**Thm:** $b, \sigma$ Lipschitz, uniformly bdd

$$dx_t^\varepsilon = b(x_t^\varepsilon)\,dt + \sigma(x_t^\varepsilon)\sqrt{\varepsilon}\,dw_t$$

$x_0^\varepsilon$ initial cond., $T$ final time. $(T=1)$

$\Rightarrow$ LDP with rate fct

$$I(x) = \inf_{h \in H^1} \int |\dot h_s^2|\,ds$$

$$\forall t \;\; x_t = x_0 + \int_0^t b(x_s)\,ds + \int \sigma(x_s)\dot h_s\,ds$$

if $\sigma$ invertible everywhere $\quad a = \sigma\sigma^T$

$$= \int_0^t \left(\dot x(s) - b(x_s)\right) a(x_s)^{-1} \left(\dot x(s) - b(x_s)\right) ds$$

**proof** $T : C_0([0,T]) \to C([0,T])$

$$w \mapsto X$$

$$T^m \quad w \longmapsto X^m \quad \text{Euler}$$

① $T^m$ cont. ✓

② $T^m \longrightarrow T$ uniformly on sublevel sets of rate fct.

③ "Exponentially good approx."

$$\overline{\lim_{m\to\infty}}\;\overline{\lim_{\varepsilon\to 0}}\; \varepsilon \log \mathbb{P}\Big(\sup_{0\le t\le 1} |x^{\varepsilon,m}(t) - x^\varepsilon(t)| \ge \delta\Big) = -\infty$$

**proof of**

② $w : |w|_{H^1} \le \alpha$

$$|x_w^m(t) - x_w(t)| \le \overbrace{\int_0^t \big|b(x_w^m(s_m)) - b(x_w(s))\big|\,ds}^{=\mathrm{I}} + \underbrace{\int_0^t \big|\sigma(x_w^m(s_m)) - \sigma(x_w(s))\big|\,|\dot w_s|\,ds}_{\mathrm{II}}$$

$$\mathrm{I} \le |b|_{Lip}\Big(\underbrace{|x_w^m(s_m) - x_w^m(s)|}_{\mathrm{III}} + |x_w^m(s) - x_w(s)|\Big)$$

$$\mathrm{II} \le |\sigma|_{Lip}\big(\text{— '' —}\big)$$

$$\mathrm{III} \le \underbrace{\int_{s_m}^s |b(x_{s_m})|\,dt}_{} + \int_{s_m}^s |\sigma(x_{s_m})|\,|\dot w_t|\,dt$$

$$\le |b|_{L^\infty}|s - s_m| + |s - s_m|^{1/2}|\sigma|_{L^\infty}\,\alpha$$

$$\Longrightarrow |x_w^m(t) - x_w(t)| \le \left(|b|_{Lip} + |\sigma|_{Lip}\right)\left(\int_0^t |x_w^m(s) - x_w(s)| + c\,(s - s_m)^{1/2}\right)$$

$\Rightarrow$ Gronwall ✓

③ $\mathbb{P}\Big(\sup_{0\le t\le 1} |x^{\varepsilon,m}(t) - x^\varepsilon(t)| \ge \delta\Big) \le\ ?$

Bound the following probabilities:

(i) $\tau = \inf\{t : x^{\varepsilon,m}(t) - x^{\varepsilon,m}(t_m) \ge \rho\}$

Bound on $\mathbb{P}(\tau < 1)$

(ii) $\mathbb{P}\big(\tau \ge 1\,;\ \sup |x^{\varepsilon,m}(t) - x^\varepsilon(t)| \ge \delta\big)$

$$|x^{\varepsilon,m}(t) - x^{\varepsilon,m}(t_m)| \leq \overbrace{|b(x_{t_m})|\underbrace{|t-t_m|}_{1/m}}^{\leq \rho/2 \text{ m large}}$$
$$+\sqrt{\varepsilon}|\sigma(x_{t_m})||w_t - w_{t_m}|$$

$$\mathbb{P}\left(\sup_t | \quad \cdots \quad | \geq \rho\right) \underset{\text{m large}}{\leq} \mathbb{P}\left[\sup_{k=1}^{m} \sup_{t\in\left[\frac{k-1}{m},\frac{k}{m}\right]} \sqrt{\varepsilon}|w_t - w_{\frac{k-1}{m}}| \geq \frac{\rho}{2|\sigma|_\infty}\right)$$
$$\leq m\,\mathbb{P}\left(\sqrt{\frac{\varepsilon}{m}} \sup_{0\leq t\leq 1} |B_t| > \frac{\rho}{2|\sigma|_\infty}\right)\Bigg]$$
$$\leq m \exp\left(c\,\frac{\rho^2 m}{\varepsilon}\right).$$

(iii) $\tau \geq 1$

$$z_t^\varepsilon = x^{\varepsilon,m}(t) - x^\varepsilon(t)$$
$$= \int_0^{t\wedge\tau} \underbrace{b(x^{\varepsilon,m}(t_m)) - b(x^\varepsilon(t))}_{\leq |b|_{Lip}\left(|z_s|^2+\rho^2\right)^{1/2}} dt$$
$$+\sqrt{\varepsilon}\int_0^t \underbrace{\left(\sigma(x^{\varepsilon,m}(t_m)) - \sigma(x^\varepsilon(t))\right)}_{\leq |\sigma|_{Lip}\left(|z_s|^2+\rho^2\right)^{1/2}} dw_t$$

$$u_t = \phi(z_t) \qquad \nabla\phi = \frac{(z_t^2+\rho^2)^{1/\varepsilon - 1}}{\varepsilon} 2z_t$$
$$= (z_t^2+\rho^2)^{1/\varepsilon}$$

$$du_t = \underbrace{\nabla\phi\, dz}\ + \varepsilon\frac{1}{2}\mathrm{tr}\left(D^2\phi\ \tilde{\sigma}\tilde{\sigma}^T\right) dt$$

BV part of this $\leq 2\dfrac{(z_t^2+\rho^2)^{1/\varepsilon-1}}{\varepsilon} z_t\,(|z_s|^2+\rho^2)^{1/2}$

$\leq \dfrac{u_t}{\varepsilon}$ (same for the other term)

$$\Rightarrow du = \underbrace{\tilde{b}}_{\leq \frac{u_t}{\varepsilon}} dt + \sqrt{\varepsilon}\tilde{\sigma}\, dW$$

(after introducing another ST) $\mathbb{E}[u_t] \leq K\int_0^t \mathbb{E}\left[\dfrac{u_t}{\varepsilon}\right]$

Gronwall $\Rightarrow$ $\mathbb{E}[u_t] \leq \phi(z_0)\, e^{K/\varepsilon\, t}$

(Maximal inequality.)

$$\Rightarrow \mathbb{P}\left(\sup |x_t^{\varepsilon m} - x_t^{\varepsilon}| \geq \delta\right)$$
$$\leq \mathbb{P}\left(\phi(\quad) \geq \phi(\delta)\right) \leq \frac{\mathbb{E}(\phi(\cdot))}{\phi(\delta)}$$

$$\Rightarrow \mathbb{P}\left(\quad\right) \leq e^{K/\varepsilon}\frac{\phi(z_0)}{\phi(\delta)}$$
$$\underbrace{\leq e^{K/\varepsilon}\frac{\rho^{2/\varepsilon}}{(\delta+\rho)^{1/\varepsilon}}}_{\varepsilon \log \text{ ''}} = K + \log\left(\frac{\rho}{\delta+\rho}\right)$$

$\longrightarrow -\infty$

$\rho \downarrow 0$!

$\square$

These bounds hold$^{\text{loc}}$ uniformly in the starting in starting point, $\mathcal{C}$ compact

$$\liminf \varepsilon \log \inf_{y \in \mathcal{C}} \mathbb{P}_y\left(X \in \mathcal{O}\right) \geq - \sup_{x \in \mathcal{C}} \inf_{\phi \in \mathcal{O}} I_y(\phi)$$

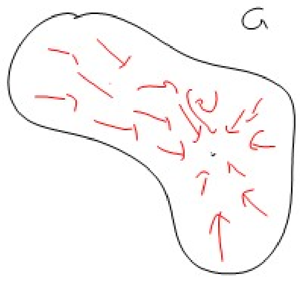
$$\limsup \varepsilon \log \sup_{y \in \mathcal{C}} \mathbb{P}_y\left(X \in F\right) \leq - \inf_{\substack{\phi \in F \\ y \in \mathcal{C}}} I_y(\phi)$$

$F$ = closed

$\square$

## Exit from a domain

$$dx_t = b(x_t)\,dt + \sqrt{\varepsilon}\,\sigma(x_t)\,dW_t$$



$G$ (smooth boundary, bdd open)

Assumptions:

* $b(0) = 0$, unique point in $G$ with this property.
* $x \in G$: look at $\dot{x}_t = b(x_t)$ started at $x$. never leaves $G$ and $x_t \longrightarrow x$.

Observation: $\mathcal{L}^\varepsilon = \frac{\varepsilon}{2} a_{ij}\partial_{ij} + b_i \partial_i$

$a = \sigma\sigma^T$

$$u(x) := \mathbb{E}_x\left[f(x_{\tau_\varepsilon}) + \int_0^{\tau_\varepsilon} g(X_s^\varepsilon)\,ds\right]$$

$$\tau_\varepsilon = \inf\{t : X_t^\varepsilon \in \partial G\}$$

solves
$$\left.\begin{aligned} \mathcal{L}^\varepsilon u &= -g && \text{in } G \\ u &= f && \text{on } \partial G \end{aligned}\right\} \circledast$$

now if $g = 1$, $f = 0$

$\Rightarrow$ solution of $\circledast$ gives $\mathbb{E}_x[\tau_\varepsilon]$

if $g = 0$, $f$ something cont. this gives $\mathbb{E}_x\left(f(X_{\tau_\varepsilon})\right)$

$\Rightarrow$ characterises the distribution on boundary.

## Pseudo-Potential

for $x, y \in \bar{G}$, $t > 0$

$$V(x,y,t) = \inf_{\substack{h \in H^1 \\ x_t = x_0 + \int_0^t b(x_s)\,ds + \int_0^t \sigma(x_s)\dot{h}_s\,ds \\ x_0 = x \\ x_t = y}} \int |\dot{h}_s|^2\,ds$$

$$V(x,y) = \inf_{t > 0} V(x,y,t)$$

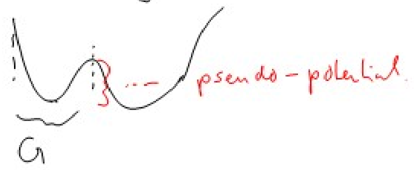
In the case $\sigma = Id$, $b = -\nabla B$
(random perturbation of a gradient flow).

Then (often) we have $V(x,y) = 2(B(y) - B(x))$

$$I(x) = \frac{1}{2}\int_0^t |\dot{x}_s + \nabla B(x_s)|^2 \, ds$$

$$= \frac{1}{2}\int_0^t \overbrace{|\dot{x}_s - \nabla B(x_s)|}^{\geq 0}{}^2 \, ds + \underbrace{\frac{4}{2}\int_0^t \underbrace{\nabla B(x_s)\,\dot{x}_s}_{\frac{d}{ds}B(x_s)} \, ds}_{= 2(B(y) - B(x))}$$

in many cases the inverse gradient flow gives the matching upper bound!



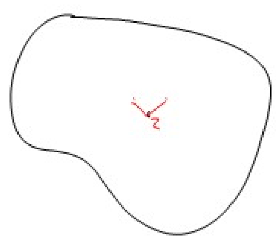

Assumptions (② dynamical system starting at boundary converges to 0)

③ $\bar{V} = \inf_{z \in \partial G} V(0,z)$ Assume that finite.

④ (Controllability at 0 and at boundary) $\exists M$ such that for $\rho > 0$ small enough $x, y, z$, $z \in \partial G \cup \{0\}$

$|x-z| + |y-z| < \rho$

$\Rightarrow \exists$ control $h$. such that $|h| \leq M$ such that the system driven by $h$ connects $x, y$ in time $T$. and $T \downarrow 0$ as $\rho \downarrow 0$.



Reason for this assumption is that it gives continuity of pseudo - potential.

Thm. Assume ① & ③ & ④ then for every $\delta > 0$ we have $\forall x \in G$

$$\lim_{\varepsilon \to 0} \mathbb{P}_x\left(e^{(\bar{V}-\delta)/\varepsilon} < \tau_\varepsilon < e^{(\bar{V}+\delta)/\varepsilon}\right) = 1$$

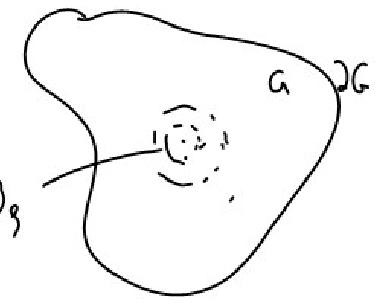
$$\tau_\varepsilon = \inf\{t : X^\varepsilon(t) \in \partial G\}$$

and also $\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}[\tau^\varepsilon] = \bar{V}$

(*) If $\mathcal{C} \subset \partial G$ closed and $\inf_{z \in \mathcal{C}} V(0,z) > \bar{V}$

$$\Rightarrow \mathbb{P}\left(x_{\tau_\varepsilon} \in \mathcal{C}\right) \to 0$$

proof of first part.




Idea for upper bound.

Lemma 1) $\eta > 0$, $\exists \rho > 0$ small enough, $\exists T > 0$

$$\underline{\lim}_{\varepsilon} \varepsilon \log \inf_{B_\rho} P_x\left(\tau^\varepsilon \le T_0\right) > -(\bar{V} + \tfrac{\eta}{2})$$

Lemma 2 $\sigma_\rho = \inf\{t : x_t \in B_\rho \cup \partial G\}$

$$\lim_{t \to \infty} \overline{\lim}\, \varepsilon \log \sup_{x \in G} P_x\left(\sigma_\rho > t\right) = -\infty$$

Lemma 3) $\forall x \in G$,

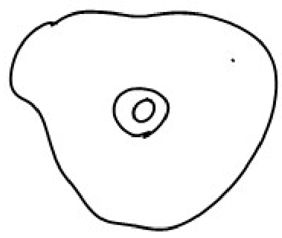
$$\lim_{\varepsilon \downarrow 0} P_x\left(x_{\sigma_\rho} \in B_\rho\right) = 1$$

Lemma 4) for any $\rho > 0$, $c > 0$ $\exists T(\rho, c) > 0$ :

$$\overline{\lim}\, \varepsilon \log \sup_{x \in G} P_x\left(\sup_{t \in T(c,\rho)} |x_t - x| > \rho\right) \le -c$$

proof of Thm upper bound,

Sketch



* $T_0$ with proba as high as we want $\sigma_\rho < T_0$

* $T_1$ with proba $e^{-(\bar{V}+\delta)/\varepsilon}$ hit boundary.

$T = T_0 + T_1$ geom series upper bound

□