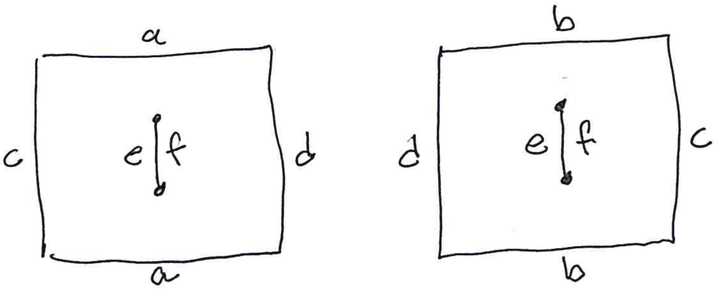
2-fold branched covers of the torus and topology.

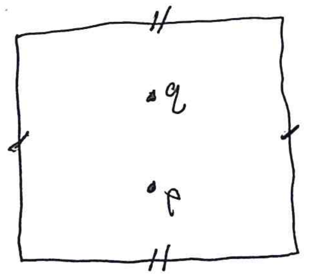
We have constructed the Z-K surface of the slit rectangle. The result is the following surface.

Example 1



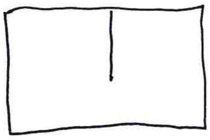
Call this surface S_1.

This surface is a 2-fold branched cover of the torus T. We can construct the covering map by mapping each square to the square below:

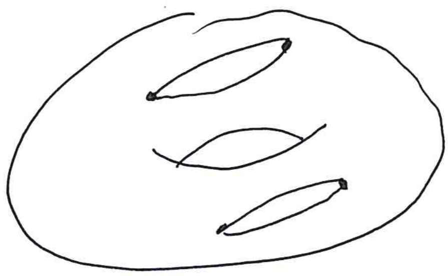


This map respects the translation structure in that it takes straight line trajectories to straightline trajectories except that it has branching over p and q.

Let $\pi_1 : S_1 \to T$ denote this map.

Let $\iota : S_1 \to S_1$ be the involution that interchanges the two squares. Note that on the original billiard table  ι corresponds to flipping along the central axis.

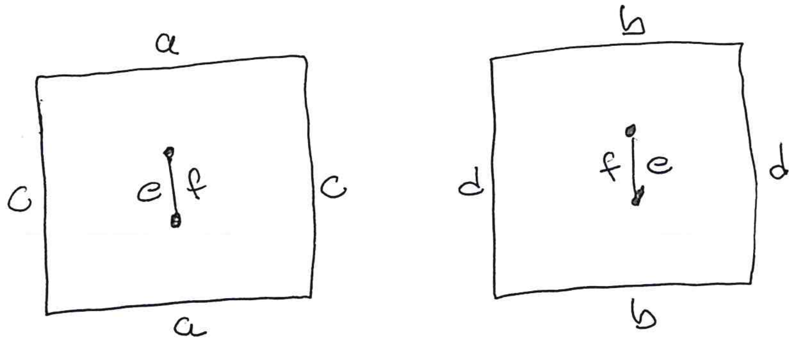
We can see that S_1 is a surface of genus 2.
We can build it in steps as



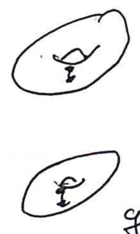
So it is a torus with two slits that are glued together.

By way of contrast we will describe a second branched cover of the torus.

Example 2.



Picture:



This construction gives rise to a second branched cover of the torus $S_2 \xrightarrow{\pi_2} T$ branched at points p and q.

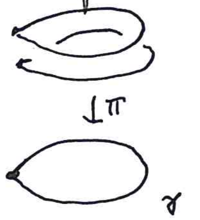
Definition. Two branched covers $\pi_1 : S_1 \to T$ and $\pi_2 : S_2 \to T$ are isomorphic if there is a homeomorphism $f : S_1 \to S_2$ so that $\pi_2 \circ f = \pi_1$.

$$\begin{tikzcd} S_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & S_2 \arrow[dl, "\pi_2"] \\ & T & \end{tikzcd}$$

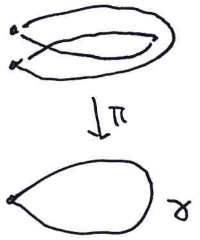
The two branched covers we have constructed are not isomorphic and we can detect this with the following invariant.

Def. Given a branched cover $\pi : S \to T$ branched over p and q we can define a homomorphism $\chi_S : \pi_1(T - \{p,q\}) \longrightarrow \mathbb{Z}_2$

(where $\mathbb{Z}_2$ denotes $\mathbb{Z}/2\mathbb{Z}$). For a loop γ $\chi_S(\gamma) = 0$ if translating the fiber along the loop brings it back to itself by the identity



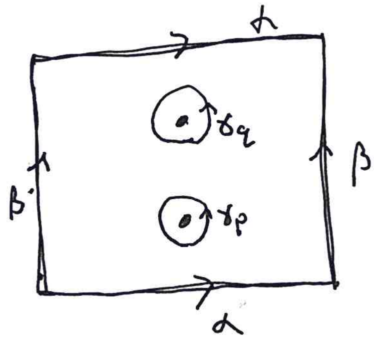
$\chi_S(\gamma)=0$

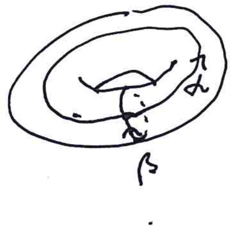


$\chi_S(\gamma)=1$

$\chi_S(\gamma)=1$ if translation along the fiber permutes the elements of the fiber.

Let $\alpha, \beta, \gamma_p, \gamma_q$ be the following loops in $T - \{p,q\}$.





χ_S is determined if we know its values on these loops. In example 1 we see that

$\chi_{S_1}(\alpha) = 1$, $\chi_{S_1}(\beta) = 0$, $\chi_{S_1}(\gamma_p) = 1$, $\chi_{S_1}(\gamma_q) = 1$. (*)

In example 2 we have:

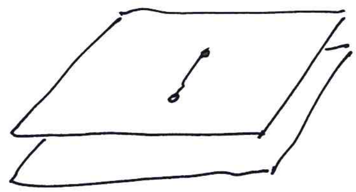
$\chi_{S_2}(\alpha) = 0$, $\chi_{S_2}(\beta) = 0$, $\chi_{S_2}(\gamma_p) = 1$, $\chi_{S_2}(\gamma_q) = 1$

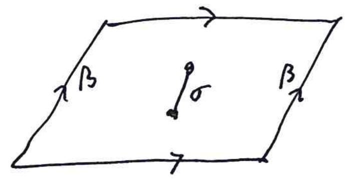
Prop. χ_S is a complete invariant for a 2-fold branched cover.

Cor. S_1 and S_2 are not isomorphic as branched covers.

Here is an alternative way to compute χ.

In the construction of example 1 think of the left hand square as the top sheet and the right hand square as the bottom sheet. Note that identifications along the ~~l~~ left and right sides interchange sheets. These sides lie above the loop β in T. Let σ denote the slit in T. We also interchange sheets in the identifications over σ.





If l is a loop in the torus we can calculate $\chi_{S_1}(l)$ by counting the number of intersections of l with $\beta \cup \sigma$ mod 2. (Here we assume that the intersections are transverse.)

We say χ_{S_1} is dual to $\beta \cup \sigma$ and write $\chi_{S_1} = [\beta] + [\sigma]$. In the same way we see that χ_{S_2} is dual to σ and write $\chi_{S_2} = [\sigma]$. This can be described in the language of homology and cohomology but for our purposes this will not be necessary.

The question we are really interested in is when do saddle connections on S_1 in a given direction divide S_1 into two invariant subsurfaces. If we have a saddle connection on S_1 it projects to a saddle connection on T which either connects p to p, q to q or p to q. In the first two cases the resulting flow on S_1 is completely periodic. We consider the third case. Let σ' be a saddle connection from p to $q + (m, n)$.

Example: $(m,n) = (2,1)$

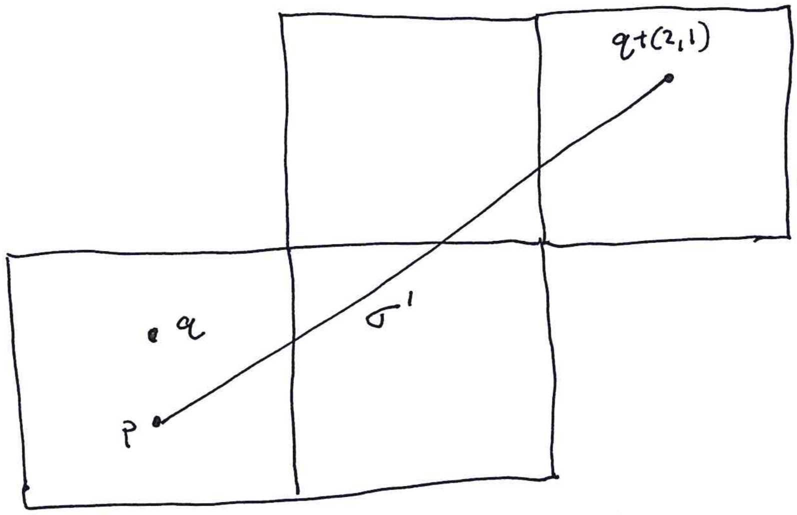
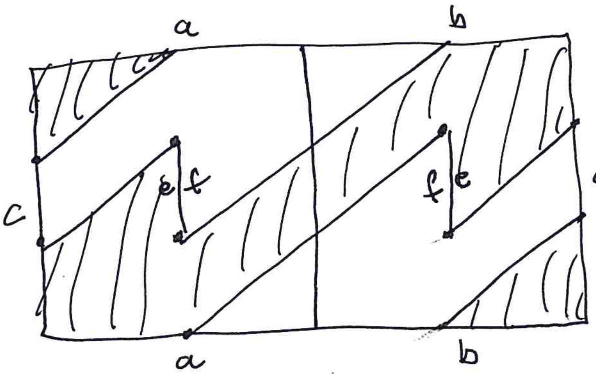
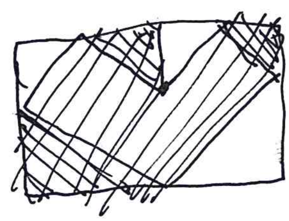


Image of σ' in S_1:





In this case σ' does divide S_1 into two subsurfaces. These are interchanged by the involution ι. Note that the picture of S_1 respects the identifications that define S_1.

We approach this problem as follows. Say we have a segment $\sigma' = \sigma'(m,n)$ from p to $q+(m,n)$ in the torus T. We can build a branched cover of T by taking two copies of T, cutting each along the slit σ' and gluing them together by connecting opposite sides of the slit.

Let's call the resulting surface $S_{\sigma'}$. $S_{\sigma'}$ clearly has an invariant decomposition into subsurfaces. If $S_{\sigma'}$ is isomorphic to S_1 then we can use this isomorphism to see that S_1 has a decomposition invariant under the flow in the direction of σ'.

Thus we want to know when $\chi_{S_{\sigma'}} = \chi_{S_1}$.

As we have seen it suffices to check equality on the loops $\alpha, \beta, \gamma_p, \gamma_q$ in $T - \{p, q\}$.

We can write $\chi_{S_{\sigma'}}$ as the class dual to σ'; so we need to count the parity of the number of intersections between σ' and the loops $\alpha, \beta, \gamma_p, \gamma_q$.

Refer to the previous example. We see that # of intersections of σ' with γ_p and γ_q is 1. The number of intersections of σ' with β is m. The number of intersections of σ' with α is n. Thus, as in equation*, m must be even and n must be odd.

Let $\sigma = \sigma(m,n)$ and $\sigma' = \sigma(m',n')$ be two saddle connections in the torus joining p to q and satisfying the parity condition that m and m' are even and n and n' are odd.

σ gives rise to a partition A, B of S_1 and σ' gives rise to a partition A', B'. Our next objective is to estimate the area of $A \Delta A'$.

It will be convenient to introduce a small amount of topology.

Let S be a surface. A 0-cochain on S is a function from S to $\mathbb{Z}_2$. A 1-cochain on S is a function from paths on S to $\mathbb{Z}_2$. There is a coboundary operator δ which takes zero-cochains to 1-cochains.

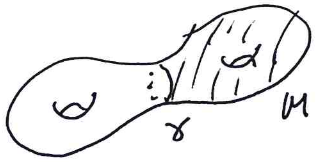
Let $f: S \to \mathbb{Z}_2$ be a function. We define $\delta(f)$ on a path p to be $f(p(1)) - f(p(0))$. Since we are dealing with $\mathbb{Z}_2$ the orientation of p is irrelevant.

The dual classes we described earlier are examples of 1-chains.

Def. A 1-cochain c is a boundary if $c = \delta(f)$ for some 0-cochain f.

~~Example: A 1-cochain is a boun~~

Example. A 1-cochain dual to a loop γ in S is a boundary iff γ bounds a submanifold $M \subset S$ in S. We can see this by taking f to be one on M and 0 outside of M.



Now we return to our question.

We can associate to a partition A, B of S_1 a 0-cochain f_A (on S_1) by setting f_A to be 0 on A and 1 on B. If A, B corresponds to $\sigma \in T$ then let σ_0 and σ_1 be the two lifts of σ to S_1. $\sigma_0 \cup \sigma_1$ is the boundary of A in S_1 so we have

$$\delta(f_A) = [\sigma_0] + [\sigma_1].$$

Since the involution $\iota : S_1 \to S_1$ interchanges A and B we have:

$$f_A(x) = f_A(\iota(x)) + 1 \mod 2.$$

Define $f_{A'}$ to be one on A' and 0 on B'. Letting σ_0' and σ_1' be the two lifts of σ' to S_1 we have $\delta(f_{A'}) = [\sigma_0'] + [\sigma_1']$.

Let g be the 0-chain on S_1 which is 1 on $A \Delta A'$ and 0 elsewhere. Observe that $g = f_A + f_{A'} \mod 2$.

Now $g(\iota(x)) = f_A(\iota(x)) + f_{A'}(\iota(x)) = f_A(x) + 1 + f_{A'}(x) + 1 = g(x) \mod 2$. So g is ι invariant and g is the pullback of some function g_T on T. That is $g(x) = g_T(\pi(x))$.

Claim that $\delta g_T = [\sigma] + [\sigma']$.

We observe that there is a correspondence between chains on T and chains on S_1 which are invariant under the involution ι. This correspondence respects the coboundary operator. Hence it suffices to check the equation after lifting everything to S_1. We need to show:

$$\delta g = [\sigma_0] + [\sigma_1] + [\sigma_0'] + [\sigma_1']$$

This follows since $\delta g = \delta f_A + \delta f_{A'} = [\sigma_0] + [\sigma_1] + [\sigma_0'] + [\sigma_1']$