Last time:

μ_n satisfies LDP $\quad (E, d)$ metric space, polish

- A "nice" $\quad \mu_n(A) = \exp\left(-n \inf_{x \in A} I(x)\right)$

rigorous : I "good rate function"
I has compact sublevel sets

- lower bound on open sets
- upper bound on closed sets.

Schilder Thm: $\sqrt{\varepsilon} B_t$, $t \in [0,1]$ satisfy LDP with respect to $C([0,1])$ with rate function

$$I(x) = \begin{cases} \frac{1}{2} \int_0^1 \dot{x}(s)^2 \, ds & \text{if } \ldots \\ +\infty & \text{else} \end{cases}$$

proof: (1) Upper bound :

$$\overline{\lim_{\varepsilon \downarrow 0}} \; \varepsilon \log \mathbb{P}\left[\operatorname{dist}(\sqrt{\varepsilon} B_t, \Phi(R)) > \delta\right] \leq -R$$

$$\Phi(R) = \{ x : I(x) \leq R \}$$

B_t^n = piecewise linearisation

$$\mathbb{P}\left[\| \sqrt{\varepsilon} B^n - \sqrt{\varepsilon} B \|_\infty > \delta\right] \overset{\circledast}{\leq} n \, \mathbb{P}\left[\sup_{0 \leq t \leq 1} |B_t| \geq \frac{\sqrt{n}\, \delta}{2\sqrt{\varepsilon}}\right]$$

Fernique: $\exists \lambda > 0$ $\quad \mathbb{E}\left(\exp\left(\lambda \sup_{t \in [0,1]} |B_t|^2\right)\right) < \infty$

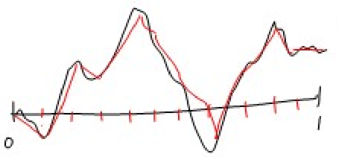
$$\circledast = n \, \mathbb{P}\left(\exp\left(\lambda \left(\sup_{0 \leq t \leq 1} |B_t|\right)^2\right) \geq \exp\left(\lambda \left(\frac{\sqrt{n}\,\delta}{2\sqrt{\varepsilon}}\right)^2\right)\right)$$

$$\leq n \cdot \mathbb{E}\left(\exp\left(\lambda \|B_t\|_\infty^2\right)\right) \exp\left(-\frac{\lambda n \delta^2}{4\varepsilon}\right)$$

$$\Rightarrow \overline{\lim} \; \varepsilon \log \mathbb{P}\left(\text{—"—} \right) \leq -\frac{\lambda n \delta^2}{4}$$

$$\leq -R \quad \text{for } n \text{ large enough.}$$

$$\mathbb{P}\left[I(\sqrt{\varepsilon} B^n) \geq R \right]$$



$$I(\sqrt{\varepsilon} B_t^n) = \frac{1}{2} \int_0^1 \left(\sqrt{\varepsilon}\, \dot{B}_t^n\right)^2 dt$$

$$= \frac{\varepsilon}{2} \sum_{i=1}^n \frac{(B_{t_i} - B_{t_{i-1}})^2}{(t_i - t_{i-1})^{\cancel{2}}} \cancel{(t_i - t_{i-1})}$$

$$= \frac{\varepsilon}{2} \sum_{i=1}^n \eta_i^2 \qquad \eta_i \overset{iid}{\sim} \mathcal{N}(0,1)$$

$$\lambda < \frac{1}{2}$$

$$\Rightarrow \mathbb{P}\left(I(\sqrt{\varepsilon} B^n) > R \right) = \mathbb{P}\left(\lambda \sum_{i=1}^n \eta_i^2 > \lambda \frac{2R}{\varepsilon} \right)$$

$$= \mathbb{P}\left(\exp\left(\lambda \sum_{i=1}^n \eta_i^2\right) \geq \exp\left(\lambda \frac{2R}{\varepsilon}\right)\right)$$

$$\leq \exp\left(-\lambda \frac{2R}{\varepsilon}\right) \underbrace{\mathbb{E}\left[\exp \lambda \eta_i^2\right]^n}_{< \infty}$$

$\overline{\lim_{\varepsilon}} \log \mathbb{P}\left(- \cdot - \right) \quad \leq -2\lambda R$

$\Rightarrow \lambda \to \frac{1}{2}$ we get the des. est.

② Lower bound.

Fix $\phi \in H$, $\delta > 0$

Want to prove

$$\underline{\lim}\ \varepsilon \log \mathbb{P}\left(\sqrt{\varepsilon} B \in B_\delta(\phi) \right) \geq -I(\phi)$$

$$\mathbb{P}\left(\sqrt{\varepsilon} B \in B_\delta(\phi) \right)$$
$$= \mathbb{P}\left(\sqrt{\varepsilon} B - \phi \in B_\delta(0) \right)$$
$$= \mathbb{P}\left(B - \frac{\phi}{\sqrt{\varepsilon}} \in B_{\delta\sqrt{\varepsilon}^{-1}}(0) \right)$$

Girsanov
$$= \mathbb{E}\left[\mathbb{1}_{\{B_{\delta\sqrt{\varepsilon}^{-1}}(0)\}}(B) \exp\left(-\frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot\phi_s \, dB_s - \frac{1}{2\varepsilon} \int_0^1 \dot\phi_s^2 \, ds \right) \right]$$

$$= \exp\left(-\frac{1}{2\varepsilon} \int_0^1 \dot\phi_s^2 \, ds \right) \underbrace{\mathbb{E}\left[\mathbb{1}_{\{B_{\delta\sqrt{\varepsilon}^{-1}}(0)\}}(B) \exp\left(-\frac{1}{\sqrt{\varepsilon}} \int \dot\phi_s \, dB_s \right) \right]}$$

$$= \mathbb{E}\left[\mathbb{1}_{\{\ldots\}} \exp\left(\frac{1}{\sqrt{\varepsilon}} \int \dot\phi_s \, dB_s \right) \right]$$

$$= \mathbb{E}\left(\mathbb{1}_{\{\ \}} \underbrace{\frac{1}{2}\left(\exp\left(-\frac{1}{\sqrt{\varepsilon}} \int \dot\phi_s B_s \right) + \exp(+\ldots) \right)}_{\underbrace{\cosh\left(\frac{1}{\sqrt{\varepsilon}} \int \dot\phi_s \, dB_s \right)}_{\geq 1}} \right.$$

$$\geq \mathbb{E}\left(\mathbb{1}_{\{B_{\varepsilon^{-1/2}\delta}(0)\}}(B) \right) \longrightarrow 1$$

$\square$

③ Properties of I

$$\Phi(R) = \left\{ x : \frac{1}{2} \int \dot{x}_s^2 \, ds < R \right\}$$

compact in $C([0,1])$ . .

Relative compact.

Arzela-Ascoli, As $I(\gamma) < \infty$
$\Rightarrow x(0) = p$ pointwise bd
clear.

Bound on modulus of cont. Assume that
$\int \dot{x}_s^2 \, ds \leq R$

$\forall t, s$

$$|x(t) - x(s)| = \left| \int_s^t \dot{x}_r \, dr \right|$$
$$\underset{\text{Schwartz}}{\overset{\text{Cauchy-}}{\leq}} (t-s)^{1/2} \underbrace{\left| \int_s^t \dot{x}_r^2 \, dr \right|^{1/2}}_{R^{1/2}}$$

<u>Lsc</u> $x_n \in \Phi(R)$

$\downarrow$ in $\|\cdot\|_\infty$

x. nts. $x \in \Phi(R)$.

$\dot{x}_n$ ~~bdd~~ bdd in $L^2(0,1) \longrightarrow y$.

$\Rightarrow \exists$ weakly conv. subsequence

$\dot{x}_n \xrightarrow{a.e} y$.

y is a derivative of x because

$$\int_0^t y(s)\,ds = \int_0^1 1\!\!1_{(0,t)}(s)\, y(s)\,ds$$

$$= \lim \int_0^1 1\!\!1_{(0,t)}\, \dot{x}_n(s)\,ds$$

$$= \lim x_n(t)$$

$$= x(t).$$

$$\int_0^1 y(s)^2\,ds \underset{\text{Fatou}}{\leq} \overline{\lim} \int \dot{x}_n(s)^2\,ds \leq 2R.$$

$\square$

<u>Comments</u> Upper bound Chebyshev, lower bd change of measure.

- Large dev. governed by rate fct, but n almost surely too.
- Similar results true for arbitrary Gaussian measures.
  Possible essay: links with Gaussian concentration.
- Possible essay: Lévy processes.

<u>An application</u> <u>Strassen's Thm</u>

Law of iterated logarithm:

$$\overline{\lim_{t \to \infty}} \frac{B_t}{\sqrt{2t \log\log t}} = 1 \quad \text{a.s.}$$

$$\underline{\lim_{t \to \infty}} \quad " \quad = -1 \quad \text{a.s.}$$

$$X^n(t) = \frac{B_{nt}}{\sqrt{2n \log\log n}} \qquad (t \in [0,1])$$

<u>Thm:</u> $X^n(t) \subseteq C([0,1])$ a.s. relatively compact. The set of limit points is a.s. $\Phi(\tfrac{1}{2}) = \{x : \tfrac{1}{2}\int \dot{x}^2\,ds \leq \tfrac{1}{2}\}$

<u>proof</u> (1) Fix $\delta > 0$. Then there exists $\lambda > 1$ such that for $n(m) = \lfloor \lambda^m \rfloor$

$$\mathbb{P}[\,\mathrm{dist}(X^n, \Phi(1/2)) > \delta \ \ \infty \text{ often}] = 0$$

<u>proof of step 1</u>

$$\Phi_\delta\left(\tfrac{1}{2}\right) = \left\{ x : \operatorname{dist}\left(x, \Phi(\tfrac{1}{2})\right) < \delta \right\}$$

$\Phi_\delta(\tfrac{1}{2})^c$ closed.

$$\inf_{x \in \Phi_\delta(\frac{1}{2})^c} I(x) > \frac{1}{2}$$

In particular we can squeeze $\frac{\mu}{2}$ in between.

$$\frac{1}{2} < \frac{\mu}{2} < \inf_{x \in \Phi_\delta} I(x).$$

$$\mathbb{P}\left[X_{m(n)} \in \Phi_\delta\left(\tfrac{1}{2}\right)^c \right]$$

$$= \mathbb{P}\left[\frac{B_{m(n)}}{\sqrt{2m(n)\log\log m(n)}} \in \Phi_\delta\left(\tfrac{1}{2}\right)^c \right]$$

$$= \mathbb{P}\left[\frac{1}{\sqrt{\log\log m(n)}} \tilde{B} \in \Phi_\delta\left(\tfrac{1}{2}\right)^c \right]$$

$$\underset{\text{upper bd.}}{\overset{\text{Schilder}}{\leq}} \exp\left(- \cancel{2} \log\log m(n) \frac{\mu}{\cancel{2}} \right)$$

$$= \left(\log m(n) \right)^{-\mu}$$

$$= \left(m \log \lambda \right)^{-\mu}$$

$\mu > 1$ $\Rightarrow$ This is summable!

$\Rightarrow$ By Borel-Cantelli this only happens finitely often

② $\forall \delta$ $\quad \overline{\lim} \operatorname{dist}\left(X_n, \Phi(\tfrac{1}{2})\right) \leq 2\delta$ a.s.

δ really small, λ very close to 1.

$\dots\dots\dots\; n(m) \; - - \; t_n \; —— \; n(m+1)$

$X_{n(m)}$ is already δ close to $\Phi(\tfrac{1}{2})$

$$\| X_n - X_{n(m+1)} \|_\infty$$

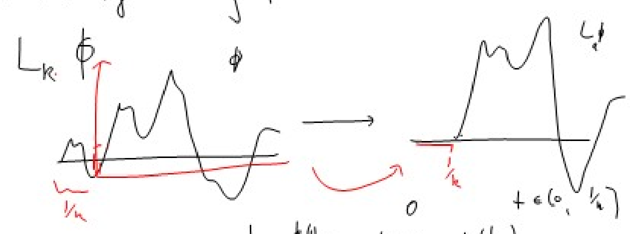
$$= \sup_{0 \leq t \leq 1} \left| \underbrace{\frac{\sqrt{2n(m)\log\log m(n)}}{\sqrt{2n\log\log n}}}_{\to 1} X_{n(m+1)}\left(\frac{nt}{n(m+1)}\right) - X_{n(m)}(t) \right|$$

$$\leq \sup \left\| \frac{\sqrt{2n(m+1)\log\log m(n)}}{\sqrt{2n\log\log n}} - 1 \right\| \, \| X_{n(m)} \|$$

$$+ \underbrace{\left\| X_{n(m+1)}\left(\frac{nt}{n(m+1)}\right) - X_{n(m)}(t) \right\|}_{\text{Detail left}}$$

③ Fix $\phi \in \Phi(\frac{1}{2})$

Fix the following operator $k \in \mathbb{N}$ fix



$L_k \phi(t) = \phi(t) - \phi(1/k)$.

k^m, look at subsequence X_{k^m}

If I look at $L_k X_{k^m}$ they are independent.

$L_k X_{k^m}$ only depends on path between $[\frac{1}{k}, 1]$.

$$X_{k^m} = \frac{B_{k^m t}}{\sqrt{2k^m \log\log k^m}}$$

Here, we only look at Brownian increments between $[k^{m-1}, k^m)$.

These are disjoint $\Rightarrow$ indep.

$$\| X_{k^m} - \phi \|_\infty \leq \sup_{0 \le t \le 1/k} |X_{k^m}(t)| + \sup_{0 \le t \le 1/k} |\phi(t)| + \| L_k (X_{k^m} - \phi) \|_\infty$$

By choosing k large enough I can ignore first two terms

$$\mathbb{P}\left(\| L_k X_{k^m} - \phi \|_\infty < \delta \right)$$

$$= \mathbb{P}\left(\sup_{0 \le t \le 1 - \frac{1}{k}} \left(\frac{B_t}{\sqrt{2 \log\log k^m}} - \tilde{\phi} \right) < \delta \right)$$

$$\geq \exp\left(-2 \log\log k^m \underbrace{\left(I(\tilde{\phi}) + \delta \right)}_{\gamma < 1/2} \right)$$

$$= (m \log k)^{-2\gamma} \quad < 1$$

not summable $\underset{\text{B.C.}}{\Rightarrow}$ □

Chapter 3 Sanov Thm

Empirical measures.

X_i iid in some polish space E

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

Birkhoff-weighted: $\frac{1}{n} \sum_{i=1}^n f(x_i) = \int f(x)\, d\mu_n(x)$

$$\frac{1}{n} \sum x_i = \int x\, \mu_n(dx)$$

<u>Thm</u> (Sanov)

X_i iid in E, μ_n empir. meas.

μ_n satisfies LDP on $\mathcal{M}_1(E)$

with scale n and rate fct.

$$I(\nu) = H(\nu|\mu)$$

if μ is the distribution of X_i.

<u>Exo</u> X_i only takes values in a finite set.
this can be proved by Stirling formula.

<u>Exo</u> Find a connection to Cramér Thm.