MA475 Example Sheet 2

26 January 2016

1. Let $f(z)$ be a meromorphic function on the complex plane. Show that an integral

$$\int \sqrt{f(z)}dz$$

determines a half-translation structure on the set U which is the complement of the zeros and poles of f in $\mathbb{C}$. (Hint: Build charts over simply connected sets by first choosing a branch of the square root and then integrating.)

2. Show that the half-translation structure associated to

$$\int \frac{-dz}{\sqrt{1-z^2}}$$

corresponds to $\mathbb{C}/\Lambda$ where Λ is group of transformations of $\mathbb{C}$ generated by $z \mapsto z + 2\pi i$ and $z \mapsto -z$. (Hint: Use the complex cosine function and its inverse.)

3. Show that a homogeneous polynomial of degree d in two variables with complex coefficients is a product of d linear terms.

4. Show that the surface defined by the equation $w^2 = \sin z$ is a Riemann surface.

5. Find the singular points for each of the following affine curves in $\mathbb{C}^2$ and the tangent lines at the singular points.

   (a) $x^2 - x = 0$

   (b) $y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy = 0$

   (c) $y^2 = x^3 - 1$

6. The multiplicity of a point (a, b, c) of a projective curve C defined by $P(x, y, z) = 0$ is the smallest integer m such that

$$\frac{\partial^m P}{\partial x^i y^j z^k}(a, b, c) \neq 0$$

for some i, j, k with $i + j + k = m$. Find the singular points for each of the following projective curves in $\mathbb{CP}^2$ and the multiplicity of the singular point.

   (a) $x^2 - xz = 0$

   (b) $xy^4 + yz^4 + xz^4 = 0$

   (c) $y^2 z = x(x-1)(x-\lambda)$

   (d) $x^n + y^n + z^n = 0$

7. Consider the equivalence relation $\sim$ on $\mathbb{C}^{n+1} - \{(0 \ldots 0)\}$ given by $(u_1 \ldots u_{n+1}) \sim (\lambda u_1 \ldots \lambda u_{n+1})$ for $\lambda \in \mathbb{C}^*$. Let $\mathbb{P}^n$ be the quotient space $\mathbb{C}^n - \{(0 \ldots 0)\}/\sim$ with the quotient topology. Let $U_j \subset \mathbb{C}^{n+1}$ be the set where $u_j \neq 0$. Define charts $\phi_j : \mathbb{C}^{n+1} \to \mathbb{C}^n$ by the formula $\phi_j((u_1 \ldots u_{n+1})) = (u_1/u_j \ldots u_{j-1}/u_j, u_{j+1}/u_j \ldots u_{n+1}/u_j)$. Thus $\phi_j^{-1} : \mathbb{C}^n \to \mathbb{C}^{n+1}$ is given by $\phi_j^{-1}((u_1 \ldots u_{n+1})) = (u_1 \ldots u_{j-1}, 1, u_{j+1} \ldots u_{n+1})$.

   (a) Compute the change of coordinate maps for two of these charts and show that $\mathbb{P}^n$ is a complex manifold of dimension n.

   (b) Show that every complex line in $\mathbb{C}^{n+1}$ intersects the unit sphere in $\mathbb{C}^{n+1}$ in a circle.

   (c) Show that $\mathbb{P}^n$ is compact.

   (d) Show that a projective curve defined by a homogeneous polynomial $P(x, y, z)$ is compact.

   (e) Show that the Riemann surface $\mathbb{P}^1$ (with the above atlas) is conformally equivalent to the Riemann sphere (defined in lecture with an atlas of two charts.)

8. Show that the linear action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$ induces an action by holomorphic automorphisms of $\mathbb{P}^{n-1}$. Show that when $n = 2$ these automorphisms are just linear fractional transformations (Möbius transformations) when expressed in terms of the variable determined by the coordinate chart U_1.

9. What is the subgroup of $GL(2,\mathbb{C})$ that fixes $\mathbb{P}^1$ pointwise? Show that every such automorphism is induced by a matrix with determinant one. How many matrices with determinant one induce the same automorphism? Identify the group of linear fractional transformations with $PSL(2,\mathbb{C})$.