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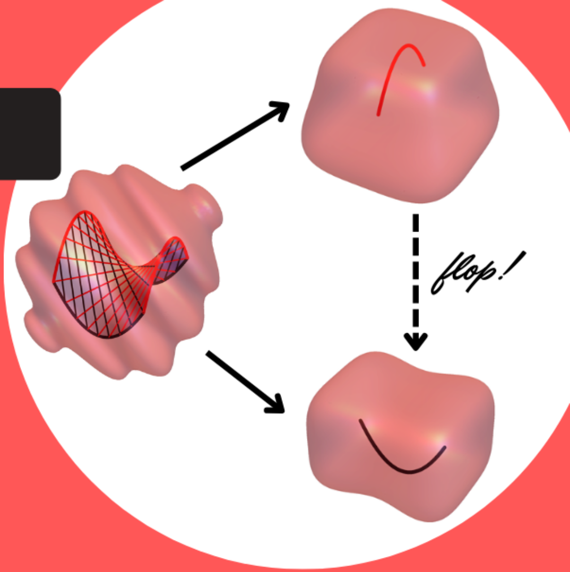
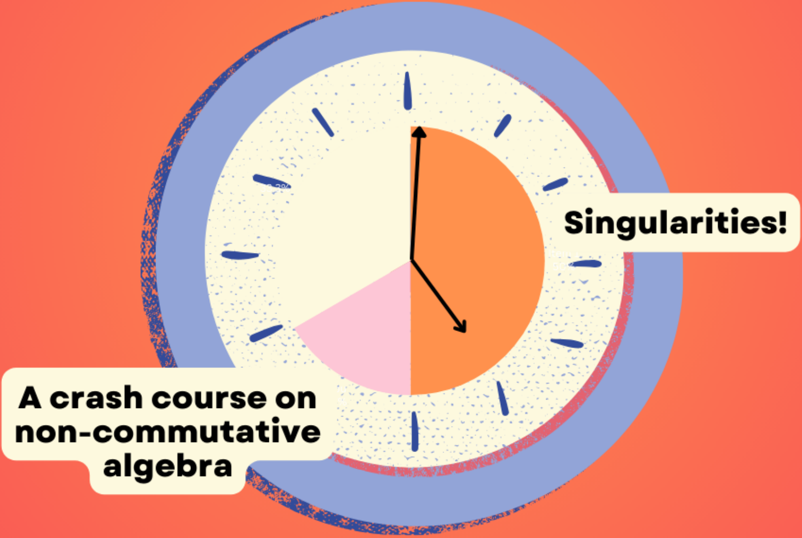


Local normal forms of non-commutative potentials

Plan for the talk



How can we study a singularity?

Analytically

Birationally

Topologically

Du Val singularity

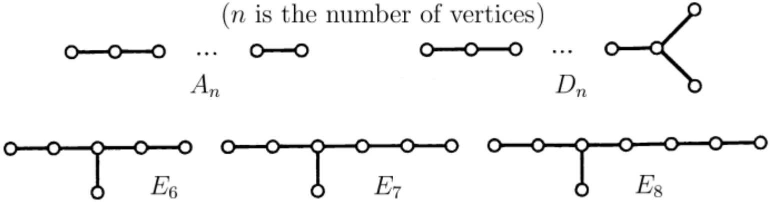
A surface has a **du Val singularity** at a point p, if locally it is analytically isomorphic to $f(x,y,z)=0$ where $f \in \mathbb{C}[[x,y,z]]$ is one of the following:

f		Type
$xy + z^{n+1}$	for $n \geq 1$	A_n
$x^2 + y^2 z + z^{m-1}$	for $m \geq 4$	D_m
$x^2 + y^3 + z^4$		E_6
$x^2 + y^3 + yz^3$		E_7
$x^2 + y^3 + z^5$		E_8

The resolution of a du Val singularity

All du Val singularities can be resolved through a finite number of blow-ups and, furthermore, **all exceptional curves are rational (-2)-curves**.

Their dual intersection diagrams are in correspondence with simply laced Dynkin diagrams:



These du Val singularities are also analytically isomorphic to the quotient of $\mathbb{C}^2$ by the action of a finite group:

Singularity type	Group
$A_n \quad (n \geq 1)$	Cyclic of order $n+1$
$D_m \quad (m \geq 4)$	Binary dihedral of order $4(m-2)$
E_6	Binary tetrahedral
E_7	Binary octahedral
E_8	Binary icosahedral

What can we do next?

Study more complicated singularities on surfaces

Generalise du Val singularities to higher dimensions

1. Compound du Val singularities

Compound du Val singularities

A 3-fold has a **compound du Val (cDV)** singularity at a point p, if locally it is analytically isomorphic to

$$f(x, y, z) + tg(x, y, z, t) = 0$$

where $f \in \mathbb{C}[[x, y, z]]$ defines a du Val singularity and $g \in \mathbb{C}[[x, y, z, t]]$ is arbitrary.

Great source:
A lockdown survey on cDV singularities, Michael Wemyss, `https://arxiv.org/abs/2103.16990`

What happens when we try to analyse cDV singularities?

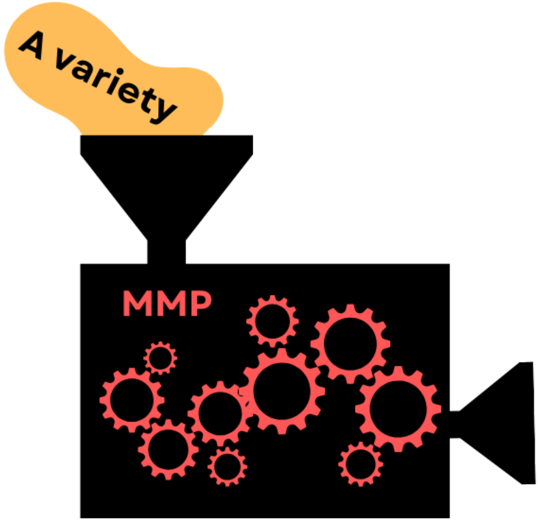
"There is no good and easy 'list' of cDV singularities, there are uncountably many of them, [...] their representation theory is almost always wild, and their birational geometry is more complicated than that for surfaces. Basically, they are just harder (than du Val singularities)."

Why do we still try, then?

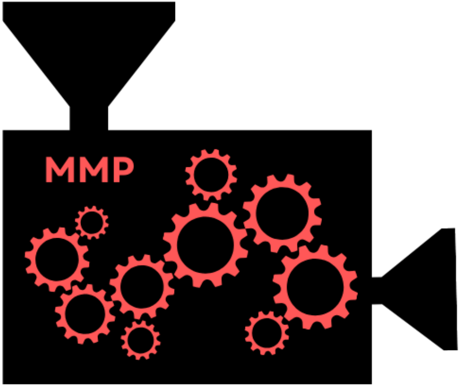
Varieties with cDV singularities are a good testing ground for a very ambitious geometric programme known as the **Minimal Model Programme**.

$$\{\text{Gorenstein terminal}\} \subset \{\text{cDV}\} \subset \{\text{Gorenstein canonical}\}$$

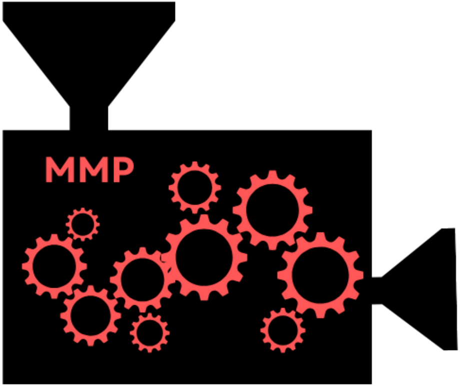
The Minimal Model Programme



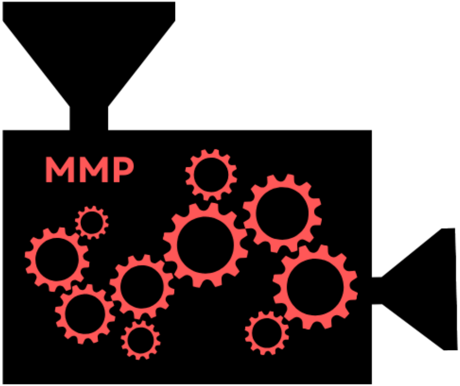
The Minimal Model Programme



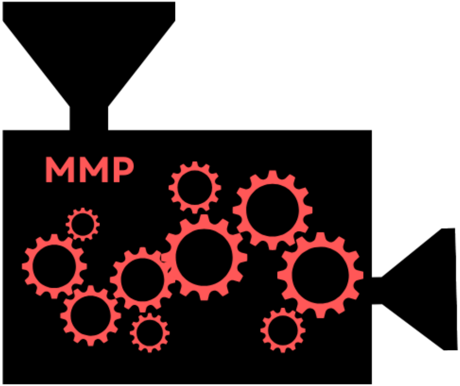
The Minimal Model Programme



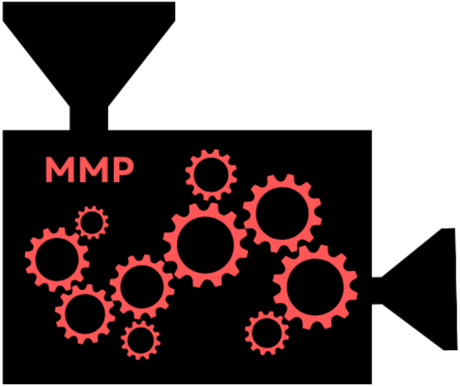
The Minimal Model Programme



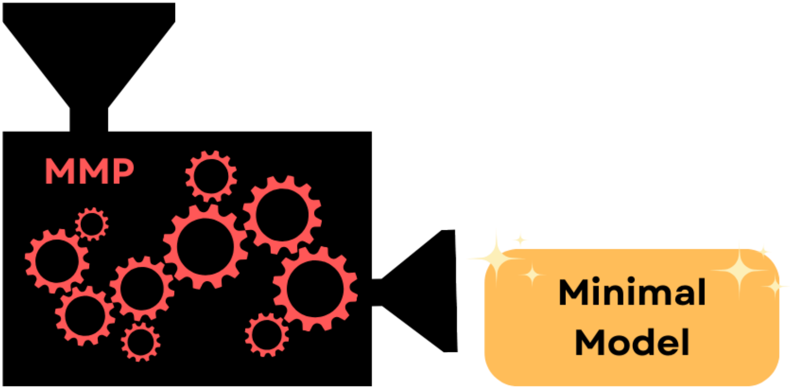
The Minimal Model Programme



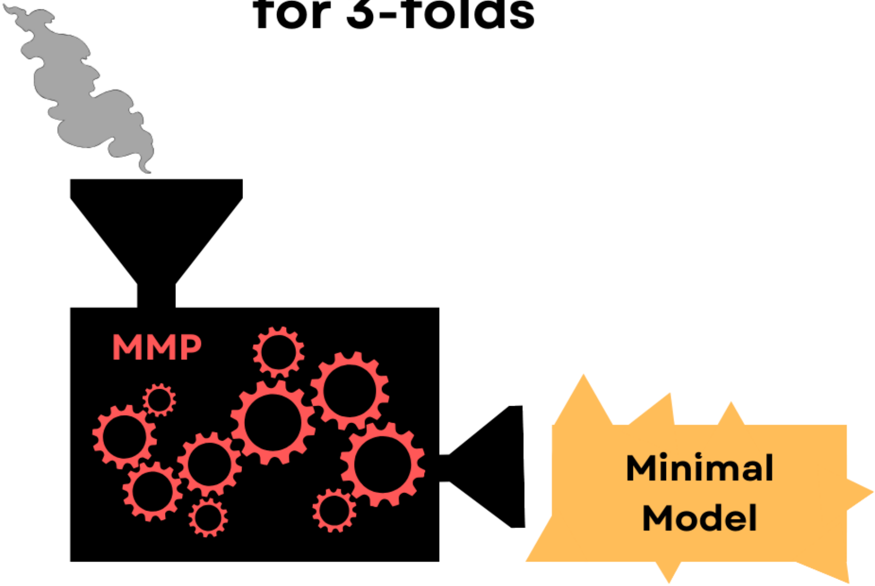
The Minimal Model Programme



The Minimal Model Programme



The Minimal Model Programme for 3-folds



Going up to dimension 3, there are some drawbacks of minimal models:

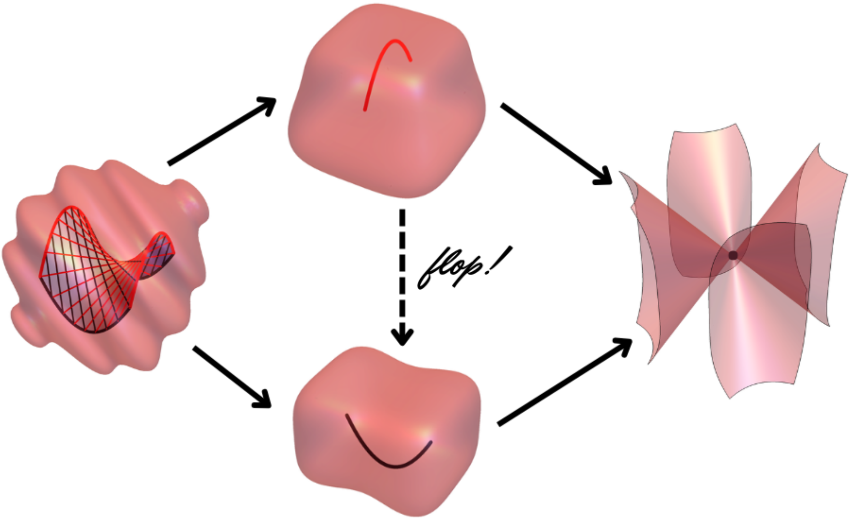
1. Minimal models can be **singular**.
2. Minimal models are not unique, but rather, they are related through a type of birational surgeries known as **flops**.

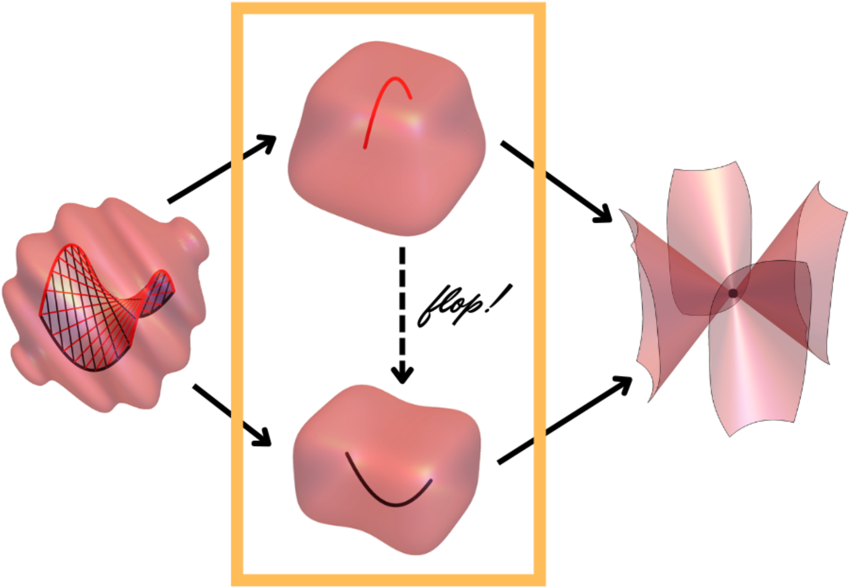
The standard example of a flop (Atiyah flop)

Consider the hypersurface $X \subset \mathbb{C}^4$ given by the equation $xy - uv = 0$ and suppose that we want to resolve the singularity at the origin. We have the 3 following possibilities:

- Blowing-up the origin.
- Blowing-up the line $y = u = 0$.
- Blowing-up the line $x = v = 0$.

Atiyah flop




flop!

Note

In the previous example, by the change of coordinates $u = z - t$, $v = z + t$, at zero the local ring is

$$\mathcal{R} = \frac{\mathbb{C}[[x, y, z, t]]}{xy - z^2 + t^2}$$

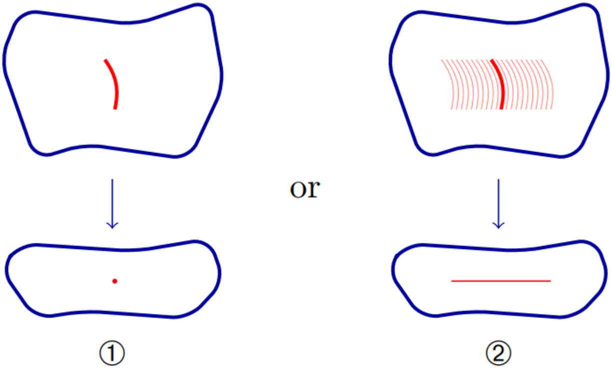
which shows that the singularity is cDV of type cA_1.

A DIFFERENT EXAMPLE

Not all cDV singularities have to be isolated. For instance, consider the variety given by $x^3 - y^3 + z^2 - xyt = 0$.

The singular subscheme of this variety is the line of the t-axis and this is a cD_4 singularity. A crepant resolution would be given by blowing-up the singular locus. The exceptional locus of the transformation would be 2-dimensional.

Contraction types



What happened when we tried to analyse cDV singularities?

"There is no good and easy 'list' of cDV singularities, there are uncountably many of them, [...] their representation theory is almost always wild, and their birational geometry is more complicated than that for surfaces. Basically, they are just harder (than du Val singularities)."

Consider a *crepant* partial resolution

$$f : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{R}$$

with

$$\mathcal{R} = \frac{\mathbb{C}[[x, y, z, t]]}{f + tg}$$

a cDV singularity.

Over the unique closed point p of $\operatorname{Spec} \mathcal{R}$, consider $C := f^{-1}(p)$ equipped with its reduced scheme structure, namely C^{red}.

From the reduced scheme C^{red}, using non-commutative deformation theory, we can construct

$$\begin{tikzcd} C^{red} \text{ in } \mathcal{X} \arrow[r, bend left, "\text{(pro)representing object}"] & \text{an algebra } A^{con} \end{tikzcd}$$

The **contraction algebra** A^{con} is a non-commutative algebra which controls all the birational geometry locally around p, allowing us for instance to tell whether two cDV singularities are non-isomorphic or what are some of their topological invariants.

We can draw the following 3 conclusions

1. cDV singularities are a class of singularities on 3-folds that play a relevant role on the **Minimal Model Programme.**
2. The minimal models of these singularities are susceptible to birational transformations named **flops** that can either come as **curve-to-point** or **divisor-to-curve** contractions.
3. These flops are regulated by mysterious non-commutative objects known as **contraction algebras**.

2. A crash course on non-commutative algebra

In some sense, the theory of contraction algebras resembles the theory of plane curves, but over a non-commutative ring

Commutative side	Non-commutative side
$\mathbb{C}[x, y]$	$\mathbb{C}\langle x, y\rangle$
$\mathbb{C}[[x, y]]$	$\mathbb{C}\langle\langle x, y\rangle\rangle$
$f(x, y) \in \mathbb{C}[[x, y]]$	$f(x, y) \in \mathbb{C}\langle\langle x, y\rangle\rangle$

For curves, a very important object is the local algebra given by the quotient by the Jacobian ideal

$$\frac{\mathbb{C}[[x,y]]}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)},$$

so let's see how to generalise it to the non-commutative case.

Let z_k denote the symbol in a monomial that is in the k-th position, which can either be x or y.

We define the following $\mathbb{C}$-linear homomorphisms acting on monomials of degree t by

$$\partial_x(z_1 z_2 \cdots z_t) = \begin{cases} z_2 \cdots z_t & \text{if } z_1 = x \\ 0 & \text{otherwise} \end{cases}$$

$$\partial_y(z_1 z_2 \cdots z_t) = \begin{cases} z_2 \cdots z_t & \text{if } z_1 = y \\ 0 & \text{otherwise} \end{cases}$$

We also define

$$\mathrm{cyc}(z_1 z_2 \cdots z_t) = \sum_{j=1}^{t} z_j z_{j+1} \cdots z_t z_1 \cdots z_{j-1}$$

Then, the linear derivatives δ_x and δ_y of a potential f are defined to be

$$\delta_x f = \partial_x(\mathrm{cyc}(f))$$
$$\delta_y f = \partial_y(\mathrm{cyc}(f))$$

Let $\mathfrak{n}$ be the two-sided maximal ideal of $\mathbb{C}$ formed by power series with zero constant term. For any subset $S \subset \mathbb{C}\langle\langle x, y\rangle\rangle$, its **closure** is defined to be

$$\overline{S} = \bigcap_{i=0}^{\infty} (S + \mathfrak{n}^i)$$

meaning, $\overline{s} \in \overline{S}$ if and only if for all $i \geq 0$, there exists $s_i \in S$ such that $\overline{s} - s_i \in \mathfrak{n}^i$.
If $(g_1, \ldots, g_n)$ is an ideal, we denote by $((g_1, \ldots, g_n))$ its closure .

Now, in analogy to the commutative case, we can define the **Jacobi algebra** of a potential f to be the quotient

$$\operatorname{Jac}(f) = \frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{((\delta_x f, \delta_y f))}.$$

In the commutative case, the dimension of the corresponding local algebra is the **Milnor number**, which gives us a measure of the the complexity of a curve at the origin:

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}.$$

For most examples of f, $\dim_{\mathbb{C}} \mathrm{Jac}(f)$ is infinite so we would also be interested in developing a relevant notion of dimension for Jacobi algebras.

Let $\mathfrak{J}$ denote the **Jacobson ideal** of $\mathrm{Jac}(f)$.

$$\begin{aligned}\mathrm{Jdim}(\mathrm{Jac}(f)) := \inf\{r \in \mathbb{R} \mid & \text{ for some } c \in \mathbb{R}, \\ & \dim \mathrm{Jac}(f)/\mathfrak{J}^n \leq cn^r \\ & \text{for every } n \in \mathbb{N}\},\end{aligned}$$

This dimension has nice properties such as for instance, that if $\mathrm{Jdim}(\mathrm{Jac}(f)) \leq 1$, either $\mathrm{Jdim}(\mathrm{Jac}(f)) = 0$ or $\mathrm{Jdim}(\mathrm{Jac}(f)) = 1$.

Connections with A^{con}

Characterisation of contraction algebras

If A^{con} is a contraction algebra associated to a crepant resolution of a cDV singularity $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, then, it is the Jacobi algebra of a potential of $\mathbb{C}\langle\langle x, y\rangle\rangle$ with $\mathrm{Jdim}(\mathrm{Jac}(f)) \leq 1$. Furthermore,

1- $\mathrm{Jdim}\, A^{con} = 0$ if and only if $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is a flop.

2- $\mathrm{Jdim}\, A^{con} = 1$ if and only if $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is a divisorial contraction to a curve.

Theorem 1 (Brown, Wemyss)

The only contraction algebras for Type A and D flops are, up to isomorphism, the Jacobi algebras of the following Type A and D potentials:

Name	Local normal form	Conditions
A_n	$x^2 + y^n$	$n \geq 2$
$D_{n,m}$	$xy^2 + x^{2n} + x^{2m-1}$	$n, m \geq 2, m \leq 2n - 1$
$D_{n,\infty}$	$xy^2 + x^{2n}$	$n \geq 2$

Theorem 2 (Brown, Wemyss)

The only contraction algebras for Type A and D divisorial contractions are, up to isomorphism, the Jacobi algebras of the following Type A and D potentials:

Name	Local normal form	Conditions
A_∞	x^2	
$D_{\infty,m}$	$xy^2 + x^{2m-1}$	$m \geq 2$
$D_{\infty,\infty}$	xy^2	

Du Val singularity

A surface has a **du Val singularity** at a point p, if locally it is analytically isomorphic to $f(x,y,z)=0$ where $f \in \mathbb{C}[[x,y,z]]$ is one of the following:

f		Type
$xy + z^{n+1}$	for $n \geq 1$	A_n
$x^2 + y^2 z + z^{m-1}$	for $m \geq 4$	D_m
$x^2 + y^3 + z^4$		E_6
$x^2 + y^3 + yz^3$		E_7
$x^2 + y^3 + z^5$		E_8

Normal forms of du Val singularities in positive characteristic

In characteristic $p = 5$.

name	normal form for $f \in k[[x,y,z]]$	
A_n, D_n, E_6, E_7	*classical forms*	
E_8	E_8^0	$z^2 + x^3 + y^5$
	E_8^1	$z^2 + x^3 + y^5 + xy^4$

In characteristic $p = 3$.

name	normal form for $f \in k[[x,y,z]]$	
A_n, D_n	*classical forms*	
E_6	E_6^0	$z^2 + x^3 + y^4$
	E_6^1	$z^2 + x^3 + y^4 + x^2y^2$
E_7	E_7^0	$z^2 + x^3 + xy^3$
	E_7^1	$z^2 + x^3 + xy^3 + x^2y^2$
E_8	E_8^0	$z^2 + x^3 + y^5$
	E_8^1	$z^2 + x^3 + y^5 + x^2y^3$
	E_8^2	$z^2 + x^3 + y^5 + x^2y^2$

In characteristic $p = 2$.

name	normal form for $f \in k[[x,y,z]]$		
A_n		$z^{n+1} + xy$	
D_{2m}	D_{2m}^0	$z^2 + x^2y + xy^m$	$m \geq 2$
	D_{2m}^r	$z^2 + x^2y + xy^m + xy^{m-r}z$	$m \geq 2,\ 1 \leq r \leq m-1$
D_{2m+1}	D_{2m+1}^0	$z^2 + x^2y + y^mz$	$m \geq 2$
	D_{2m+1}^r	$z^2 + x^2y + y^mz + xy^{m-r}z$	$m \geq 2,\ 1 \leq r \leq m-1$
E_6	E_6^0	$z^2 + x^3 + y^2z$	
	E_6^1	$z^2 + x^3 + y^2z + xyz$	
E_7	E_7^0	$z^2 + x^3 + xy^3$	
	E_7^1	$z^2 + x^3 + xy^3 + x^2yz$	
	E_7^2	$z^2 + x^3 + xy^3 + y^3z$	
	E_7^3	$z^2 + x^3 + xy^3 + xyz$	
E_8	E_8^0	$z^2 + x^3 + y^5$	
	E_8^1	$z^2 + x^3 + y^3 + xy^3z$	
	E_8^2	$z^2 + x^3 + y^5 + xy^2z$	
	E_8^3	$z^2 + x^3 + y^5 + y^3z$	
	E_8^4	$z^2 + x^3 + y^5 + xyz$	

How to get a classification for all local normal forms of non-commutative potentials in positive characteristic?

Thank you!

Any questions?