Course Outline

MA475 Riemann Surfaces

1. Riemann Surfaces and Atlases
   (a) Holomorphic and smooth atlases
   (b) Atlases and geometric structures
   (c) Translation structures, integration and holomorphic 1-forms
   (d) Cut and paste constructions: torus, octagon, pillowcase and tetrahedron
2. Algebraic Curves
   (a) Holomorphic functions in two complex variables
   (b) Affine algebraic curves
   (c) Atlases for non-singular algebraic curves and the holomorphic implicit function theorem
   (d) Construction of $\mathbb{CP}^2$ and its charts
   (e) Homogeneous polynomials and projective curves
   (f) Asymptotes and points at infinity
3. Holomorphic Maps Between Riemann Surfaces
   (a) The holomorphic inverse function theorem
   (b) Local form of holomorphic maps
   (c) Open mapping theorem, non-constant holomorphic functions
   (d) Proper maps, local homeomorphisms, covering maps and branched covers
   (e) The degree formula for proper maps
   (f) Meromorphic functions on $\mathbb{CP}^1$
4. Topology and holomorphic maps
   (a) Definition and properties of Riemann surfaces of finite type
   (b) Review of the topology of surfaces, Euler characteristic and genus, surfaces with boundaries

(c) The Riemann-Hurwitz Theorem

(d) Torus and octagon examples

(e) Review of covering space theory, regular covers

(f) The monodromy homomorphism and monodromy around punctures

(g) Irregular cover example

(h) Elliptic and hyperelliptic examples

(i) Meromorphic functions on $\mathbb{CP}^2$ and the Fermat Quartic example

5. The Weierstrass $\mathcal{P}$-function and Elliptic Integrals

   (a) Meromorphic functions on tori of the form $\mathbb{C}/\Lambda$

   (b) Using meromorphic functions on tori to parametrise cubic curves

   (c) Elliptic integrals, translation structures and holomorphic 1-forms

   (d) Determining a lattice from a cubic curve

   (e) Holomorphic 1-forms and topology (degree-genus formula)

6. Hyperbolic Surfaces

   (a) Simply connected Riemann surfaces and the Uniformization Theorem

   (b) Hyperbolic geometry and the disk

   (c) Examples of hyperbolic surfaces

7. Interesting results that are not in the course

8. References and Sources

   (a) "Riemann Surfaces" by Simon Donaldson (Parts I and II)

   (b) "Complex Algebraic Curves" by Frances Kirwan (Chapters 1-6) Chapter 6 includes a discussion of holomorphic 1-forms without a discussion of the geometry. This is available electronically through the Warwick Library.

   (c) "Complex Analysis" by Lars Ahlfors (Chapter 7)

(d) Curt McMullen online notes for Math 213b. (Pages 1-22) He discusses the regular octagon surface on page 8 and the irregular cover example on page 14.

(e) "A primer on Riemann Surfaces" by Alan Beardon. Pages 41-42 contain a discussion of polygon gluing constructions without connecting them to holomorphic 1-forms. Chapter 9 proves the Uniformization Theorem.

(f) "Part IID Riemann Surfaces" by Alexei Kovalev (Cambridge) online. On page 26 he discusses meromorphic differentials.

(g) "Flat Surfaces" by Anton Zorich (Chapters 1-3). This reference includes a geometric discussion of holomorphic 1-forms with applications to the dynamics of billiard flows.