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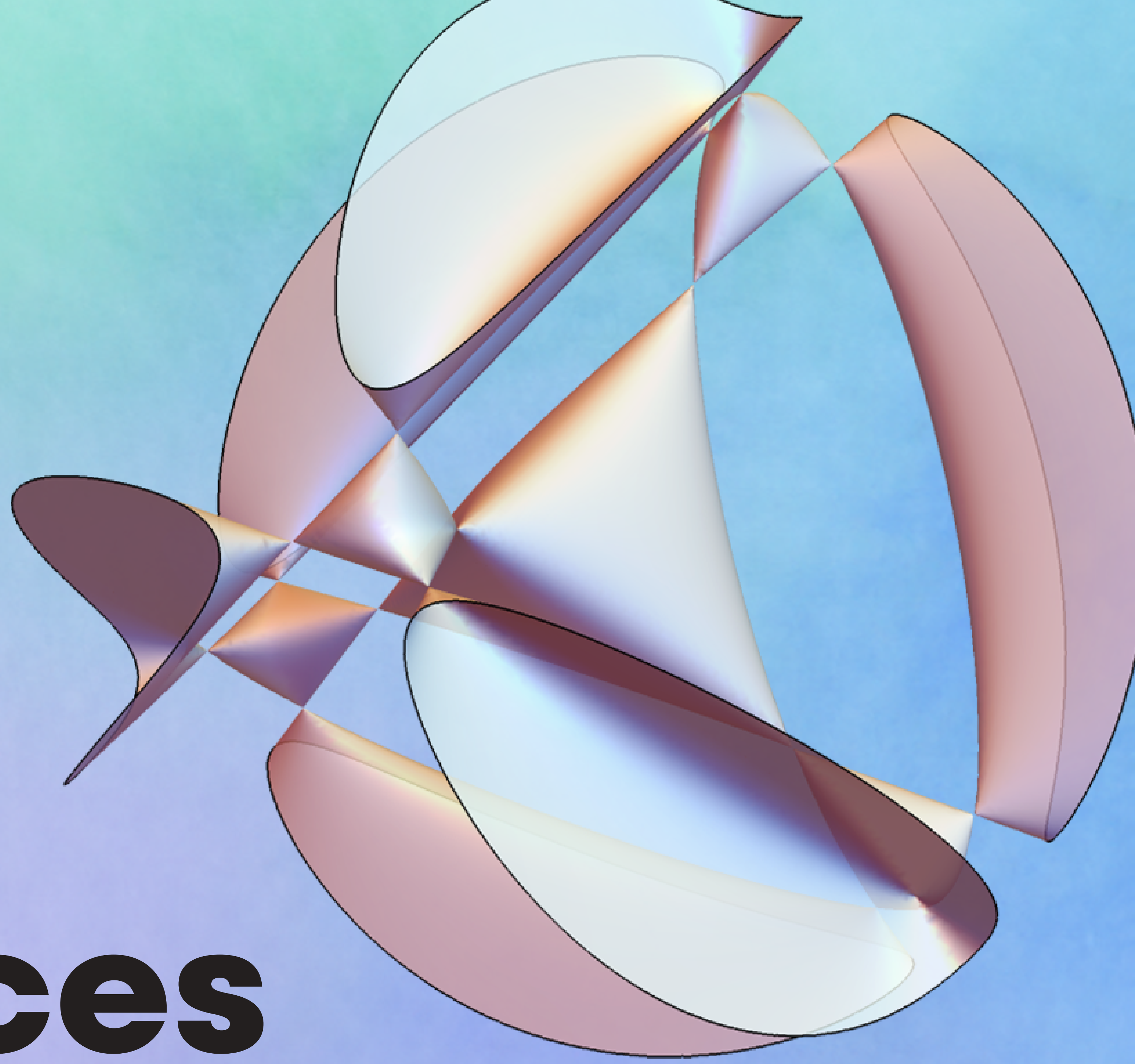
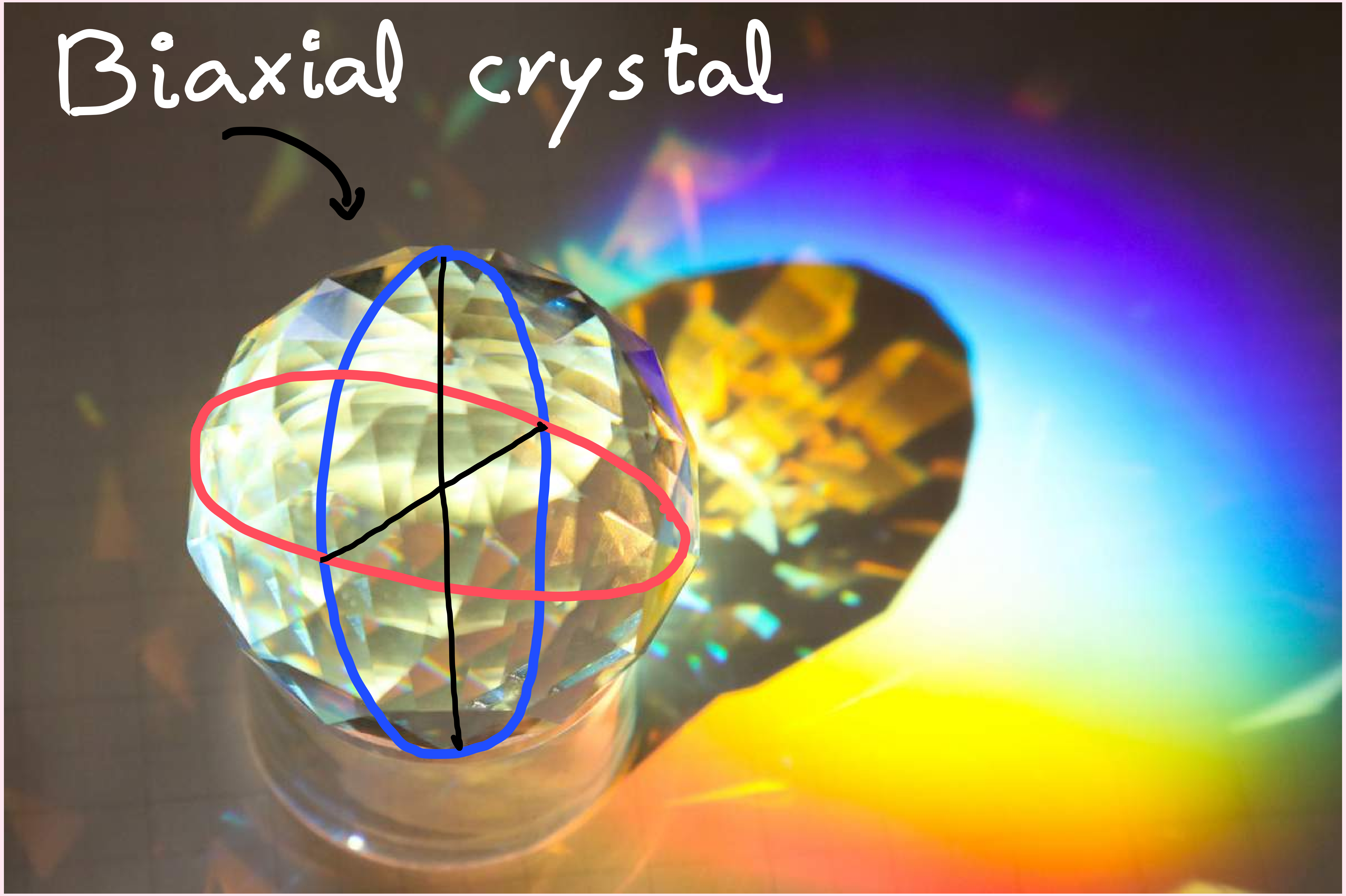


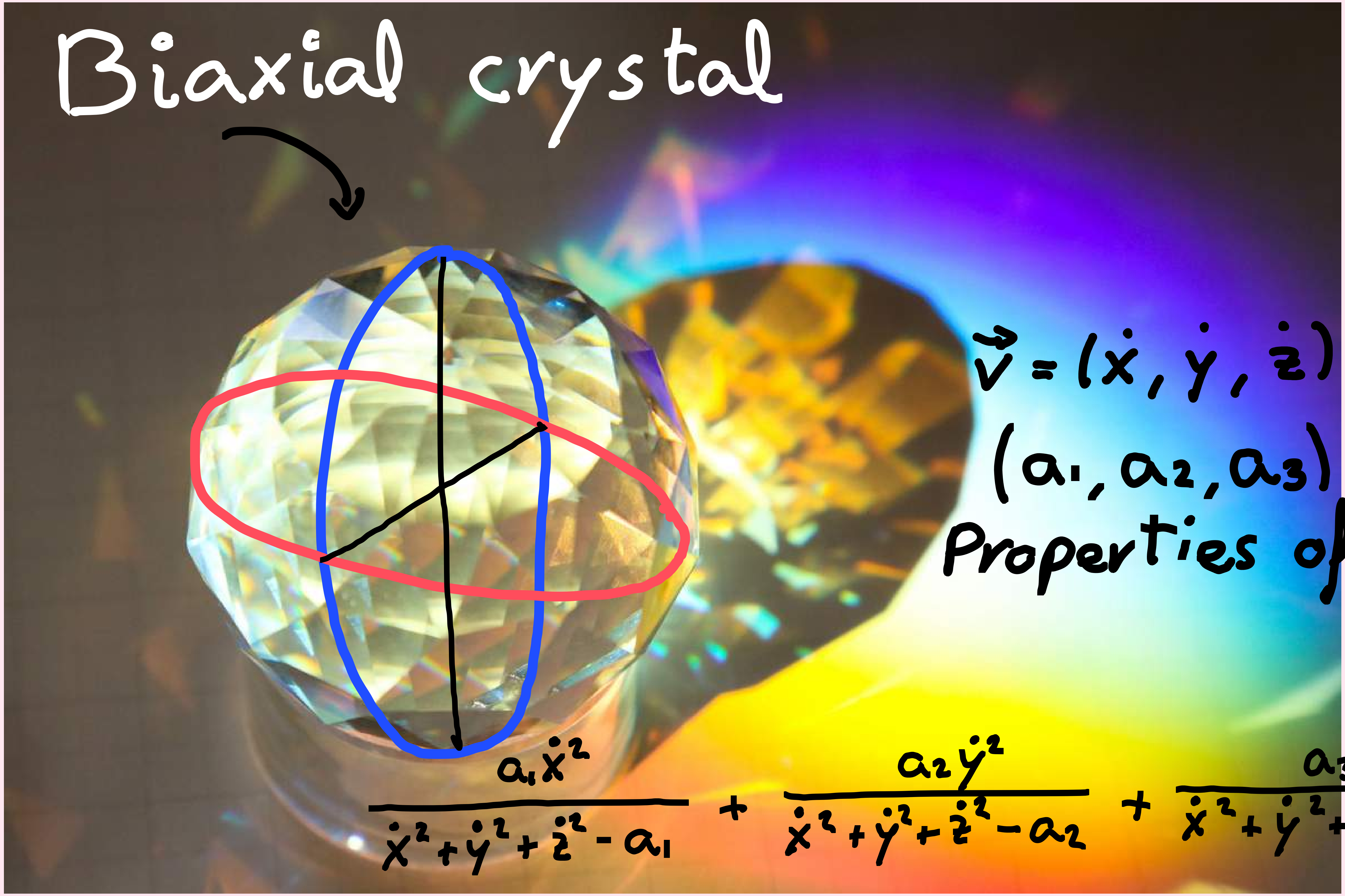
Kummer surfaces

BRIDGING THE GAP BETWEEN GEOMETRY AND NUMBER THEORY

Fresnel (1822)



Fresnel (1822)



Biaxial crystal

$\vec{v} = (\dot{x}, \dot{y}, \dot{z})$

(a_1, a_2, a_3)

Properties of the crystal

$$\frac{a_1 \dot{x}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - a_1} + \frac{a_2 \dot{y}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - a_2} + \frac{a_3 \dot{z}^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - a_3} = 0$$

A bit of Algebra

$$\frac{a_1 \dot{x}^2}{\dot{x}^2+\dot{y}^2+\dot{z}^2-a_1} + \frac{a_2 \dot{y}^2}{\dot{x}^2+\dot{y}^2+\dot{z}^2-a_2} + \frac{a_3 \dot{z}^2}{\dot{x}^2+\dot{y}^2+\dot{z}^2-a_3} = 0$$

Remove denominators + homogenise

$$\boxed{\text{Quartic surface in } \mathbb{P}^3}$$

Hamilton (1833)

The quartic surface has

4 real singularities

$$\left(\pm a_3\sqrt{\frac{a_1^2-a_2^2}{a_1^2-a_3^2}},\ 0,\ \pm a_1\sqrt{\frac{a_2^2-a_3^2}{a_1^2-a_3^2}},\ 1\right)$$

\+ 12 complex singularities

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16 singularities! (isn't that too much?)

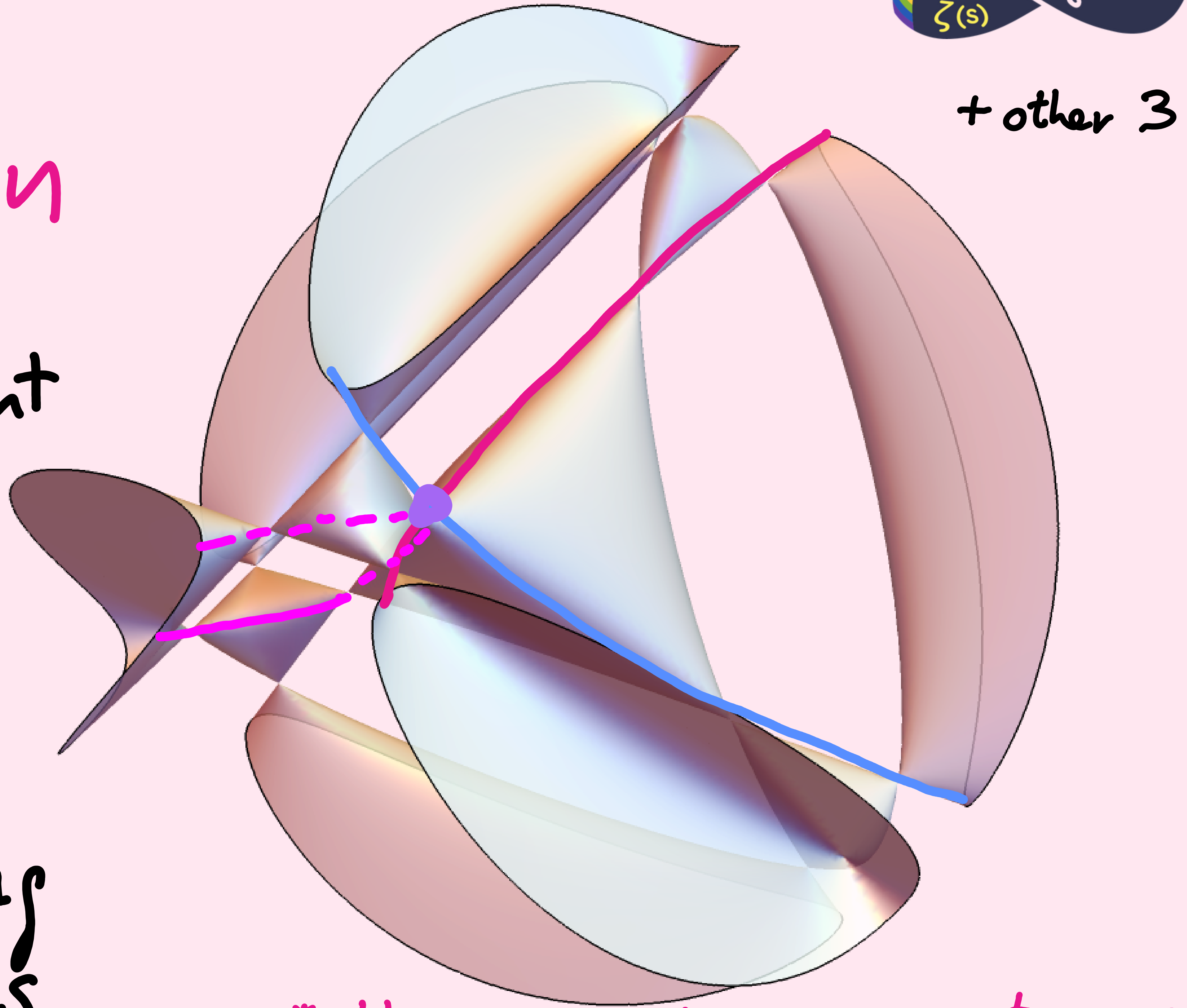
Cauchy, Conley, Darboux, Sylvester... et al

There is something special about this kind of surfaces

1- 16 is the maximum number of singularities that a quartic surface can have

2- There is a 16_6 configuration

There are 16 different planes such that through each singularity there is 6 planes containing 5 other singularities



* these are known as tropes

Kummer (1864)

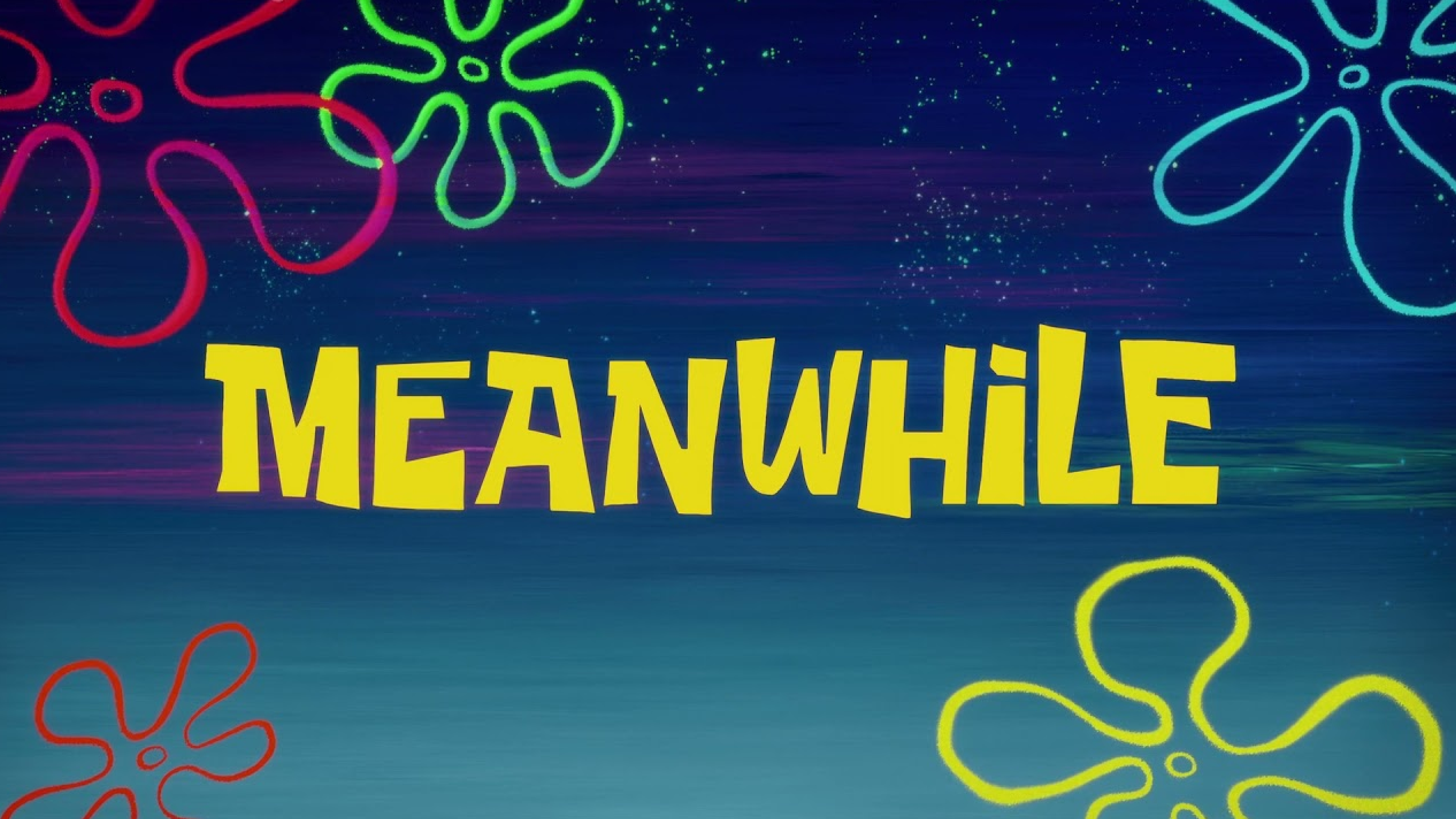
"Any of these surfaces come from a member of the 3-parameter family:

$$(x^2+y^2+z^2+w^2+a(xy+zw)+b(xz+yw)+c(xw+yz))^2+K\,xyzw=0$$

where $K=a^2+b^2+c^2-2abc-1$ "

let's then call them...

"Kummer surfaces"

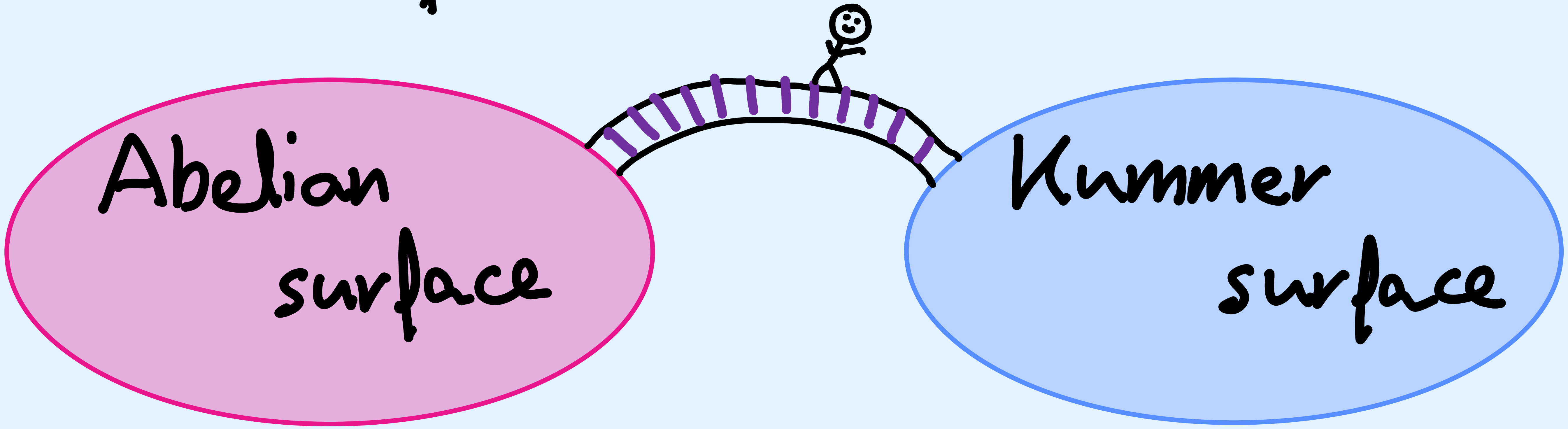

MEANWHILE

RIEMANN (1840s)

Described the theory of Abelian varieties and found out that under certain conditions (principal polarisations) one can find embeddings into projective space by considering special functions (theta functions)

Göpel (1847)

Discovered that some of the theta functions of an Abelian surface satisfied a quartic relation that corresponds to the equation of a Kummer surface



LET'S EXPLORE THE CONNECTION

Let's consider a nice example of Abelian surface, the Jacobian variety of a curve of genus 2

ATTENTION: Hand-waviness upcoming!

A genus 2 curve $\mathcal{C}$ over a field of characteristic $\neq 2, \boxed{3, 5}$ (not really an issue) and algebraically closed can be written as

$$y^2 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

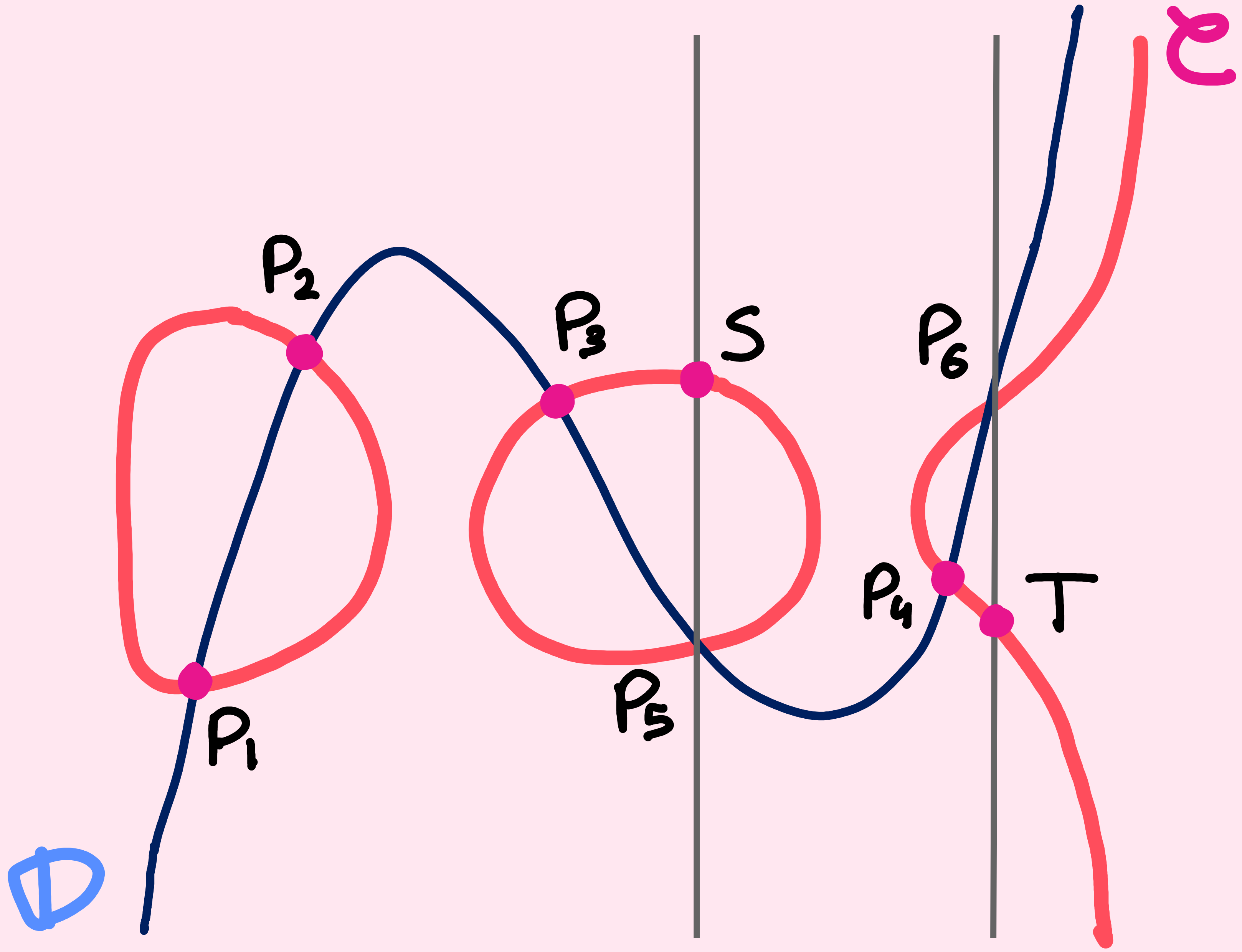
$+$ point of infinity (∞)

These curves have a special hyperelliptic involution

$$\boxed{(x,y) \overset{\iota}{\longleftrightarrow} (x,-y) \quad \text{and} \quad \infty \text{ is fixed}}$$

The Jacobian of $\mathcal{C}$ can be "loosely" described as

$$\mathrm{Jac}(\mathcal{C}) = \{[P_1 + P_2 - 2\infty]\} \qquad P_1 \neq \iota(P_2)$$



$$[P_1 + P_2 - 2\infty] \oplus [P_3 + P_4 - 2\infty] = [S + T - 2\infty]$$

where $\begin{cases} S = \iota(P_5) \\ T = \iota(P_6) \end{cases}$

and all the $P_i \in \mathcal{C} \cap \mathcal{D}$ where $\mathcal{D}$ is the cubic going through $\{P_1, P_2, P_3, P_4\}$

The Projective Embedding of Jac($\mathcal{C}$)

Problem

For a general curve $\mathcal{C}$ there is not an easy description of Jac($\mathcal{C}$) as a projective variety

Why?

To get such embedding we need to consider specific functions in the symmetric product of $\mathcal{C}$ with itself. This is doable but it gives us

AN EMBEDDING IN $\mathbb{P}^{15}$ GIVEN BY THE INTERSECTION OF 72 QUADRICS

SOLUTION

We consider only the functions that are invariant under the involution $\iota : (x,y) \mapsto (x,-y)$

These are 4 functions that satisfy a quartic relation.

Sounds familiar? ... actually...

Kummer surfaces are precisely the quotients of Abelian surfaces by the involution that sends a point to its inverse.

How does this connect with the special traits of the Kummer?

1- 16 singular points

MOTTO IN GIT

FIXED POINTS under a group action $\Longleftrightarrow$ SINGULAR POINTS in the quotient variety

Given that the Kummer surface is the quotient of an Abelian surface by the action that sends a point to its inverse:

FIXED POINTS: Points that are equal to their inverse, i.e. 2-torsion points!

$$\mathrm{Jac}(\mathcal{C})[2]^{*} \cong (\mathbb{Z}/2\mathbb{Z})^{4}$$

$2g$

↳ 16 points!

* the characteristic of the ground field must be $\neq 2$

② – 16 tropes

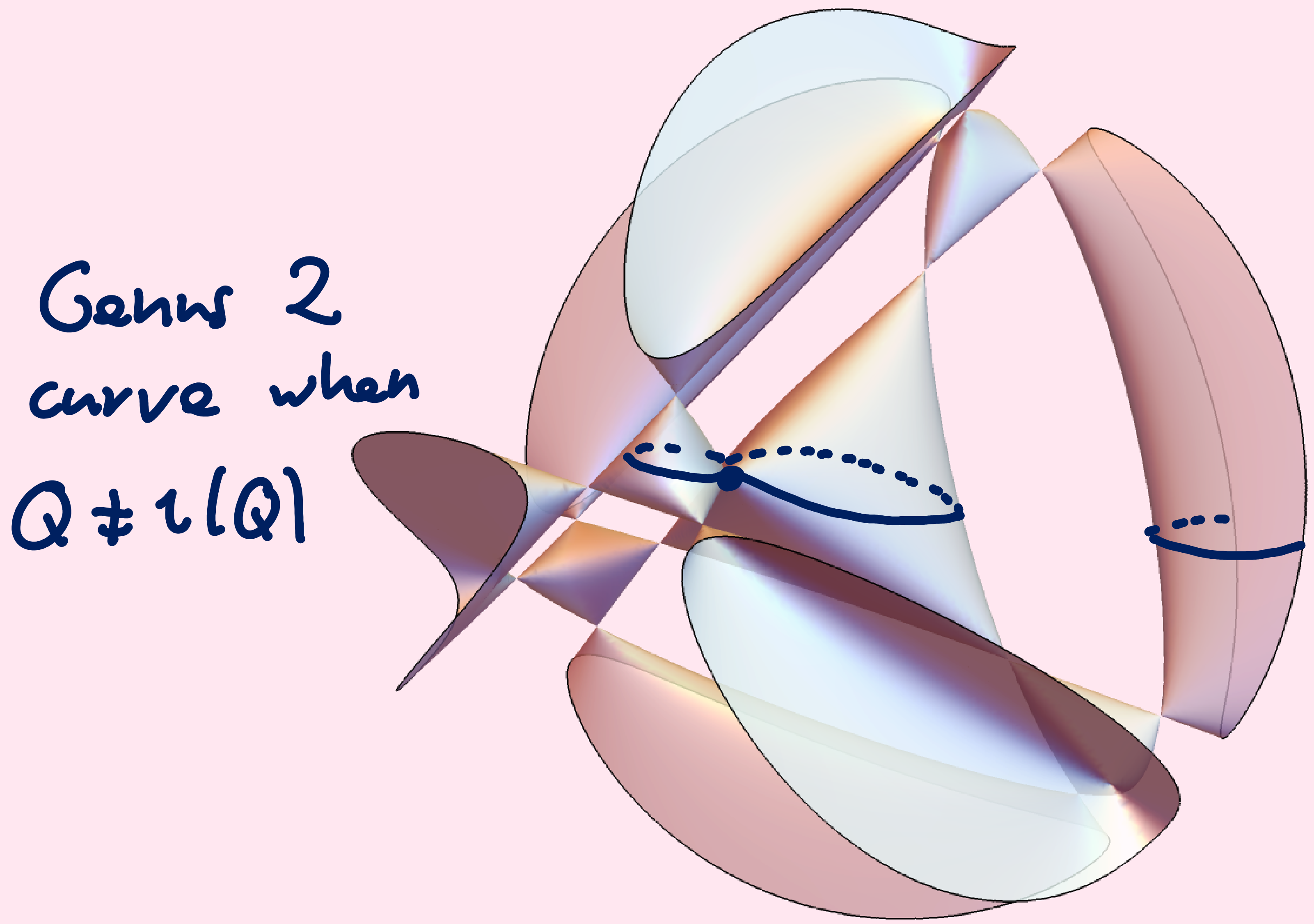
Fix a point $Q \in \mathcal{C}$, we can embed $\mathcal{C} \hookrightarrow \mathrm{Jac}(\mathcal{C})$ through the map $P \longmapsto [P+Q-2\infty]$ (Abel-Jacobi)

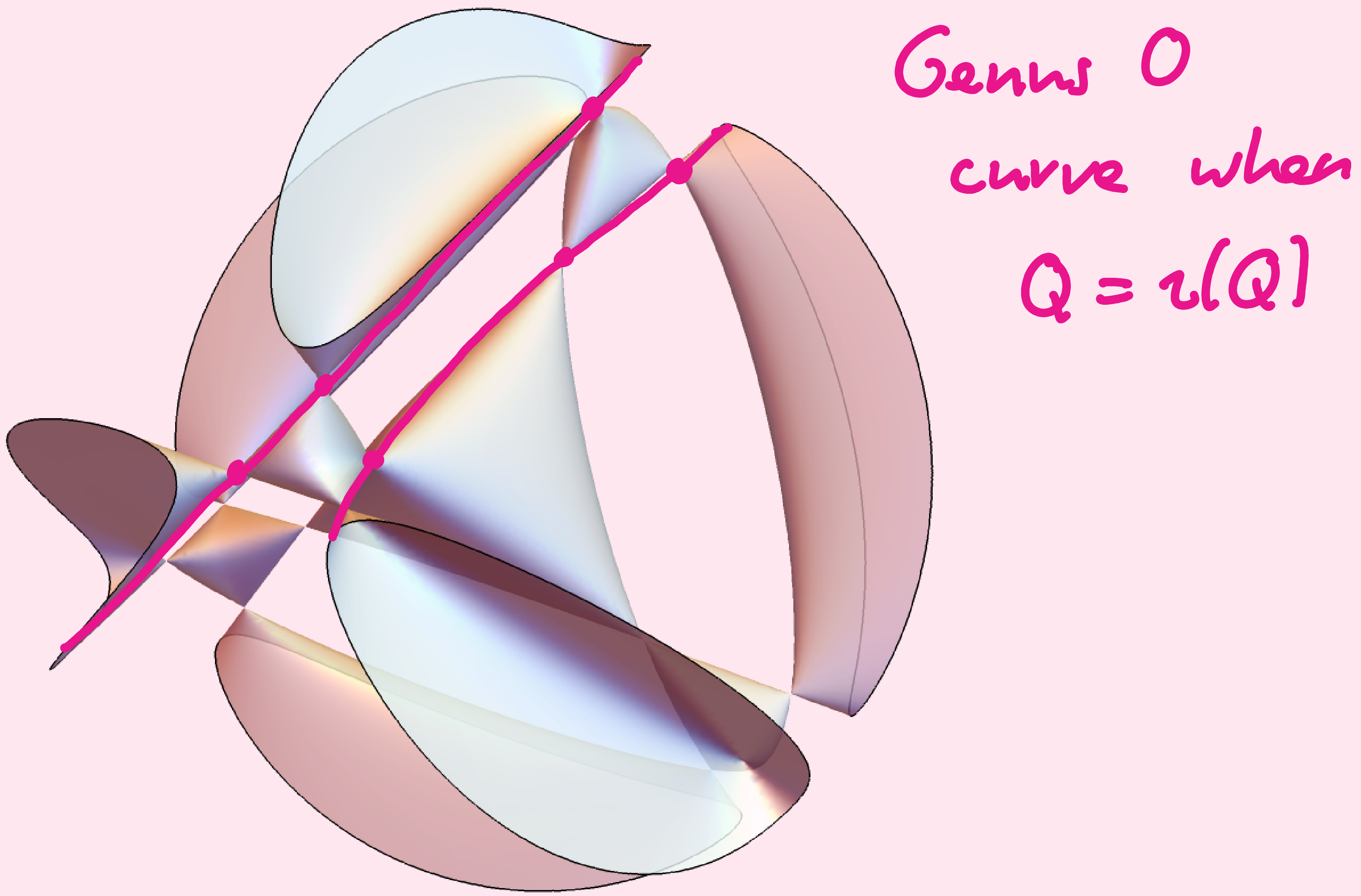
Composing with the quotient, we get a morphism

$$\mathcal{C} \xrightarrow{\varphi_Q} \mathrm{Kum}(\mathcal{C})$$

Let $Q \in \mathcal{C}$ such that $\iota(Q) \neq Q$. Then φ_Q is an embedding.

Now, suppose $Q = \iota(Q)$. Then, it is easy to check that φ_Q has degree 2 and $\varphi_Q(C)$ is one of the tropes.





The Kummer does not have a group law...

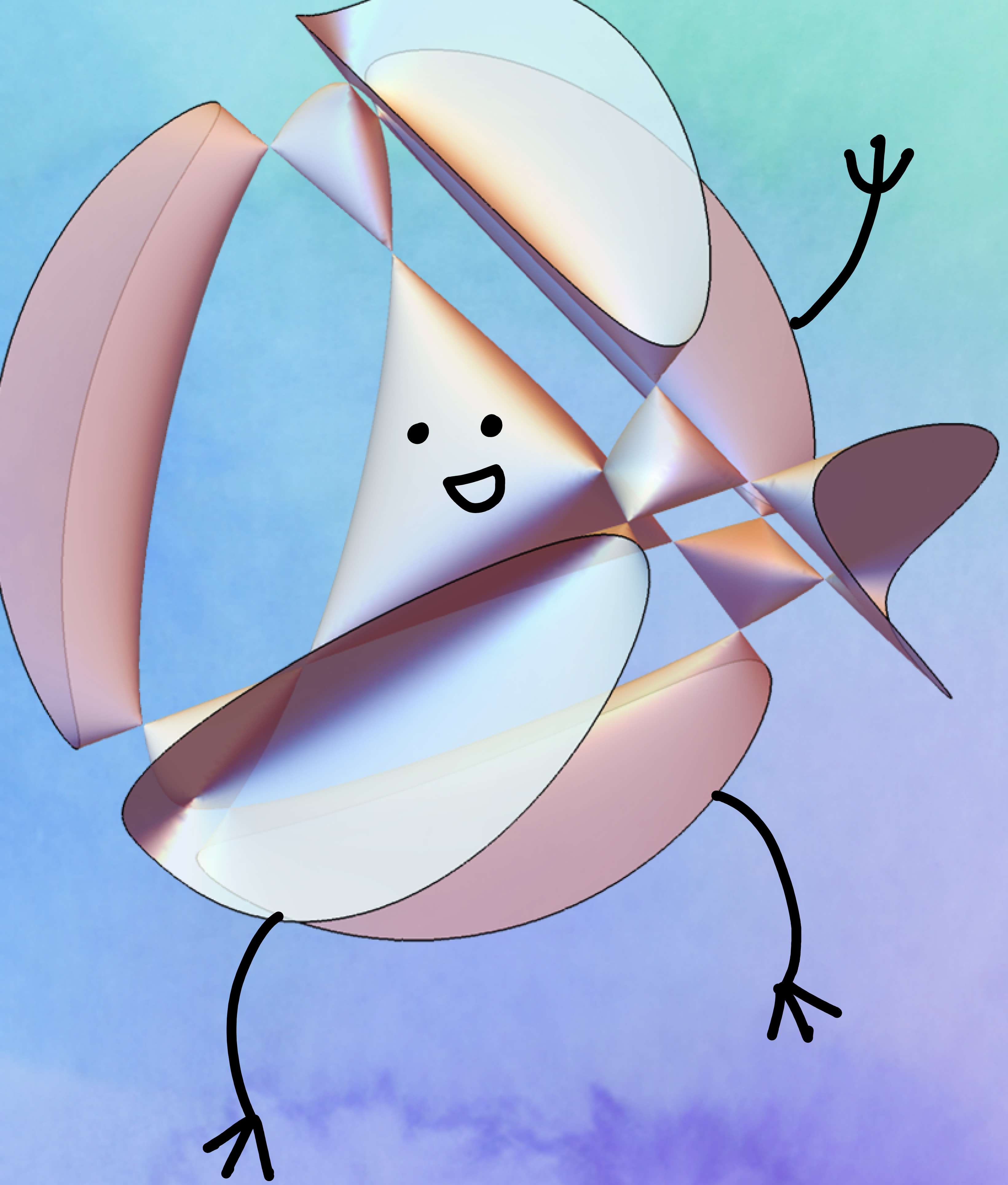
... but we can define linear transformations which correspond to addition by a 2-torsion point in the Abelian surface

The 16 tropes are the orbits of any trope by these linear transformations

What do I do ?

- Kummer surfaces in characteristic 2
- Desingularisation of Kummer surfaces
- Computation of explicit models

(yes, the 72 equations)



Thank you!

Any questions?