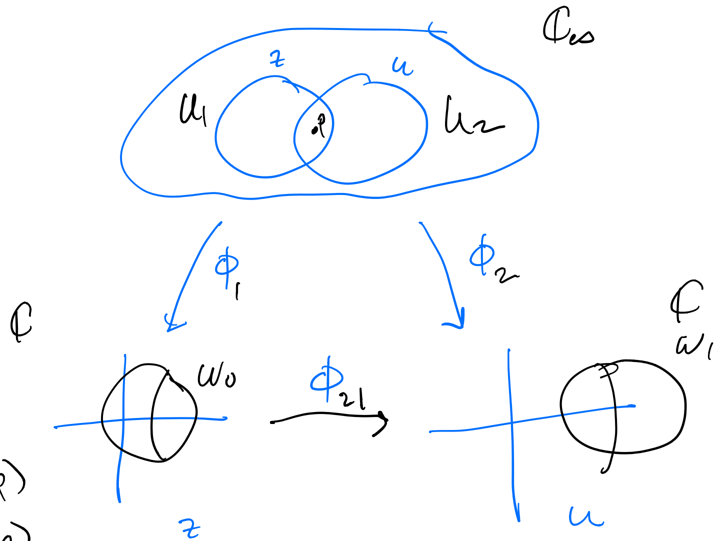
Have been surreptitiously switching variable names. Gives a more automatic way of keeping track of changes of variable.



Two different copies of $\mathbb{C}$. Different names for the variable in each copy

$z = \phi_1(p)$

$u = \phi_2(p)$

$u = \phi_{21}(z)$ $\quad$ $z = \phi_{12}(u)$ where defined

"$u = \frac{1}{z}$" $\quad$ "$z = \frac{1}{u}$"

ϕ_{12}

Computing ϕ_{01} means solving for z in terms of u.

$$\phi_2(p) = \phi_{21}(\phi_1(p))$$

$$u = \phi_{21}(z)$$

$$\text{"}\ u = \frac{1}{z}\ \text{"}$$

(This notation will be useful when we talk about 1-forms.)

Write $\bar{V}$ for the closure of V in $\mathbb{C}_\infty \times \mathbb{C}_\infty$. ②

Claim. $\bar{V} = V \cup$ one point.

$\mathbb{C}_\infty \times \mathbb{C}_\infty$ is a 2 complex dimensional manifold and we have not discussed these. We proceed despite this. We have charts ϕ_1, ϕ_2 for $\mathbb{C}_\infty$ so we can get charts (ϕ_1, ϕ_1) (ϕ_1, ϕ_2) (ϕ_2, ϕ_1) (ϕ_2, ϕ_2) for $\mathbb{C}_\infty \times \mathbb{C}_\infty$. The relevant chart for the point (∞, ∞) is (ϕ_2, ϕ_2). Introduce variables $u = \frac{1}{z}$ and $v = \frac{1}{w}$.

$u=\frac{1}{z} \quad v=\frac{1}{w}$ encodes the transition map

(ϕ_{21},ϕ_{21}) and (ϕ_{12},ϕ_{12}).

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C}_\infty \times \mathbb{C}_\infty \bigcirc \arrow[dl, "(\phi_1,\phi_1)"'] \arrow[dr, "(\phi_2,\phi_2)"] & \\
\underset{(z,w)}{\mathbb{C}^2} \arrow[rr, "(\phi_{21},\phi_{21})"] & & \underset{(u,v)}{\mathbb{C}^2}
\end{tikzcd}$$

$u=\frac{1}{z}$
$v=1/w$

$z=\frac{1}{u}$
$w=1/v$

equation for V is $w^2=P(z)$

equation for V in these coordinates is $\left(\frac{1}{v}\right)^2=P\left(\frac{1}{u}\right)$

$$\frac{1}{v^2}=a_0+a_1\left(\frac{1}{u}\right)+\dots a_n\left(\frac{1}{u^n}\right)$$

$$\frac{1}{v^2}=\frac{1}{u^n}\underbrace{\left(a_0u^n+\dots a_n\right)}_{Q(u)}$$

$$v^2=\frac{u^n}{Q(u)} \qquad Q(0)\neq 0.$$

rational function

We can treat this as we did before with

$$h(\gamma)=\exp\left(\frac{1}{2}\int_\gamma \frac{R'}{R}\,du\right) \qquad R(u)=\frac{u^n}{Q(u)}$$

a <u>rational function</u> instead of a polynomial.

(Not the only way of approaching this.) ④

Recall that if γ is a loop and $P(z) = \prod (z - z_j)$ then

$$h(\gamma) = (-1)^{\sum_j \text{wind}(\gamma, z_j)}$$

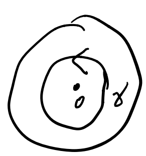
$R(u)$ differs from $P(z)$ in that $R(u) = \prod (u - u_j)^{n_j}$ where n_j can be negative.

Same proof gives

$$h(\gamma) = (-1)^{\sum_j n_j \cdot \text{wind}(\gamma, u_j)}$$

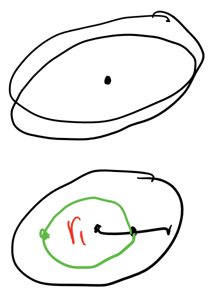
Now we are only considering one point

Choose a disk D around 0 not containing any roots of Q.

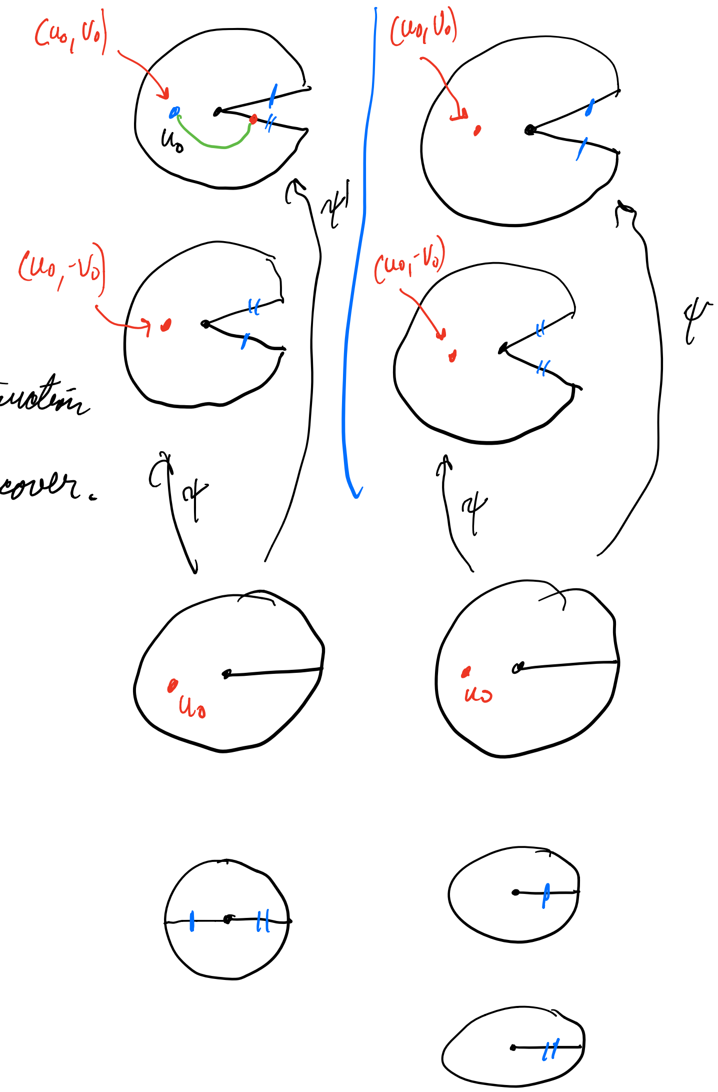


$h(\gamma) = +1$ if d even
$ -1$ if d odd

Slit disk:

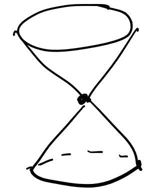


Odd case: Even case:



Explicit construction of a 2-fold branched cover.

Centers are connected:



Clearly this is a singular point.

In the odd case (∞,∞) has a disk nbd. in $\bar{V}$.

In the even case (∞,∞) has a nbd. which is the union of two disks joined at a point.

Claim that in the odd case $\bar{V}$ has a Riemann surface structure and in the even case we "can pull the two disks apart" to create a topological surface with a Riemann surface structure.

This is the topological picture but we are interested in the "holomorphic" picture.

We want to construct charts.

This is automatic in the even case. Both functions ψ and ψ' extend to the disk.

The only problem is that the two functions take the same value at $u=0$.

$$\psi(u) = w_0 \cdot \exp\left(\frac{1}{2}\int_{u_0}^{u} \frac{R'}{R}\,du\right)$$

$$\psi'(u) = -w_0 \cdot \exp\left(\frac{1}{2}\int_{u_0}^{u} \frac{R'}{R}\,du\right)$$

Key point is that the result does not depend on the path chosen from u_0 to u because we are working in a slit disk

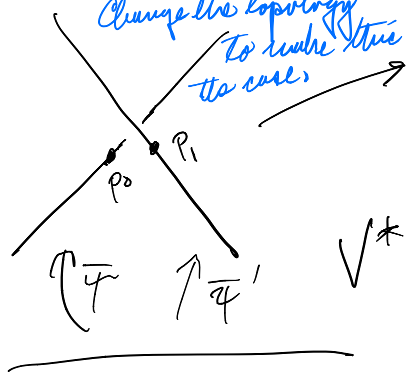
In the even case the result is independent of the path in the entire disk.

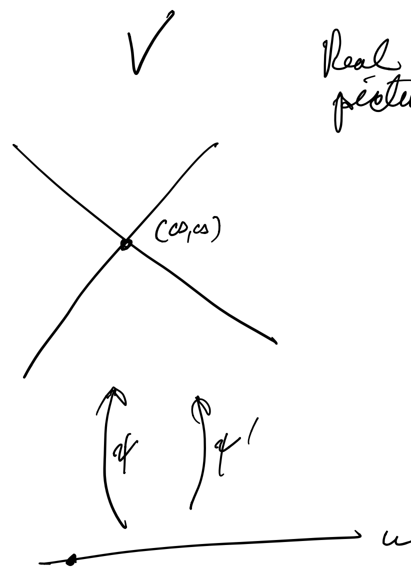
(In the odd case the answer depends on which way the path goes around 0.)

If there is no zero at ∞

We have inverse coordinate charts but their images are not open sets.

Change the topology to make this the case,





Real pictures

In the even case we have two functions which would like inverse coordinate charts.

(Both give holomorphic maps into $\mathbb{C}^2$.)

(ψ, ψ' parametrize branches of $\sqrt{z^4 \cdot Q(z)}$.)

The only problem is that their images intersect so their images are not open in $\bar{V}$.

The solution is to create a new abstract Riemann surface V^* as follows.

$$V^* = (V - \{(\infty,\infty)\}) \cup \{p_0, p_1\}.$$

Remove the point (∞,∞) $^{\text{from } \bar{V}}$ and replace it by p_0, p_1. Nbds of p_0 are images of nbds of zero under ψ_0. Nbds of p_1 are images of nbds of zero under ψ_1.

We can define an atlas for $\bar{V}$ using functions $\bar{\psi}$, $\bar{\psi}'$ where $\bar{\psi}(0)=p_0$, $\bar{\psi}_1(0)=p_1$

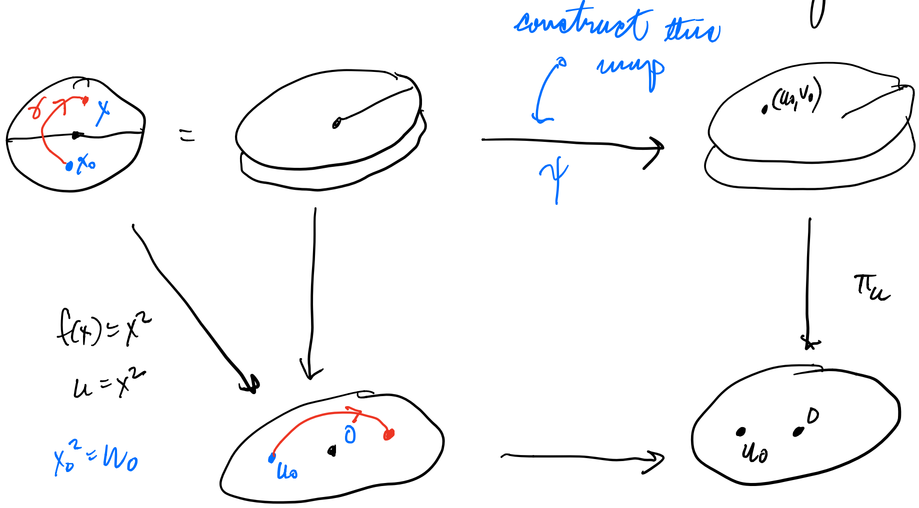
$$\bar{\psi}(u) = \begin{cases} (0,0) & \text{if } u=0 \\ \psi(z) & u\neq 0. \end{cases} \qquad \bar{\psi}'(z) = \begin{cases} (0,0) & \text{if } u=0 \\ \psi_1(z) & \text{if } u\neq 0. \end{cases}$$

The inclusion from $V^* \hookrightarrow V \subset \mathbb{C}_\infty \times \mathbb{C}_\infty$ is holomorphic.

In the odd case we are starting with the right topology we just need to build a chart at ∞.

The trick is the following.

Instead of constructing a map from the disk to $\bar{V}$ we construct a map from the "branched double cover" of the disk.



Define $\psi(x)$ by

$$\psi(x) = v_0 \exp\left(\frac{1}{2}\int_{f(\gamma)} \frac{R'}{R}\, du\right)$$

Why is this unambiguous?

Say γ_0 and γ_1 are paths from x_0 to x.

Need to check $v_0 \exp\left(\frac{1}{2}\int_{f(\gamma_0)} \frac{R'}{R}\, du\right) = v_0 \exp\left(\frac{1}{2}\int_{f(\gamma_1)} \frac{R'}{R}\, du\right)$

or $h(f(\gamma_0)\cdot f(\gamma_1^{-1})) = 1$.

But $f(\gamma_0)\cdot f(\gamma_1^{-1}) = f(\gamma_0\cdot\gamma_1^{-1}) = (\gamma_0\cdot\gamma_1^{-1})^2$ goes around zero twice as many times as $\gamma_0\cdot\gamma_1^{-1}$ so

$$h(f(\gamma_0)\cdot f(\gamma_1^{-1})) = h(\gamma_0\gamma_1^{-1})^2 = 1.$$

Want to construct a chart so that the function π_u has valence 2.

$z \mapsto z^2$ is the "model map" for having valence 2.

Given a point $\tilde{u}$ let γ be a path from $\tilde{u}_0$ to u.

Consider two paths $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ from $\tilde{u}_0$ to $\tilde{u}$.

$\alpha_0 \cdot \alpha_1^{-1}$

The winding number of the loop in the punctured disk is necessarily even. It follows that $h(\beta_0 \cdot \beta_1^{-1}) = 1$ or $h(\beta_0) = h(\beta_1)$. We have successfully built a hol. fcn.

Solving topological problems has analytical consequences.

Proposition. Any construction which produces an atlas on the <u>topological</u> surface V^* so that the inclusion of

V^* into $\mathbb{C}^2$ is holomorphic produces a surface conformally equivalent to V^*.

Proof. Any V' is the alternate Riem. surface structure on V^*. We get a homeomorphism $h: V^* \to V'$ which is holomorphic away from a finite set of points. By removable sing., h is holomorphic everywhere. Similarly h^{-1} is hol. everywhere.