposition

$$\deg_C(\mathcal{O}(D)|_C) = \chi(\mathcal{O}_Y(D)|_C) - \chi(\mathcal{O}_C)$$

Proof

By Riemann-Roch, we have that for $\mathcal{O}_{\mathcal{Y}}(\mathcal{D})$ in C, we have the equality

$$\chi(\mathcal{O}_Y(D)) = \deg(\mathcal{O}(D)) + 1 - g(C) = \deg(\mathcal{O}(D)) + \chi(\mathcal{O}_C) \qquad \square$$

Characterisation of the intersection number in terms of χ

Theorem

For any two divisors $D_1, D_2 \in \mathrm{Div}(Y)$,

$$D_1 \cdot D_2 = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_Y(-D_1)) - \chi(\mathcal{O}_Y(-D_2)) + \chi(\mathcal{O}_Y(-D_1 - D_2))$$

Behaviour of intersections under maps

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Proposition

Let $\varphi : Y \to Z$. For $C \in \mathrm{Div}(Y)$ and $D \in \mathrm{Div}(Z)$,

$$C \cdot \varphi^*(D) = \varphi_*(C) \cdot D$$

Numerical equivalence

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Numerical equivalence

We say that two divisors D_1 and D_2 of Y are **numerically equivalent** if $D_1 \cdot C = D_2 \cdot C$ for every $C \in \text{Div}(Y)$.

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Néron-Severi group... according to Raskin

The **Néron-Severi group** of Y, $\mathrm{NS}(Y)$ is defined to be the set of all divisors of Y up to **numerical** equivalence.

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Néron-Severi group

The **Néron-Severi group** of Y, $\mathrm{NS}(Y)$ is defined to be the set of all divisors of Y up to **algebraic** equivalence.

Numerical equivalence

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Numerical equivalence

We say that two divisors D_1 and D_2 of Y are **numerically equivalent** if $D_1 \cdot C = D_2 \cdot C$ for every $C \in \mathrm{Div}(Y)$.

Now...

Let $\mathrm{Num}(Y)$ denote the set of all divisors of Y up to numerical equivalence. Then, $\mathrm{Num}(Y)$ is the quotient of $\mathrm{NS}(Y)$ by its torsion subgroup.

Why do we care about this group?

The intersection form descends to give a nondegenerate symmetric bilinear form $\mathrm{Num}(Y) \times \mathrm{Num}(Y) \to \mathbb{Z}$. In particular, $\mathrm{Num}(Y)$ is a torsion-free abelian group that has a lattice associated to it that gives us lots of information about Y.


That's all Folks!