$$W_1 \subset \mathbb{R}^n \xrightarrow{\;G\;} W_2 \subset \mathbb{R}^m$$

Summary of the pullback

$$G^*: \Lambda^k(W_2) \to \Lambda^k(W_1)$$
$(\forall k)$.

Then $\theta \mapsto G^*(\theta)$ is an algebra homomorphism

① It is linear: $G^*(\theta + \psi) = G^*(\theta) + G^*(\psi)$.

② $G^*(f) = f \circ g$ ← Functions pull back by composing.

③ $G^*(\theta \wedge \psi) = G^*(\theta) \wedge G^*(\psi)$ ← Respects wedge product

④ $G^*(d\theta) = d(G^*\theta)$ ← Natural with respect to the exterior derivative.

⑤ $(G \circ F)^*(\theta) = G^*(F^*\theta)$.
$$\begin{tikzcd} O \arrow[r, "G"] \arrow[rr, bend right, "G \circ F"'] & O \arrow[r, "F"] & O \end{tikzcd}$$

As an algebra the collection of forms is generated by functions and 1-forms: $\theta = \sum_I f_I \, dx_{i_1} \wedge \dots dx_{i_k}$.

$$G(\vec{x}) = \begin{bmatrix} G_1(\vec{x}) \\ \vdots \\ \vdots \\ G_m(\vec{x}) \end{bmatrix} \qquad DG = \begin{bmatrix} DG_1 \\ \vdots \\ \vdots \\ DG_m \end{bmatrix} = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial G_m}{\partial x_1} & \cdots & \frac{\partial G_m}{\partial x_n} \end{bmatrix}$$

$$G^*(\theta) = \sum_I G^*(f_I) \cdot G^*(dx_{i_1}) \wedge \dots G^*(dx_{i_k})$$

$$= \sum_I f_I \circ G \; dG_1 \wedge \dots dG_m$$

↖ Expand this expression

where $dG_j = \sum \frac{\partial G_j}{\partial x_k} dx_k$.

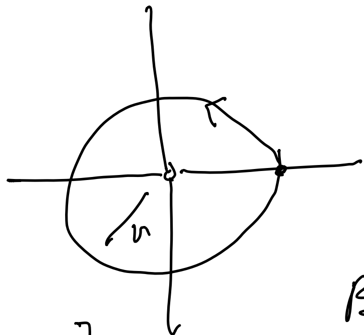
Also read.

Integration examples.

$$\omega_2 = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

is called this w/ form?

$$\omega_1 = \frac{x\,dx + y\,dy}{x^2+y^2}$$




Both defined on $\mathbb{R}^2 - \{0\}$.

$$\gamma(t) = \begin{bmatrix} r\cdot\cos t \\ r\cdot\sin t \end{bmatrix}$$

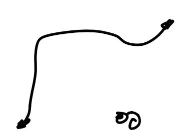
Exercise: Both forms are satisfy $d\omega = d\psi = 0$.

(We say they are closed.)

$$\int_\gamma \omega_2 = \int_0^{2\pi} \gamma^* \omega_2 = \int_0^{2\pi} dt = 2\pi.$$

Exercise: $\omega_1 = d\,\overbrace{\underbrace{\log\sqrt{x^2+y^2}}_{\frac{1}{2}\log(x^2+y^2)}}^{f}.$

$\rho : [a,b] \to \mathbb{R}^2 - \{0\}$



For a general path $\rho$ we have:

$$\int_\rho \omega_1 = \int_a^b \rho^*\omega_1 = \int_a^b \rho^* df = \int_a^b d(\rho^* f)$$

$$= \int_a^b d(f \circ p) = \int_a^b (f \circ p)' dt$$

$$= \int_a^b \frac{d}{dt} f \circ p \, dt = \Big|_{f(p(a))}^{f(p(b))}$$

$$= f(p(b)) - f(p(a)).$$

Note in particular that $\int_p \omega_2 = 0$ for any loop. $\left(\oint_p \omega_2 = 0\right)$ since for a loop $p(a) = p(b)$.

In particular $\omega_1$ and $\omega_2$ behave differently.

---

The formalism of a graded algebra with a differential $d$ arises in cohomology theory (which I am not assuming!) The following language comes from cohomology theory.

Def. A form $\theta$ is <u>closed</u> if $d\theta = 0$.

A form $\theta$ is <u>exact</u> if $\theta = d\eta$.

Note. Every exact form is closed since $d\theta = d^2\eta = 0$. But not every closed form is exact as we will see.

Remark: $\omega_1$ is <u>exact</u>. $\omega_2$ is closed but not exact.

Criterion for exactness for a 1-form:

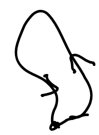
Shifrin Math 3110. Time:

Day 31 32:00.

Theorem. For a 1-form $\theta$ the following are equivalent.

① path independence $\int_\gamma \theta$


② vanish on loops $\oint \theta = 0$



③ $\theta = df$

Related to the property of a conservative vector field having a potential function.

Interaction between the geometry of the problem we want to study and the theory of forms.

Example: If we use the metric on $\mathbb{R}^n$ (or the dot product) we can turn 1-forms into vector fields

$$\theta = f_1\, dx_1 + \dots f_n dx_n \longmapsto \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ \vdots \\ f_n \end{bmatrix}}_{F}$$

$$\theta_p(V) = F_p \cdot V.$$

In our setting we want are interested in the connection between forms and complex valued functions so it is useful to introduce complex valued forms.

We define <u>complex</u> <u>valued</u> $k$-form to be $\Lambda^k_{\mathbb{C}}(U)$

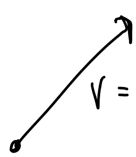
$$\theta = \sum f_I(p)\, dx_I$$

where $f_I : U \to \mathbb{C}$ are complex valued functions. (Algebra over the ring of smooth $\mathbb{C}$-valued functions on $U$.)

$\Lambda_{\mathbb{C}}(U)$

Example $\quad dz \stackrel{\text{def}}{=} dx + i\,dy$

$dx(v) = v_1 \;,\; dy(v) = v_2$



$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad dz[v] = v_1 + i v_2$

So $dz$ applied to a vector $v$ in $\mathbb{R}^2$ gives the vector interpreted as a complex number.

Exercise: $\underbrace{\frac{1}{z}\,dz}_{\substack{\text{complex} \\ \text{valued 1-form}}} \left(\text{or } \frac{dz}{z}\right) = \omega_1 + i\omega_2.$

(In particular $\frac{dz}{z}$ is closed but not exact.)

*Example.* Any $f = u + iv$ is a holomorphic function then

$$
\begin{aligned}
df &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\frac{\partial v}{\partial x}dx + i\frac{\partial v}{\partial y}dy \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + i\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad \updownarrow \text{CR} \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}\right)dy \\
&= f'\,dx + i \cdot f'\,dy \\
&= f'(dx + i\,dy) = f'\,dz.
\end{aligned}
$$

CR:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

---

$w = f(z)$ $\qquad$ $f^*(g(w)\,dw) = g(f(z)) \cdot f'(z)\,dz$

Same formula as the real case but $f'$ has a different interpretation.

$f$ holomorphic.

(fdz is closed.)

Theorem. $d(f\,dz) = 0$ iff $f$ satisfies the Cauchy-Riemann equations.

$$
\begin{aligned}
d(f\,dz) &= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\wedge dx \\
&\quad + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dx \\
&\quad + i\left(\frac{\partial u}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dy \\
&\quad - \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dy \\
&= \frac{\partial u}{\partial y}\,dy\wedge dx \\
&\quad + i\,\frac{\partial v}{\partial y}\,dy\wedge dx \\
&\quad + i\,\frac{\partial u}{\partial x}\,dx\wedge dy \\
&\quad - \frac{\partial v}{\partial x}\,dx\wedge dy \\
&= \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)dx\wedge dy
\end{aligned}
$$

Special case of the "fundamental theorem of calculus in $\mathbb{R}^n$".

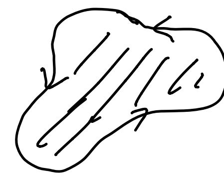
**Gauss' Theorem.** Let $U \subset \mathbb{R}^2$ be a subsurface with boundary and $\omega$ a 1-form then:

$$\int_U d\omega = \int_{\partial U} \omega$$

inherits an orientation

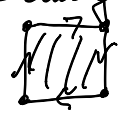
Example:

Choose a direction on $\partial U$ so that when facing forward your left hand points into $U$.



Corollary: homotopy invariance of the integral of a closed form.

apply to rectangle



Second criterion for exactness:

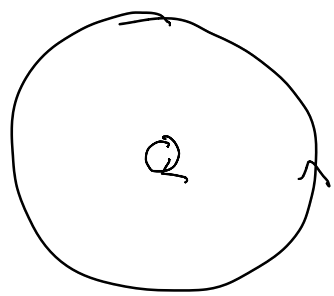
Thm,

If $U$ is simply connected (any homeo. to a disk) then and $\omega$ is a 1-form with $d\omega = 0$ on $U$ then $\omega = df$.

$\square$

Cauchy Integral Formula.

(In 1-complex variable want a proof from first principles since it is this result that leads to power series representation.)



$$\int \frac{f(z)}{z}\,dz$$

$f$ is holomorphic in $D-\{0\}$, $\frac{1}{z}$ is hol. in $D-\{0\}$. $\frac{f(z)}{z}$ is hol in $D-\{0\}$.

$$\int_{\substack{U \\ \shortparallel \\ D_R - D_r}} d\left(\frac{f(z)}{z}\,dz\right) = \int_{\partial U} \frac{f(z)}{z}\,dz$$

$$= \int_{\partial D_R} \frac{f(z)}{z}\,dz - \int_{\partial D_r} \frac{f(z)}{z}\,dz.$$

$$t \mapsto r\cos t + i\cdot r\sin t = \gamma(t) \qquad \gamma'(t) = -r(\sin t + i\cos t)$$

$$\int_{\partial D_r} \frac{f(z)}{z}\,dz = \int_t \frac{f(\gamma(t))}{\gamma(t)}\cdot \gamma'(t)\,dt$$

$=$

Write $f(z) = f(0) + (f(z) - f(0))$

$$\int_t O(r) \cdot r \, dt$$

$$= o(r) \to 0.$$

$$\frac{1}{2\pi i}\int \frac{f(0)}{z} dz + \int \frac{f(z)-f(0)}{z-0} dz$$

tends to 0.

$n \cdot f(0)$

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0} \to f'(0).$$

so this quotient is bounded for $r$ small.

---

Remarks: Winding number

$df$ is not a function.

Does not transform like a function.

Residue is a ~~property~~ of a holomorphic 1 form.

In 1-complex variable we identify $f$ and $f dz$ whenever it is convenient.

End here?