MA475 Example Sheet 1

14 January 2020

1. A consequence of Cauchy's theorem is that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^{n+1}}dw.$$

   Deduce from this that power series expressions for holomorphic functions can be differentiated term by term.

2. As sequence if functions f_n converges locally uniformly to f in a domain D if $f_n \to f$ on each compact subset of D. Morera's theorem shows that if the holomorphic functions f_n converge locally uniformly then the limit is holomorphic.

   Show that the the series

$$\sum_{|n|>N}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

   converges uniformly in the disc $\{z : |z| < N\}$.

   Deduce that

$$\frac{1}{z}+\sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

   converges to a holomorphic function in $\mathbb{C}-\mathbb{Z}$. Find the derivative of this function.

3. Let D be an open disc divided by a diameter into two half discs D_1 and D_2. Use Morera's theorem to show that if f is holomorphic in each open half-disk and f is continuous on the entire disk then f is holomorphic on the entire disk. (This argument shows that overlap functions are holomorphic in the construction of atlases for polyhedra.)

4. Give $\mathbb{C}$ two atlases $\mathcal{A}_1$ consisting only of the identity map and $\mathcal{A}_2$ consisting of the map $h(z) = \bar{z}$. Show that $\mathcal{A}_1$ and $\mathcal{A}_2$ determine different conformal structures on $\mathbb{C}$ but that $h : (\mathbb{C}, \mathcal{A}_1) \to (\mathbb{C}, \mathcal{A}_2)$ is a holomorphic equivalence.

5. Let V be a real vector space and let $J : V \to V$ be a real linear transformation satisfying $J^2 = -I$. Consider the sub-ring of the matrix ring generated by real linear combinations of I and J. Show that this subring is isomorphic to $\mathbb{C}$ and that, as a module over this subring, V becomes a complex vector space.

6. Let S^2 be the unit sphere in $\mathbb{R}^3$. Let $p \in S^2$. Let T_p be the tangent space to S^2 at p. Define $J : T_p \to T_p$ by $J(v) = p \times v$. Show that J defines a complex structure on T_p. Show that the notion of angle that comes from this complex structure is the same as the notion of angle that comes from restricting the metric on $\mathbb{R}^3$ to T_p.