Rigidity and the construction of the dual abelian variety

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Study group on abelian varieties

October 17, 2023

1 Introduction

The main sources for this talk are the notes of James S. Milne [Mil08a] and the book of David Mumford [Mum70] on abelian varieties.

Definition 1.1. *Recall that an* ***abelian variety*** *is*

- *A* ***group variety****, i.e. an algebraic variety A over k endowed with regular maps*

$$\begin{aligned} + : \quad & A \times_k A \longrightarrow A, \\ \mathrm{inv} : \quad & A \longrightarrow A, \\ 0 : \quad & \mathrm{Spec}(k) \longrightarrow A. \end{aligned}$$

  behaving like the ***addition****, the map that sends any element to its* ***inverse*** *and the* ***identity element*** *0 of an additive group, respectively, in a way that is consistent with the group axioms.*

- ***Geometrically connected*** *i.e. it remains connected when we extend scalars to the algebraic closure.*

- ***Complete*** *i.e. for any variety B, the projection morphism $A \times B \to B$ is a closed map (i.e. maps closed sets onto closed sets).*

For a point $a \in A$, we can define an isomorphism $t_a : A \to A$ called the **right translation by** a as the composite

$$A \xrightarrow{1\times a} A \times A \xrightarrow{+} A$$

$$x \longmapsto (x, a) \longmapsto x + a$$

2 The rigidity theorem

A surprising fact about abelian varieties is that their group law is commutative. This is a consequence from the fact that regular maps from complete varieties are in some sense *rigid*, as demonstrated by the following theorem:

Theorem 2.1 (**Rigidity Theorem**). *Let $\alpha : V \times W \to U$ be a regular map and assume that the variety V is complete and that $V \times W$ is geometrically irreducible. If there exist points $u_0 \in U(k)$, $v_0 \in V(k)$ and $w_0 \in W(k)$ such that*

$$\alpha(V \times \{w_0\}) = \{u_0\} = \alpha(\{v_0\} \times W)$$

then,

$$\alpha(V \times W) = \{u_0\}.$$

An intuitive explanation of this statement is that if we are able to find a map α that contracts a copy of V and a copy of W to the same point $\{u_0\}$, then, α contracts the whole space into that same point.

Proof. Since the hypotheses continue to hold after extending scalars from k to $\bar{k}$, we can assume k is algebraically closed.

Let U_0 be an open affine neighbourhood of u_0 and let $\pi_2 : V \times W \to W$ be the projection into the second component. Let Z denote the set of second coordinates of points of $V \times W$ not mapping into U_0. This is defined as $Z := \pi_2(\alpha^{-1}(U \setminus U_0))$ and it is closed in W, as V is complete.

A point $w \in W$ lies outside of Z if and only if $\alpha(V \times \{w\}) \subset U_0$. In particular, w_0 lies outside of Z so $W \setminus Z$ is non-empty.

It is a fact that whenever we have a regular map from a complete and connected variety into an affine variety, the image is always a point [Mil08b, Theorem 7.5]. As $V \times \{w\}$ is complete and connected (it is irreducible) and U_0 is affine, $\alpha(V \times \{w\})$ must be a point whenever $w \in W \setminus Z$. In fact,

$$\alpha(V \times \{w\}) = \alpha(v_0, w) = \{u_0\}.$$

As a consequence, we deduce that α is constant on $V \times (W \setminus Z)$. As $V \times (W \setminus Z)$ is non-empty and open in $V \times W$ and $V \times W$ is irreducible, $V \times (W \setminus Z)$ is dense in $V \times W$. As U is separated, α must agree with the constant map on the whole of $V \times W$. □

A corollary of this theorem is the following:

Corollary 2.2. *Every regular map $\alpha : A \to B$ of abelian varieties is the composite of a homomorphism with a translation.*

Proof. Let $b = \alpha(0)$. Then, $\alpha = t_b \circ \alpha_0$ where $\alpha_0(0) = 0$, and the only thing left that we need to show is that α_0 is a homomorphism. Consider the map

$$\begin{aligned} \varphi : A \times A &\longrightarrow B \\ (a, a') &\longmapsto \alpha_0(a + a') - \alpha_0(a) - \alpha_0(a') \end{aligned}$$

The goal is to show that φ is constantly 0. It is easy to see that φ is a regular map as it is the difference of the two regular maps $(\alpha_0 \circ +)$ and $(+ \circ (\alpha_0 \times \alpha_0))$ in the following commutative diagram:

$$\begin{array}{ccc} A \times A & \xrightarrow{+} & A \\ \downarrow{\scriptstyle \alpha_0 \times \alpha_0} & & \downarrow{\scriptstyle \alpha_0} \\ B \times B & \xrightarrow{+} & B \end{array}$$

As $\varphi(A \times 0) = 0 = \varphi(0 \times A)$, from the previous theorem, we deduce that $\varphi = 0$ and α_0 is a homomorphism. □

It is an easy exercise to the reader to show the fact that commutative groups are characterised by the property that the map inv that sends every element to its inverse is a homomorphism. As $\mathrm{inv}(0) = 0$, from the last corollary we deduce that inv is a homomorphism and therefore,

Corollary 2.3. *The group law on an abelian variety is commutative.*

Now that we have proven one of the main properties of abelian varieties, we can start working on setting the foundations to prove another important result, which is the **existence of a dual abelian variety**.

3 Invertible sheaves

These definitions can all be found in chapter 13 of Milne's notes on algebraic geometry [Mil08b].

Suppose we have an affine variety $V = \mathrm{Spec}(R)$ and let M be a finitely generated R-module. Then, there is a unique sheaf of $\mathcal{O}_V$-modules $\mathcal{M}$ on V such that for all $f \in R$,

$$\Gamma(D(f), \mathcal{M}) = R_f \otimes_R M,$$

where R_f is the localisation of R at f.

This $\mathcal{O}_V$-module $\mathcal{M}$ is said to be **coherent**.

There is, in fact, a **fully faithful functor**, from the category of finitely generated R-modules to the category of coherent $\mathcal{O}_V$-module defined by $M \mapsto \mathcal{M}$ with a quasi-inverse $\mathcal{M} \mapsto \Gamma(V, \mathcal{M})$.

Now, consider a (not necessarily affine variety) V. An $\mathcal{O}_V$-module $\mathcal{M}$ is said to be **coherent** if, for every open affine subset U of V, $\mathcal{M}|_U$ is coherent.

The easiest example of a coherent $\mathcal{O}_V$-module is $\mathcal{O}_V^n$. For that reason, it is interesting to study the modules that look, at least locally like $\mathcal{O}_V^n$.

Definition 3.1. *An $\mathcal{O}_V$-module is said to be **locally free of rank** n if it is locally isomorphic to $\mathcal{O}_V^n$, that is, if every point $P \in V$ has an open neighbourhood such that $\mathcal{M}|_U \cong \mathcal{O}_V^n$.*

It is important to remark that every locally free $\mathcal{O}_V$-module of rank n is coherent.

Among this modules, we will mostly work with ones with a particular rank:

Definition 3.2. *An **invertible sheaf** on V is a locally free $\mathcal{O}_V$-module $\mathcal{L}$ of rank 1.*

A nice property of these sheaves is that the tensor product of any two invertible sheaves is again an invertible sheaf. This allows us to endow the set of isomorphism classes of invertible sheaves with a product structure given by $[\mathcal{L}] \cdot [\mathcal{L}'] := [\mathcal{L} \otimes \mathcal{L}']$. This product structure is associative and commutative (because tensor products are associative and commutative, up to isomorphism) and $[\mathcal{O}_V]$ is the identity element.

If we define

$$\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_V)$$

it is easy to check that if $\mathcal{L}$ is invertible, $\mathcal{L}^\vee$ is also an invertible sheaf. Moreover, there is a canonical isomorphism

$$\begin{aligned} \mathcal{L}^\vee \otimes \mathcal{L} &\longrightarrow \mathcal{O}_V \\ (f, x) &\longmapsto f(x) \end{aligned}$$

from which we deduce that $[\mathcal{L}^\vee] \cdot [\mathcal{L}] = [\mathcal{O}_V]$.

From these remarks, we deduce that the set of isomorphism classes of invertible sheaves on V form a group known as the **Picard group**, $\mathrm{Pic}(V)$ of V.

Some people may have already seen another definition of the Picard group of V linked to the group of divisors of V. We will now see what is the connection between these two definitions in the case where V is irreducible and smooth.

3.1 Connection with the group of divisors

Definition 3.3. *Let V be an irreducible and smooth variety. For a divisor D on V we can define the **Riemann-Roch space** associated to D as the vector space*

$$L(D) = \{f \in k(V)^\times \mid \operatorname{div}(f) + D \geq 0\}.$$

We can then define a sheaf $\mathcal{L}(D)$ on V by defining the sections for each open subset U as

$$\Gamma(U, \mathcal{L}(D)) = \{f \in k(V)^\times \mid \operatorname{div}(f) + D \geq 0 \text{ on } U\} \cup \{0\}.$$

With a bit of work, one can show that $\mathcal{L}(D)$ is an invertible sheaf and that the canonical map

$$\begin{aligned} \mathcal{L}(D) \otimes \mathcal{L}(D') &\longmapsto \mathcal{L}(D + D') \\ f \otimes g &\longmapsto fg \end{aligned}$$

is an isomorphism. Therefore, there is a homeomorphism

$$\begin{aligned} \operatorname{Div}(V) &\longrightarrow \operatorname{Pic}(V) \\ D &\longmapsto [\mathcal{L}(D)] \end{aligned}$$

which is zero on $\operatorname{PDiv}(V)$, the set of principal divisors of V. In fact, this same map defines an isomorphism

$$\begin{aligned} \operatorname{Div}(V)/\operatorname{PDiv}(V) &\longrightarrow \operatorname{Pic}(V) \\ [D] &\longmapsto [\mathcal{L}(D)] \end{aligned}$$

Defining the inverse of this map, is difficult and the construction is not necessarily explicit, so whenever we are working with smooth varieties (and this is indeed our case, as abelian varieties are smooth), it is often easier to work with elements in $\operatorname{Pic}(V)$ in terms of divisors.

4 Pic^0 and the definition of the dual abelian variety

Probably the most famous construction of an abelian variety is the Jacobian variety associated to a (smooth) curve C, which I will now explain.

Given a divisor D of a smooth variety C, we can associate a quantity known as the **degree** of D defined as the sum of the coefficients of the prime divisors of D. It is easy to check that the degree induces a well-defined group homomorphism

$$\begin{aligned} \deg : \mathrm{Pic}(C) &\longrightarrow \mathbb{Z} \\ [D] &\longmapsto \deg(D) \end{aligned}$$

whose kernel is

$$\mathrm{Pic}^0(C) = \{[D] \in \mathrm{Pic}(V) \mid \deg(D) = 0\}.$$

The Jacobian of C is defined to be

$$\mathrm{Jac}(C) = \mathrm{Pic}^0(C).$$

For an abelian variety A over k we want to be able to replicate this process: to find a well-defined notion of $\mathrm{Pic}^0(A)$ in such a way that we can canonically construct another abelian variety $\hat{A}$ such that we have that

$$\hat{A}(k) = \mathrm{Pic}^0(A).$$

This is what we will call the **dual abelian variety** of A, and its construction and properties will be discussed in later talks.

References

[Mil08a] James S. Milne. Notes on Abelian Varieties, 2008.

[Mil08b] James S. Milne. Notes on Algebraic Geometry, 2008.

[Mum70] David Mumford. *Abelian Varieties*. London: Oxford University Press, 1970.